



Tata McGraw Hill's

A COURSE IN

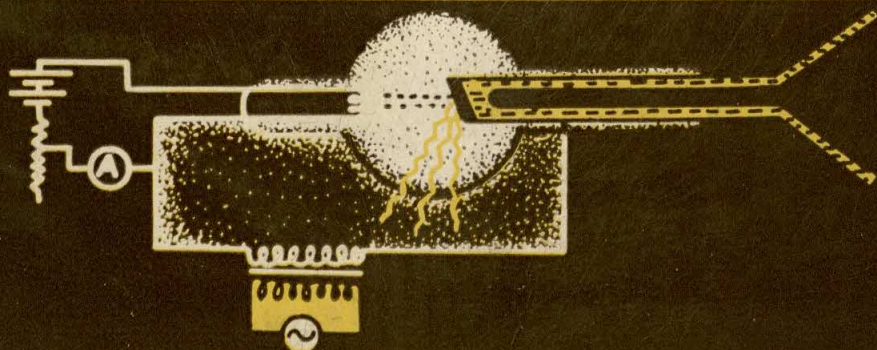
PHYSICS

FOR

IIT

Joint Entrance Examination

**AND OTHER ENGINEERING COLLEGE
ENTRANCE EXAMINATIONS**



A COURSE IN
PHYSICS

FOR THE STUDENT
AND THE TEACHER

955

A COURSE IN PHYSICS

For IIT-Joint Entrance Examination
and Other Engineering College
Entrance Examinations



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A Note to the Reader

This is an era of fierce competition, where only the best survive. A majority of our students sit for the Joint Entrance Examination (JEE) for admission to the most coveted Indian Institutes of Technology (IITs) located at New Delhi, Bombay, Kanpur, Madras and Kharagpur and Banaras Hindu University Institute of Technology at Varanasi. These institutes enjoy the highest reputation as centres of excellence in the fields of engineering and technology. The degrees awarded by the IITs are considered equivalent to those awarded by the leading engineering institutes of the world.

IIT-JEE is a highly competitive examination. A large number of students sit for this examination but only a small percentage of them get through. It is a fact that IIT-JEE is a very difficult examination and an organised effort is required to succeed in it. Keeping this in view, we have ventured to bring out this comprehensive physics book to help candidates prepare for the Physics paper of the IIT-JEE.

It must be recognized that a thorough preparation of the 10+2 pattern of examination is not sufficient for IIT-JEE which relies heavily on the application of the basic principles of the subject. Hence the need and importance of this book cannot be over-emphasized. This book is written bearing in mind the syllabus and pattern of the IIT-JEE.

The book is divided into 20 chapters which cover the entire Physics syllabus of IIT-JEE. The format of each chapter is as follows:

1. Review of basic concepts, laws, principles and definitions.
2. SI units of the relevant physical quantities.
3. Formulae to be used to solve numerical problems.
4. A large number of numerical problems with complete solutions including those from previous years' papers.
5. True-false statements with reasons. Multiple choice questions with answers key.
6. Numerical exercises with answers. These exercises are carefully designed to test the candidates' comprehension and application of the basic principles in each topic.

Candidates are advised to carefully go through the solved numerical problems and then solve all the numerical exercises at the end of each chapter. This will give them the necessary training and confidence so vital for IIT-JEE. To boost their confidence, candidates are advised to read Tata McGraw-Hill's Model Solutions to IIT-JEE.

This book is an outcome of the combined efforts of teachers who have had considerable experience and success in guiding students for IIT-JEE. It is hoped that candidates appearing for this examination will find this

book extremely useful for their preparation, and achieve the desired success. Since the IIT-JEE is an extremely tough examination, this book will help the student prepare for a lot of other engineering college-entrance examinations also.

All suggestions from teachers and students will be gratefully acknowledged.

THE PUBLISHERS

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Units and Dimensions

1.1 REVIEW OF BASIC CONCEPTS

1.1.1 The Measurement Process The process of measurement is basically a *comparison* process. To measure a physical quantity, we have to find out *how many times* a *standard amount* of that physical quantity is present in the quantity being measured. The number thus obtained is known as the *magnitude* and the standard chosen is called the *unit* of the physical quantity.

1.1.2 The Unit The unit of a physical quantity is a value of it in terms of which other quantities of that kind are expressed. It is chosen arbitrarily but then agreed upon universally.

1.1.3 The Standard The actual *physical embodiment* of the unit of a physical quantity is known as a *standard* of that physical quantity.

1.1.4 The General Conference of Weights and Measures The Conference General des Poids et Mesures (CGPM) or the General Conference of Weights and Measures was founded in 1876. It has its central office at Sevres near Paris in France. It is an international body that provides a common international forum for all scientists and engineers concerned with measurement.

1.1.5 The Syst  me Internationale d'Unit  s (The SI Units) The SI units were approved by the CGPM in 1960 and endorsed by both the International Organisation for Standardisation (ISO) and the International Electrotechnical Commission (IEC) in 1962. They possess a logical superiority over all other systems and are now used the world over. The SI units are a *coherent* system of units, i.e. in them the derived units are obtained from the basic or fundamental units by multiplication or division without introducing any numerical factors.

1.1.6 The Fundamental SI Units There are six primary or fundamental units in the SI. They are for length, mass, time, electric current, temperature and luminous intensity. In addition there is an additional unit for the amount of substance, two supplementary units for the angle in a plane and the solid angle and a large number of derived or secondary units—all logically and coherently based on the primary units.

1.1.7 Dimensions of a Physical Quantity The dimensions of a physical quantity are the *powers* to which the fundamental units must be raised to represent it.

Table 1.1 Fundamental SI Units

<i>Physical Quantity</i>	<i>Name of the Unit</i>	<i>Symbol</i>
Length	metre	m
Mass	kilogram	kg
Time	second	s
Electric current	ampere	A
Temperature	kelvin	K
Luminous intensity	candela	cd
Amount of substance	mole	mol
Angle in a plane	radian	rad
Solid angle	steradian	sr

1.1.8 Dimensional Formula The dimensional formula of a physical quantity is an expression telling us how and which of the fundamental quantities enter into the unit of that quantity.

It is customary to express the fundamental quantities by a capital letter, e.g. length (L), mass (M), time (T), electric current (I), temperature (K) and luminous intensity (C). We write appropriate powers of these capital letters within square brackets to get the dimensional formula of any given physical quantity.

1.1.9 Uses of Dimensions The concept of dimensions and dimensional formulae are put to the following uses:

- (i) Checking the results obtained
- (ii) Conversion from one system of units to another
- (iii) Deriving relationships between physical quantities
- (iv) Scaling and studying of models.

The underlying principle for these uses is the principle of homogeneity of dimensions. According to this principle, the 'net' dimensions of the various physical quantities on both sides of a permissible physical relation must be the same; also only dimensionally similar quantities can be added to or subtracted from each other.

1.1.10 Limitations of Dimensional Analysis

(i) The method of dimensional analysis does not give any information about pure numbers (like $1/2$, π , etc.) and non-dimensional constants. They have to be determined from experiments.

(ii) This method gives no information about dimensional constants such as the universal constant of gravitation (G) or Planck's constant (h), and where they have to be introduced.

(iii) This technique is useful only for deducing and verifying 'power relations'. Relationships involving exponential, trigonometric functions, etc. cannot be obtained or studied by this technique.

(iv) In this method, we compare the powers of the fundamental quantities (like M, L, T, etc.) to obtain a number of independent equations for finding the unknown powers. Since the total number of such equations cannot exceed the number of fundamental quantities, we cannot use this method to obtain the required relation if the quantity of interest depends upon more parameters than the number of fundamental quantities used.

(v) In many problems, it is difficult to guess the parameters on which the quantity of interest may depend. This requires a trained, subtle and intuitive mind.

1.2 SI UNITS OF DERIVED QUANTITIES

Table 1.2 gives the SI units of some derived physical quantities.

Table 1.2 SI Units of Some Derived Physical Quantities

<i>Physical Quantity</i>	<i>Name of SI Unit</i>	<i>Symbol</i>
Force	newton	N
Work, Energy	joule	J
Power	watt	W
Frequency	hertz	Hz
Electric charge	coulomb	C
Electric potential	volt	V
Electric capacitance	farad	F
Electric resistance	ohm	Ω
Magnetic flux	weber	Wb
Magnetic flux density	tesla	T
Inductance	henry	H
Illumination	lux	lx
Luminous flux	lumen	lm
Quantity of heat	joule	J
Velocity	metre per second	ms^{-1}
Acceleration	metre per second squared	ms^{-2}
Pressure	newton per square metre	Nm^{-2}
Moment of inertia	kilogram metre squared	kgm^2
Momentum	kilogram metre per second	kg ms^{-1}
Surface tension	newton per metre	Nm^{-1}
Elastic moduli	newton per square metre	Nm^{-2}
Specific heat capacity	joule per kilogram kelvin	$\text{J kg}^{-1} \text{K}^{-1}$
Specific latent heat	joule per kilogram	J kg^{-1}
Thermal conductivity	watt per metre kelvin	$\text{W m}^{-1} \text{K}^{-1}$
Quantity of light	lumen second	lm s
Luminance	candela per square metre	cd m^{-2}
Sound intensity	watt per square metre	W m^{-2}
Absolute permeability	henry per metre	H m^{-1}
Magnetic field strength	ampere per metre	A m^{-1}
Magnetic vector potential	weber per metre	Wb m^{-1}
Intensity of magnetisation	ampere per metre	A m^{-1}
Magnetic energy density	joule per cubic metre	J m^{-3}
Reluctance	ampere per weber	A Wb^{-1}
Absolute permittivity	farad per metre	F m^{-1}
Electric dipole moment	coulomb metre	C m
Electric field intensity	volt per metre	V m^{-1}
Electric polarisation	coulomb per square metre	C m^{-2}
Electrochemical equivalent	kilogram per coulomb	kg C^{-1}
Poynting vector	watt per metre squared	W m^{-2}

1.3 FORMULAE

Table 1.3 gives the dimensional formulae of some important derived physical quantities.

Table 1.3 Dimensional formulae of some Physical Quantities

Physical Quantity	Dimensional Formula	Physical Quantity	Dimensional Formula
Area	L^2	Capacitance	$M^{-1} L^{-2} T^2 Q^2$
Volume	L^3	Electric current	I or $Q T^{-1}$
Density	$M L^{-3}$	Electric potential	$M L^2 T^{-2} Q^{-1}$
Velocity	$L T^{-1}$	or	$M L^2 T^{-3} I^{-1}$
Acceleration	$L T^{-2}$	Electric field	$M L T^{-2} Q^{-1}$
Momentum	$M L T^{-1}$	or	$M L T^{-3} I^{-1}$
Force	$M L T^{-2}$	Inductance	$M L^2 Q^{-2}$
Energy, work	$M L^2 T^{-2}$	or	$M L^2 T^{-2} I^{-2}$
Power	$M L^2 T^{-3}$	Resistance	$M L^2 T^{-1} Q^{-2}$
Frequency	T^{-1}	or	$M L^2 T^{-3} I^{-2}$
Pressure	$M L^{-1} T^{-2}$	Magnetic flux	$M L^2 T^{-1} Q^{-1}$
Torque, couple	$M L^2 T^{-2}$	or	$M L^2 T^{-2} I^{-1}$
Moment of inertia	$M L^2$	Magnetic field	$L^{-1} T^{-1} Q$
Temperature	K	or	$L^{-1} I$
Heat energy	$M L^2 T^{-2}$	Magnetic induction	$M T^{-1} Q^{-1}$
Entropy	$M L^2 T^{-2} K^{-1}$	or	$M T^{-2} I^{-1}$
Specific heat capacity	$L^2 T^{-2} K^{-1}$	Permeability	$M L Q^{-2}$
Specific latent heat	$L^2 T^{-2}$	or	$M L T^{-2} I^{-2}$
Thermal conductivity	$M L T^{-3} K^{-1}$	Permittivity	$M^{-1} L^{-3} T^2 Q^2$
Electric charge	Q or IT	or	$M^{-1} L^{-3} T^4 I^2$

1.4 NUMERICAL EXAMPLES

Example 1: Given that 1 pound = 0.4536 kg and 1 foot = 0.3048 m prove by the method of dimensions that 1 HP = 746 W. You are given that 1 HP = 550 ft-pound/s = 550 × 32 ft-poundal/s.

Solution: The HP (horse power) is a unit of power whose primary unit in the FPS system (foot-pound-second system) is ft-poundal/s. We are required to convert this unit to the SI unit of power, the watt.

Now the dimensions of power are $M L^2 T^{-3}$. Let M_1 , L_1 and T_1 stand for the units of mass, length and time in the SI and let M_2 , L_2 and T_2 represent the corresponding units in the FPS system.

We assume that

$$1 \text{ HP} = x \text{ watt}$$

$$\text{i.e.} \quad 550 \times 32 \text{ ft-poundal/s} = x \text{ watt}$$

$$\text{or} \quad 550 \times 32 M_2 L_2^2 T_2^{-3} = x M_1 L_1^2 T_1^{-3}$$

$$\begin{aligned} \therefore x &= 550 \times 32 \left(\frac{M_2}{M_1} \right) \left(\frac{L_2}{L_1} \right)^2 \left(\frac{T_2}{T_1} \right)^{-3} \\ &= 550 \times 32 \left(\frac{0.4536 \text{ kg}}{1 \text{ kg}} \right) \left(\frac{0.3048 \text{ m}}{1 \text{ m}} \right)^2 \left(\frac{1 \text{ s}}{1 \text{ s}} \right)^{-3} \\ &= 550 \times 32 \times 0.4536 \times (0.3048)^2 \\ &\approx 746 \end{aligned}$$

Thus

$$1 \text{ HP} = 746 \text{ W}$$

Example 2: If the velocity of light (c), the constant of gravitation (G) and Planck's constant (h) be chosen as the fundamental units, find the dimensions of mass, length and time in the new system.

Solution: We first need to know the dimensions of c , G and h in terms of M , L and T .

$$c = \text{velocity of light} = LT^{-1}$$

Also from Newton's law:

$$F = G \frac{m_1 m_2}{r^2}$$

$$\begin{aligned} \therefore [G] &= \frac{[F][r^2]}{[m_1][m_2]} = \frac{[MLT^{-2}] \times [L^2]}{[M] \times [M]} \\ &= [M^{-1} L^3 T^{-2}] \end{aligned}$$

Finally, we know that the energy E of a photon of frequency ν is given by

$$E = h \nu$$

$$\begin{aligned} \therefore [h] &= \frac{[E]}{[\nu]} = \frac{[ML^2T^{-2}]}{[T^{-1}]} \\ &= [ML^2T^{-1}] \end{aligned}$$

Thus $LT^{-1} = c$, $M^{-1} L^3 T^{-2} = G$ and $ML^2 T^{-1} = h$. From these we have

$$\therefore h/(G/c) = M^2$$

$$\therefore M = c^{1/2} h^{1/2} G^{-1/2}$$

Using this value of M we have

$$L^2 T^{-1} = h/M = c^{-1/2} h^{1/2} G^{1/2}$$

$$\text{and } L T^{-1} = c$$

$$\therefore L = c^{-3/2} h^{1/2} G^{1/2}$$

Using this value of L along with $LT^{-1} = c$, we get

$$T = c^{-5/2} h^{1/2} G^{1/2}$$

Example 3: An apprentice engineer found that the volume V of water which passes any point of a canal during time t is related to the cross-sectional area A of the canal and the velocity u of the water by the relation $V = k A t u$ where k is a dimensionless constant. Test by the method of dimensions if the relation is correct.

Solution: We are given that

$$V = k A t u$$

The dimensions of the LHS are $[V] = [L^3]$

The dimensions of the RHS are

$$\begin{aligned} [A][t][u] &= [L^2][T][LT^{-1}] \\ &= [L^3] \end{aligned}$$

Since the dimensions of the two sides are the same, the given relation is dimensionally correct.

Example 4: By the method of dimensions, show that the energy per unit volume is equal to the pressure.

Solution: We know that the dimensions of energy, volume and pressure are respectively:

$$[\text{energy}] = [M L^2 T^{-2}]$$

$$[\text{Volume}] = [L^3]$$

$$[\text{Pressure}] = [\text{force}]/[\text{area}] = [M L T^{-2}]/[L^2] \\ = [M L^{-1} T^{-2}]$$

$$\text{Also } \frac{[\text{Energy}]}{[\text{Volume}]} = \frac{[M L^2 T^{-2}]}{[L^3]} = M L^{-1} T^{-2}$$

Thus it follows that dimensionally, energy per unit volume is equal to pressure.

Example 5: Show dimensionally that the frequency n of transverse waves in a string of length l and mass per unit length m under a tension T is given by

$$n = \frac{k}{l} \sqrt{\frac{T}{m}}$$

Solution: We are given that

$$n = f(l, T, m)$$

We assume that

$$n = k l^a T^b m^c$$

where k is a dimensionless constant and a , b and c are unknown powers to be determined. Writing the dimensions of all the quantities involved, we have

$$[T^{-1}] = [L]^a [M L T^{-2}]^b [M L^{-1}]^c$$

$$\text{or } T^{-1} = L^{a+b-c} M^{b+c} T^{-2b}$$

Equating powers of L , M and T on both sides, we have

$$a + b - c = 0$$

$$b + c = 0$$

$$-2b = -1$$

These give $b = 1/2$, $c = -1/2$ and $a = -1$

$$\therefore n = k l^{-1} T^{1/2} m^{-1/2}$$

$$= \frac{k}{l} \sqrt{\frac{T}{m}}$$

This is the required relation.

Example 6: Assuming that the mass m of the largest stone that can be moved by a flowing river depends upon the velocity v of water, its density ρ and

acceleration due to gravity g , show that m varies as the sixth power of the velocity of water in the river.

Solution: We are given that

$$m = f(v, \rho, g)$$

We assume that

$$m = k v^a \rho^b g^c$$

where k is a dimensionless constant and a , b and c are unknown powers to be determined. Substituting the dimensions of all the quantities involved, we have

$$[M] = [LT^{-1}]^a [M L^{-3}]^b [LT^{-2}]^c$$

Equating powers of M , L and T on both sides, we have

$$1 = b$$

$$0 = a - 3b + c$$

$$0 = -a - 2c$$

$\therefore c = -a/2$, so that

$$0 = a - 3 \times 1 - a/2$$

or $0 = a/2 - 3$

$\therefore a = 6$ and $c = -3$

Thus $m = k v^6 \rho g^{-3} = \frac{k \rho v^6}{g^3}$

or $m \propto v^6$, i.e. the mass of the largest stone moved is proportional to the sixth power of the velocity of water in the river.

Example 7: Given that the time period T of oscillation of a gas bubble from an explosion under water depends upon P , d and E where P is the static pressure, d the density of water and E is the total energy of explosion, find dimensionally a relation for T . (IIT 1981)

Solution: We are given that

$$T = f(P, d, E)$$

Assuming that $T = k P^a d^b E^c$ and substituting dimensions of all the quantities involved, we have

$$[T] = [M L^{-1} T^{-2}]^a [M L^{-3}]^b [M L^2 T^{-2}]^c$$

Equating powers of M , L and T on both sides, we have

$$a + b + c = 0$$

$$-a - 3b + 2c = 0$$

$$-2a - 2c = 1$$

Solving these equations, we get

$$a = -5/6, b = 1/2 \text{ and } c = 1/3$$

$$\therefore T = k P^{-5/6} d^{1/2} E^{1/3}$$

$$\text{or } T = k \left(\frac{\sqrt{d}^3 \sqrt[3]{E}}{\sqrt[6]{P^5}} \right)$$

This is the required relation for T .

Example 8: Experiments show that the frequency (n) of a tuning fork depends upon the length (l) of the prong, the density (d) and the Young's modulus (Y) of its material. From dimensional considerations, find a possible relation for the frequency of a tuning fork.

Solution: We are given that

$$n = f(l, d, Y)$$

Assuming that

$$n = k l^a d^b Y^c$$

and substituting dimensions of all the quantities involved, we have

$$[T^{-1}] = [L]^a [M L^{-3}]^b [M L^{-1} T^{-2}]^c$$

Equating powers of M , L and T on both sides, we have

$$b + c = 0$$

$$a - 3b - c = 0$$

$$-2c = -1$$

These give $c = 1/2$, $b = -1/2$ and $a = -1$

$$\therefore n = k l^{-1} d^{-1/2} Y^{1/2}$$

$$\text{or } n = \frac{k}{l} \sqrt{\frac{Y}{d}}$$

This is the required relation for the frequency of a tuning fork.

Example 9: It is known that the period T of a magnet of magnetic moment M vibrating in a uniform magnetic field of intensity H depends upon M , H and I where I is the moment of inertia of the magnet about its axis of oscillations. Show that

$$T = 2\pi \sqrt{\frac{I}{MH}}$$

Solution: We first note that the dimensions of I are $M L^2$. Also the magnetic moment has the units $A\text{-m}^2$ so that its dimensions can be written as $A L^2$ where A stands for the dimensions of the electric current. Finally the magnetic field intensity H has the units newton per ampere metre so that its dimensions can be written as

$$[H] = [M L T^{-2}] / [A] [L] = [M T^{-2} A^{-1}]$$

We now assume that

$$T = k I^a M^b H^c$$

Substituting dimensions of all the quantities involved, we have

$$\begin{aligned} [T] &= [M L^2]^a [A L^2]^b [M T^2 A^{-1}]^c \\ &= M^{a+c} L^{2a+2b} T^{-2c} A^{b-c} \end{aligned}$$

Equating powers of M, L and T and A on both sides, we have $a + c = 0$, $2(a + b) = 0$, $-2c = 1$, $b - c = 0$. From the first three, we get $c = -1/2$, $a = 1/2$ and $b = -1/2$,

These values are consistent with the fourth equation. Thus

$$a = 1/2, b = -1/2 \text{ and } c = -1/2$$

$$\therefore T = k I^{1/2} M^{-1/2} H^{-1/2} = k \sqrt{I/MH}$$

Experiments show that $k = 2\pi$. Therefore

$$T = 2\pi \sqrt{\frac{I}{MH}}$$

Example 10: The height h to which a liquid rises in a capillary tube of radius r depends, in addition to r , on (i) the surface tension σ of the liquid, (ii) the density ρ of the liquid and (iii) the acceleration due to gravity g . Is it possible to obtain dimensionally a relation for h without the experimental information that h is inversely proportional to r ? What is this relation?

Solution: We are given that

$$h = f(r, \sigma, \rho, g)$$

$$\text{Let } h = k r^a \sigma^b \rho^c g^d$$

Substituting the dimensions of all the quantities on both sides, we get

$$[L] = [L]^a [M T^{-2}]^b [M L^{-3}]^c [L T^{-2}]^d$$

Equating powers of M, L and T, we have

$$b + c = 0$$

$$a - 3c + d = 1$$

$$-2b - 2d = 0$$

We thus have only *three* equations for determining *four* unknowns. A complete solution is, therefore, not possible. We do need information about the value of one of the four unknowns a , b , c , and d .

The experimental information given to us is

$$h \propto 1/r$$

The power of r in the required relation is, therefore, -1 . Hence

$$a = -1$$

Using this value of a we have

$$b + c = 0$$

$$d - 3c = 2$$

$$b + d = 0$$

whence $b = 1, c = -1$ and $d = -1$.

$$\text{Thus } h = k r^{-1} \sigma^1 \rho^{-1} g^{-1} = \frac{k \sigma}{r \rho g}$$

which is the required relation.

1.5 TRUE-FALSE STATEMENTS WITH REASONS

In this section, we give below some statements. We have to decide whether they are *true* or *false* giving the reason in brief in support of our answer.

1. The size (u) of the unit of a physical quantity and its numerical magnitude (n) are related to each other by the relation

$$n u = \text{constant}$$

2. There is no way of defining a length standard in terms of a time standard and vice-versa.
3. On a logarithmic scale, the radius of the earth (6.4×10^6 m) is approximately half-way between: the radius of the nucleus ($\sim 10^{-15}$ m) and the radius of the observable universe ($\sim 10^{26}$ m).
4. All the following pairs of quantities have identical dimensions:
 - (a) Work and torque
 - (b) Planck's constant and angular momentum
 - (c) Pressure and Young's modulus
 - (d) Linear momentum and moment of a force
5. The universal gas constant R , occurring in the equation of state

$$PV = nRT$$

for n moles of an ideal gas, is a dimensional constant.

6. The relation

$$p = \frac{8 \eta l V}{\pi r^4}$$

is dimensionally permissible. Here p is the pressure difference at the ends of a capillary tube of radius r and length l through which a volume V of a liquid of coefficient of viscosity η is flowing per second.

7. The constant a in the Van der Waal equation of state for a real gas

$$\left(P + \frac{a}{V^2}\right)(V - b) = RT$$

must have the dimensions of pressure multiplied by the sixth power of the dimensions of length.

8. A pressure of 10^6 dyne/cm² is equivalent to 10^5 N/m².
9. The quantity $1/\sqrt{\mu_0 \epsilon_0}$ has the same dimensions as those of acceleration.
10. Out of (i) acceleration due to gravity, (ii) the surface tension of water, (iii) the weight of a standard kilogram mass and (iv) the velocity of light in vacuum, only the last one qualifies to be called a dimensional constant.

ANSWERS

1. True. The numerical magnitude (n) assigned to a given physical quantity is inversely proportional to the size (u) of the unit chosen, i.e.

$$n \propto 1/u$$

or

$$n u = \text{constant}$$

2. False. We can, for example, define a standard length as the length of a pendulum having a time period equal to the standard of time chosen.
3. True. The logarithms of 10^{-15} and 10^{26} are -15 and 26 respectively so that their midpoint is

$$\frac{-15 + 26}{2} = 5.5 \approx 6$$

Also the logarithm of the radius of the earth is of the order of 6.

4. False. The first three pairs have identical dimensions but the last pair, namely, linear momentum = $[M L T^{-1}]$ and moment of a force = $[M L^2 T^{-2}]$ have different dimensions.
5. False. R is not dimensionless; its dimensions are $[M L^2 T^{-2} K^{-1}]$.
6. True. The dimensions of the LHS are $[M L^{-1} T^{-2}]$ while those of the RHS are
$$\left[\frac{M L^{-1} T^{-1} \times L \times L^3 T^{-1}}{L^4} \right] \quad \text{or} \quad [M L^{-1} T^{-2}]$$
7. True. The quantity a/V^2 must have the dimensions of pressure as it is being added to pressure. The dimensions of a are, therefore, those of pressure multiplied by $(L^3)^2$, i.e. the sixth power of the dimensions of length.
8. True.

$$1 \text{ dyne} = 10^{-8} \text{ N and } 1 \text{ cm} = 10^{-2} \text{ m. Hence}$$

$$\begin{aligned} 10^6 \text{ dyne/cm}^2 &= 10 \text{ N}/10^{-4} \text{ m}^2 \\ &= 10^5 \text{ N/m}^2 \end{aligned}$$

9. False. The units of μ_0 (absolute permeability of vacuum) and ϵ_0 (absolute permittivity of vacuum) are henry per metre and farad per metre respectively. The units of $1/\sqrt{\mu_0 \epsilon_0}$ are then $(\text{m}/\sqrt{\text{henry} \times \text{farad}})$. Hence the dimensions of this quantity are

$$\begin{aligned} \frac{L}{(M L^2 T^{-2} I^{-2} \times M^{-1} L^{-2} T^4 I^2)^{1/2}} &= \frac{L}{\sqrt{T^2}} \\ &= L T^{-1} \end{aligned}$$

the same as those of velocity, not acceleration.

10. True. Out of the four dimensional quantities given, only the velocity of light in vacuum is invariable.

MULTIPLE CHOICE QUESTIONS

Select the best alternative in each of the following

1. Out of the following the only one that is not a unit of length is
 - (a) angstrom
 - (b) micron
 - (c) parsec
 - (d) radian
2. The unit of angular acceleration in the SI system is
 - (a) N kg^{-1}
 - (b) m s^{-2}
 - (c) rad s^{-2}
 - (d) N m kg^{-1}
3. Cesium clocks have an accuracy of one part in 10^{11} . Two such clocks operating at this accuracy for 5000 years will record a difference of
 - (a) practically zero second
 - (b) one minute
 - (c) one second
 - (d) one mean solar day
4. The dimensions $M L^{-1} T^{-2}$ can correspond to
 - (a) moment of a force
 - (b) surface tension
 - (c) modulus of elasticity
 - (d) coefficient of viscosity
5. A dimensionally consistent relation for the volume V of a liquid of coefficient of viscosity η flowing per second through a tube of radius r and length l and having a pressure difference p across its end is
 - (a) $V = \pi p r^4 / 8 \eta l$
 - (b) $V = \pi \eta l / 8 p r^4$
 - (c) $V = 8 p \eta l / \pi r^4$
 - (d) $V = \pi p \eta / 8 l r^4$
6. Out of the following the only pair that does not have identical dimensions is
 - (a) angular momentum and Planck's constant
 - (b) moment of inertia and moment of a force
 - (c) work and torque
 - (d) impulse and momentum

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7. The SI unit of Stefan's constant is
 (a) $\text{W s}^{-1} \text{m}^{-2} \text{K}^{-4}$ (b) $\text{J s m}^{-2} \text{K}^{-4}$
 (c) $\text{J s}^{-1} \text{m}^{-2} \text{K}^{-1}$ (d) $\text{W m}^{-2} \text{K}^{-4}$
8. The SI unit of magnetic permeability is
 (a) A m^{-1}
 (b) A m
 (c) H m^{-1}
 (d) No unit; it is a dimensionless number
9. The ratios L/R and RC (where L , C and R stand for inductance, capacitance and resistance respectively) have the same dimensions as those of
 (a) velocity (b) acceleration
 (c) time (d) force
10. The dimensions of electric susceptibility are
 (a) $\text{M}^0 \text{L}^{-1} \text{T}^0 \text{A}$
 (b) $\text{M}^0 \text{L}^{-2} \text{T}^0 \text{A}^2$
 (c) $\text{M L T}^{-2} \text{A}$
 (d) no dimensions; it is a dimensionless number.

NUMERICAL EXERCISES

1. Show that the SI unit of torque is 10^7 times its CGS unit.
2. Find the value of an impulse of 500 dyne/sin a system based on the metre, the kilogram and the minute as fundamental units.
3. If the density of mercury at 0°C ($= 13.6 \times 10^3 \text{ kg m}^{-3}$), the acceleration due to gravity ($= 9.81 \text{ m s}^{-2}$) and the normal atmospheric pressure ($= 1.01 \times 10^5 \text{ N m}^{-2}$) are taken as the fundamental units, what will be the units of mass, length and time?
4. The basic mass, length and time standards in two different systems of units are related by constant factors such that
 (velocity) $v' = v \lambda^2 / \tau$
 (acceleration) $a' = a \lambda \tau$
 (force) $F' = F / \tau \lambda$

Find the relation between the mass, length and time standards of the two systems.

5. By the method of dimensions, test the correctness of the equation

$$\delta = M g l^3 / 4 b d^3 Y$$

where δ is the depression produced in the middle of a beam of length l , breadth b and thickness d placed symmetrically on two knife edges at its ends and loaded in the middle by a mass M . Y stands for the Young's modulus of the material of the beam.

6. Prove dimensionally that the retarding force F on a spherical body of radius a moving with a velocity v in a viscous medium of coefficient of viscosity η is given by

$$F = k a \eta v$$

7. The torsional constant τ (i.e. couple per unit twist) of a wire depends on (i) its length l , (ii) its radius r and (iii) the modulus of rigidity η of its material. Obtain dimensionally a relation for the torsional constant.
8. A student gave the following formula for the time of oscillation t of a small drop of a liquid of radius r

$$t = k \sqrt{\sigma / \rho} r^3$$

where ρ and σ refer to the density and surface tension of the liquid. Is this relation dimensionally correct? If not, deduce the correct relation assuming that t depends on these very factors.

9. Reynold's number N_R is known to be a dimensionless number that determines the conditions of steady or laminar flow of a viscous liquid through a pipe. Given that N_R depends on the density ρ , the coefficient of viscosity η and the average speed v of the liquid as well as the radius r of the pipe, is it possible to obtain dimensionally a relation for N_R ? Given the additional information that $N_R \propto r$, show that

$$N_R = k \frac{r \rho v}{\eta}$$

10. The magnitude of the restoring torque $\vec{\tau}$ acting on an electric dipole of dipole moment $\vec{p}$ kept in uniform electric field $\vec{E}$ is found to depend on (i) the magnitude of $\vec{p}$ and $\vec{E}$ and (ii) $\sin \theta$ where θ is the angle between $\vec{p}$ and $\vec{E}$. Obtain dimensionally a relation for the magnitude of $\vec{\tau}$.

ANSWERS

MULTIPLE CHOICE QUESTIONS

- | | |
|--------|---------|
| 1. (d) | 2. (c) |
| 3. (c) | 4. (c) |
| 5. (a) | 6. (b) |
| 7. (d) | 8. (c) |
| 9. (c) | 10. (d) |

NUMERICAL EXERCISE

- 0.3 new units
- $1.20 \times 10^3 \text{ kg}$, 0.757 m , 0.278 s
- $l' = l \lambda / \tau^3$, $t' = t \lambda \tau^2$, $m' m / \lambda^2 \tau^2$
- $\tau = k \eta r^4 / 2l$
- No, $t = k \sqrt{\rho r^3 / \sigma}$
- No
- $\tau = p E \sin \theta$

Kinematics in One and Two Dimensions

2.1 REVIEW OF BASIC CONCEPTS

2.1.1 Particle An object of negligible mass is known as a particle.

2.1.2 Motion When a body's position with respect to its immediate surroundings changes with time, it is said to be in motion. The concept of motion is a relative one and a body that may be in motion relative to one reference system, may be at rest relative to another.

2.1.3 Reference Systems The motion of a particle is always described with respect to a reference system. A reference system is 'made' by taking an arbitrary point as origin and imagining a co-ordinate system to be attached to it. This co-ordinate system chosen for a given problem constitutes the reference system for it.

We generally choose a co-ordinate system attached to the earth as the reference system for most of the problems.

2.1.4 Position Vector The position vector of a particle at any instant is the directed line segment drawn from the origin to its position in space at that instant.

2.1.5 Trajectory The path or trajectory (rectilinear or curved) of a particle is the curve described by it as it moves in space with respect to a chosen reference system. We call the motion of a particle as *rectilinear* if its trajectory is a straight line.

2.1.6 Displacement The displacement of a particle between two given time instants is a vector having a magnitude equal to the (shortest) distance moved by the particle between the two time instants. It is given by the vector equal to the difference between its position vectors at the 'final' and the 'initial' instants. We usually denote it by s (Fig. 2.1).

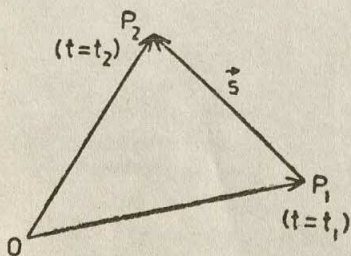


Fig. 2.1

2.1.7 Velocity The time rate of change of the displacement vector of a particle gives its velocity (v). Thus

$$v = \frac{ds}{dt}$$

The magnitude of the velocity vector gives the speed of the particle. Thus the velocity tells us about the speed and the direction of motion of a particle.

2.1.8 Acceleration A particle is said to be accelerated if its velocity changes with time. The acceleration vector (a) equals the time rate of change of the velocity of the particle

$$a = \frac{dv}{dt}$$

The magnitude of the acceleration vector equals the time rate of change of the speed of the particle.

2.1.9 One-Dimensional Motion With Constant Acceleration A particle is said to be moving with a constant acceleration if its acceleration does not change with time. A particle moving along only *one axis* of a co-ordinate system with uniform acceleration is said to be having a one-dimensional motion with constant acceleration.

Equations of Motion The equations of motion in one-dimension with constant acceleration are

$$v = u + at$$

$$s = ut + \frac{1}{2}at^2$$

$$v^2 - u^2 = 2as$$

where u = initial velocity of the particle at $t = 0$

v = final velocity of the particle after a time t

a = constant acceleration of the particle

and s = distance moved by the particle in time t

Kinematical Graphs The 'displacement-time' and the 'velocity-time' graphs of a particle are often used to provide us with a visual representation of the motion of the particle. The 'shape' of the graphs depends on the initial 'co-ordinates' and the 'nature' of the acceleration of the particle (Fig. 2.2).

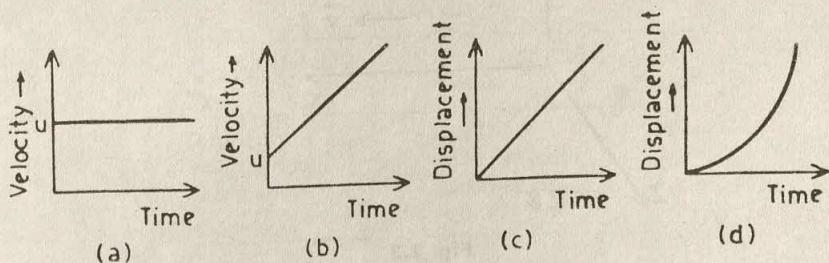


Fig. 2.2 Curves (a) and (c) represent motion with a constant speed u . Curves (b) and (d) represent motion with a uniform acceleration a starting with an initial speed u .

The following general results are always valid

- (i) The slope of the displacement-time graph at any instant gives the speed of the particle at that instant.

- (ii) The slope of the velocity-time graph at any instant gives the acceleration of the particle at that instant.
- (iii) The area enclosed by the velocity-time graph, the time-axis and the two ordinates at time instants t_1 and t_2 gives the distance moved by the particle in the time interval from t_1 to t_2 .

These results are just the equivalents of the mathematical relations:

$$v = \frac{ds}{dt}, a = \frac{dv}{dt}$$

$$S_{t_1 \rightarrow t_2} = \int_{t_1}^{t_2} v dt$$

2.1.10 Relative Motion We know that the concepts of 'rest' and 'motion' are relative and the description of the motion of a particle depends on the reference system used. In the domain of Newtonian mechanics (i.e. where all speeds involved are much less than c , the speed of light in vacuum), we have a very simple transformation equation relating the results of observations of one observer with those of another when the reference systems of the two observers are accelerated with respect to each other. Thus, for two observers O and O' (where O is moving with a constant speed v along the x -axis relative to O'), we have

$$x'' = x - vt$$

$$y' = y$$

$$z' = z$$

and $V = V' + v$

where (x, y, z) and (x', y', z') are the co-ordinates of a particle with respect to the reference systems of observers O and O' (Fig. 2.3).

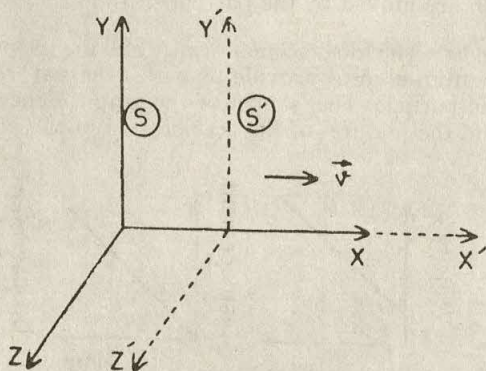


Fig. 2.3

2.1.11 Projectile Motion The projectile is a general name given to an object that is given an initial velocity and which subsequently follows a path determined by the gravitational force acting on it and by the frictional resistance of the air. The path followed by a projectile is called its *trajectory*. If the resistance of the air is not negligible, the trajectory of a projectile is quite complex. However, we usually neglect the air resistance and assume that motion takes place in empty space.

Equations of Projectile Motion The general case of projectile motion corresponds to that of an object that has been given an initial velocity v_0 , at some angle θ above (or below) the horizontal. The horizontal and vertical displacements x and y are given by

$$x = (v_0 \cos \theta) t$$

and
$$y = (v_0 \sin \theta) t - \frac{1}{2} g t^2$$

The equation of the trajectory of the projectile is obtained by eliminating t from these equations

$$y = (\tan \theta) x - \frac{g}{2 (v_0 \cos \theta)^2} x^2$$

This being the equation of a parabola, the trajectory of a projectile is a *parabolic curve*.

Time of Flight The time taken by a projectile to return to its initial elevation after projection is known as its time of flight (T). It is given by

$$T = \frac{2 v_0 \sin \theta}{g}$$

Horizontal Range The horizontal displacement undergone by a projectile when it returns to its initial elevation is called its horizontal range (R).

$$R = \frac{v_0^2 \sin 2\theta}{g}$$

For a given angle of elevation, the horizontal range is proportional to the square of the initial velocity. Also, in the absence of air resistance, the maximum horizontal range is attained with an angle of elevation of 45° .

Maximum Height The maximum height h of a projectile is reached at a time when its vertical velocity component has decreased to zero.

$$h = \frac{v_0^2 \sin^2 \theta}{2g}$$

2.2 SI UNITS

Table 2.1

Physical Quantity	SI Unit	
	Name	Symbol
Displacement	metre	m
Velocity, speed	metre per second	ms^{-1}
Acceleration	metre per second squared	ms^{-2}

2.3 FORMULAE

$$1. \quad v = \frac{ds}{dt} \quad a = \frac{dv}{dt}$$

s = displacement, v = velocity, a = acceleration

- The equations of motion in one-dimension with constant acceleration are (the symbols have their usual meaning)

$$v = u + a t$$

$$s = u t + \frac{1}{2} a t^2$$

$$v^2 - u^2 = 2 a s$$

Also $S_{t_1 \rightarrow t_2} = \int_{t_1}^{t_2} v dt$

3. The horizontal and vertical displacements x and y of a projectile at a time t are

$$x = (v_0 \cos \theta) t$$

$$y = (v_0 \sin \theta) t - \frac{1}{2} g t^2$$

where v_0 = initial velocity and θ = angle of projection

Equation of trajectory

$$y = (\tan \theta) x - \frac{g}{2 (v_0 \cos \theta)^2} x^2$$

Time of flight

$$t = \frac{2 v_0 \sin \theta}{g}$$

Horizontal range

$$R = \frac{v_0^2 \sin 2\theta}{g}$$

Maximum height

$$h = \frac{v_0^2 \sin^2 \theta}{2g}$$

2.4 NUMERICAL EXAMPLES

Example 1: A steel sphere is held by an electromagnet and when the current is switched off, it sets 'on' a stroboscopic photography timing set-up. It is found that the steel sphere, starting from rest, takes 450 millisecond to fall through one metre. What value does this give to the acceleration due to gravity? How far does the sphere fall in the first 200 millisecond?

Solution: Let ' g ' be the value of the acceleration due to gravity. We are given that the steel sphere falls 1 m, from rest, in 450 millisecond. Thus

$$u = 0, a = g, t = 450 \times 10^{-3} \text{ s and } s = 1 \text{ m}$$

Using $s = ut + \frac{1}{2} at^2$

we have $1 \text{ m} = 0 \times t + \frac{1}{2} g \times (450 \times 10^{-3} \text{ s})^2$

$$\therefore g = \frac{2}{(0.450)^2} \text{ ms}^{-2}$$

$$= 9.876 \text{ ms}^{-2}$$

The distance moved in the first 200 millisecond is

$$\begin{aligned} s &= 0 \times 0.200 + \frac{1}{2} \times 9.876 \times (0.200)^2 \text{ m} \\ &= 0.1975 \text{ m} \end{aligned}$$

Example 2: A stone is dropped from the top of a tall cliff and ' n ' second later another stone is thrown vertically downwards with a velocity of ' u '

m/s. How far below the top of the cliff will the second stone overtake the first?

Solution: The second stone will 'catch up' with the first stone when the distance covered by it in $(t - n)$ second will equal the distance covered by the first stone in t second.

Now distance covered by the first stone in t second $= \frac{1}{2}gt^2$ and distance covered by the second stone in $(t - n)$ second

$$= u(t - n) + \frac{1}{2}g(t - n)^2$$

$$\therefore \frac{1}{2}gt^2 = u(t - n) + \frac{1}{2}g(t - n)^2$$

$$\text{or } \frac{1}{2}g[t^2 - (t - n)^2] = u(t - n)$$

$$\text{or } \frac{1}{2}g[(2t - n)n] = u(t - n)$$

$$\text{or } gnt - \frac{1}{2}gn^2 = ut - un$$

$$\text{or } t(gn - u) = (\frac{1}{2}gn - u)n$$

$$\text{or } t = \frac{n(\frac{1}{2}gn - u)}{(gn - u)}$$

The distance covered by the first stone in this time is

$$h = \frac{1}{2}gt^2 = \frac{1}{2}g \left[\frac{n(\frac{1}{2}gn - u)}{(gn - u)} \right]^2$$

Thus the second stone will overtake the first a distance

$$\frac{1}{2}g \left[\frac{n(\frac{gn}{2} - u)}{(gn - u)} \right]^2$$

below the top of the cliff.

Example 3: A smooth frictionless track PQ making an angle θ with the vertical is made between the highest point P and any other point Q of a vertical circle of radius R . Assuming a particle to start from P with a velocity u , calculate the time taken by it to reach Q . What happens if $u = 0$?

Solution: If we drop a perpendicular ON from O on PQ ,

we have $\frac{PN}{OP} = \cos \theta$ so that PN

$$= a \cos \theta$$

The length of the track PQ is, therefore, $2a \cos \theta$. Also the acceleration of the particle down the track is $g \cos \theta$. If t is the time taken by the particle to go down the track, we have

$$2a \cos \theta = ut + \frac{1}{2}(g \cos \theta)t^2$$

$$\therefore g \cos \theta t^2 + 2ut - 4a \cos \theta = 0$$

$$\therefore t = \frac{-2u \pm \sqrt{4u^2 + 16ag \cos^2 \theta}}{2g \cos \theta}$$

$$= \frac{-u \pm \sqrt{u^2 + 4ag \cos^2 \theta}}{g \cos \theta}$$

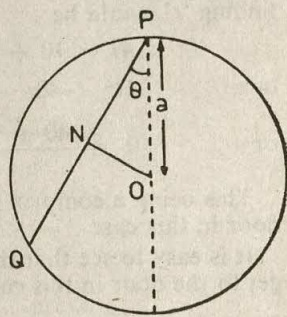


Fig. 2.4

Neglecting the $-ve$ sign, the required time taken by the particle to go down the track is

$$t = \frac{-u + \sqrt{u^2 + 4ag \cos^2 \theta}}{g \cos \theta}$$

If $u = 0$, $t = t_0 = \frac{\sqrt{4ag \cos^2 \theta}}{g \cos \theta} = 2 \sqrt{\frac{a}{g}}$

This is independent of θ , i.e. the position of the point Q . Thus the time taken by a particle to slide down a frictionless track originating from the highest point of a vertical circle is independent of the 'chord' followed.

Example 4: A man runs at a speed of 4 m/s to overtake a standing bus. When he is 6 m behind the door (at $t = 0$), the bus moves forward and continues with a constant acceleration of 1.2 m/s^2 . How long does it take for the man to 'get to' the door? If he was initially 10 m behind the door, can he catch the bus?

Solution: Let the man take t s to get at the door. The distance moved by the man in t s should therefore be 6 m more than the distance moved by the bus in the same time.

Now distance moved by the man in ' t ' second $= 4t$ and distance moved by the bus in ' t ' second $= 0 + \frac{1}{2} \times 1.2 \times t^2 = 0.6 t^2$

$$\therefore 4t = 6 + 0.6 t^2$$

$$\text{or } 6t^2 - 40t + 60 = 0$$

$$\text{or } t = \frac{40 \pm \sqrt{1600 - 1440}}{12}$$

$$= \frac{40 \pm \sqrt{160}}{12}$$

$$= 4.387 \text{ s or } 2.27 \text{ s}$$

The man will therefore get to the door after 2.27 s.

If the man was initially 10 m behind the door, the required equation for finding ' t ' would be

$$4t = 10 + 0.6 t^2$$

$$\text{or } 6t^2 - 40t + 100 = 0$$

$$\text{or } t = \frac{40 \pm \sqrt{1600 - 2400}}{12}$$

This being a complex number, it means that the man cannot get to the door in this case.

It is easy to see that the 'maximum distance' from which the man can get to the door in this case is nearly 6.67 m.

Example 5: From the top of a tower of height 50 m, a ball is thrown vertically upwards with a certain velocity. It hits the ground 10 s after it is thrown up. With what velocity is the ball projected? How much time does it take to cover a distance AB where A and B are two points 20 m and 35 m below the edge of the tower?

Solution: We take the upward direction as positive. If u is the velocity with which the ball is projected upwards, we have

$$\text{initial velocity of the ball} = +u$$

(uniform) acceleration acting on the ball = $-g = -9.8 \text{ m/s}^2$; distance to be moved by the ball before it hits the ground = -50 m and time taken = 10 s .

Using $s = ut + \frac{1}{2}at^2$, we have

$$-50 = u \times 10 + \frac{1}{2}(-9.8)10^2$$

$$\therefore u = \frac{490 - 50}{10} \text{ ms}^{-1} = 44 \text{ ms}^{-1}$$

Now calling t_1 and t_2 as the times taken by the ball to reach points A and B, we have

$$-20 = 44 t_1 + \frac{1}{2}(-9.8) t_1^2$$

$$-35 = 44 t_2 + \frac{1}{2}(-9.8) t_2^2$$

Solving for t_1 and t_2 , we get

$$t_1 = \frac{44 \pm \sqrt{44^2 + 4 \times 20 \times 4.9}}{9.8} \approx 9.4 \text{ s}$$

$$t_2 = \frac{44 \pm \sqrt{44^2 + 4 \times 35 \times 4.9}}{9.8} \approx 9.7 \text{ s}$$

Time taken to cover the distance AB

$$= (9.7 - 9.4) \text{ s}$$

$$= 0.3 \text{ s}$$

Example 6: In an experimental demonstration of the concept of terminal velocity, a small steel ball is dropped from a height of 0.5 m into a tall glycerine jar. It hits the glycerine with a certain velocity and then sinks to the bottom with the same constant velocity. It reaches the bottom 0.5 s after it is dropped. What is the height of glycerine in the jar? What is the average velocity of the ball over the whole journey?

Solution: The 'terminal velocity' acquired by the ball as a result of its 'freefall' through 0.5 m is given by

$$v^2 - 0^2 = 2 \times 9.8 \times 0.5$$

or $v = \sqrt{9.8} \text{ ms}^{-1}$

$$= 3.13 \text{ ms}^{-1}$$

Also the time taken by the ball to make this 'fall' is given by

$$v = 0 + 9.8 t$$

or $t = \frac{v}{9.8} \text{ s} = \frac{3.13}{9.8} \text{ s}$

$$\approx 0.32 \text{ s}$$

Let 'h' be the height of glycerine in the jar. The time taken by the ball to reach the bottom is then

$$\frac{h}{v} = \frac{h}{3.13} \text{ s}$$

we thus have $\frac{h}{3.13} + 0.32 = 0.5$

$$\therefore h = 0.18 \times 3.13 \text{ m} \approx 0.56 \text{ m}$$

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Thus the height of glycerine in the jar is nearly 0.56m. Now,

$$\begin{aligned}\text{Average velocity} &= \frac{\text{Total distance covered}}{\text{Total time taken}} \\ &= \frac{0.5 + 0.56}{0.5} \text{ ms}^{-1} = 2.12 \text{ ms}^{-1}\end{aligned}$$

Example 7: A ball of mass 100 g is projected vertically upwards from the ground with a velocity of 49 m/s. At the same time another identical ball is dropped from a height of 98 m to fall freely along the same path as followed by the first ball. After sometime, the two balls collide and stick together and finally fall together. Find the time of flight of the masses.

(IIT 1985)

Solution: We first find *when* and where the two balls collide. Let them collide at an instant t seconds after they start their respective motion. Clearly the two balls are at the same height above the ground at this instant.

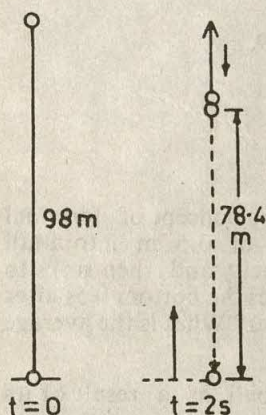


Fig. 2.5

The height of the first ball after t seconds

$$= 49t - \frac{1}{2} \times 9.8 t^2 = 4.9t(10 - t)$$

Also the height of the second ball after t seconds

$$= 98 - \text{downward distance moved by it in } t \text{ seconds}$$

$$= 98 - \frac{1}{2} \times 9.8 t^2 = 4.9(20 - t^2)$$

$$\therefore 4.9t(10 - t) = 4.9(20 - t^2)$$

$$\text{or } 10t - t^2 = 20 - t^2$$

$$\text{or } t = 2 \text{ s}$$

The balls thus collide two seconds after the start of their motion. Their velocities at this instant are

$$\begin{aligned}\text{First ball: } v_1 &= (49 - 9.8 \times 2) \text{ m/s} \\ &= 29.4 \text{ m/s directed upwards}\end{aligned}$$

$$\begin{aligned}\text{Second ball: } v_2 &= (0 + 9.8 \times 2) \text{ m/s} \\ &= 19.6 \text{ m/s directed downwards}\end{aligned}$$

If V is the velocity of the combined mass of the two balls after they stick together following their collision, we have, by principle of conservation of momentum,

$$200 \times V = 100 \times 29.4 - 100 \times 19.6$$

$$\therefore V = 4.9 \text{ m/s}$$

The 'combined mass' thus moves upward, after collision, with a velocity of 4.9 m/s. Its height above the ground at this instant is (considering the position of either of the two balls before collision)

$$(98 - \frac{1}{2} \times 9.8 \times 2^2) \text{ m} = (98 - 19.6) \text{ m} = 78.4 \text{ m}$$

We can now find the time t' taken by the 'combined mass' of the two balls to fall to ground. We have for this 'combined mass'

$$u = 4.9 \text{ m/s, } s = -78.4 \text{ m}$$

$$a = -g = -9.8 \text{ m/s}^2$$

$$\therefore -78.4 = 4.9t' + \frac{1}{2}(-9.8)t'^2$$

$$\text{or } t'^2 - t' - 16 = 0$$

$$\begin{aligned} \text{or } t' &= \frac{1 \pm \sqrt{1+64}}{2} = \frac{1 \pm 8.06}{2} \\ &= 4.53 \text{ s (leaving out the negative solution)} \end{aligned}$$

The 'combined mass' thus takes 4.53 s to fall to the ground. Since the balls collided 2 s after they started their motion, their total time of flight is

$$(2 + 4.53) \text{ s} = 6.53 \text{ s}$$

Example 8: A stone is dropped from the top of a cliff of height H . ' n ' seconds later, a second stone is projected downwards from the same cliff with a vertically downward velocity u . Show that the two stones will reach the bottom of the cliff together if

$$8H(u - gn)^2 = gn^2(2u - gn)^2$$

What can you say about the limiting value of ' n '?

Solution: The time taken by the first stone to reach the bottom of the cliff is $\sqrt{2H/g}$

Since the second stone is projected ' n ' seconds later, it must cover the same distance in a time $\left(\sqrt{\frac{2H}{g}} - n\right)$. We thus have

$$\begin{aligned} H &= u \left[\sqrt{\frac{2H}{g}} - n \right] + \frac{1}{2} g \left[\sqrt{\frac{2H}{g}} - n \right]^2 \\ &= u \sqrt{\frac{2H}{g}} - un + \frac{1}{2} g \left[\frac{2H}{g} + n^2 - 2n \sqrt{\frac{2H}{g}} \right] \\ &= (u - gn) \sqrt{\frac{2H}{g}} + \left(H + \frac{gn^2}{2} - un \right) \end{aligned}$$

$$\text{or } (u - gn) \sqrt{\frac{2H}{g}} = \frac{n}{2} (2u - gn)$$

$$\text{or } \sqrt{\frac{2H}{g}} = \frac{n(2u - gn)}{2(u - gn)}$$

$$\text{or } \frac{2H}{g} = \frac{n^2}{4} \left(\frac{2u - gn}{u - gn} \right)^2$$

$$\text{or } H = \frac{gn^2}{8} \left(\frac{2u - gn}{u - gn} \right)^2$$

$$\text{or } 8H(u - gn)^2 = gn^2(2u - gn)^2$$

The two stones thus reach the bottom together if

$$8H(u - gn)^2 = gn^2(2u - gn)^2$$

Since the second stone has to reach the bottom in a time

$$\left(\sqrt{\frac{2H}{g}} - n \right)$$

$$\text{we must have } n \leq \sqrt{\frac{2H}{g}}$$

Thus the limiting value of n is $\sqrt{\frac{2H}{g}}$

Example 9: At time $t = 0$, the driver of a train moving at a speed v_1 sights another train at a distance x ahead of him moving at a speed v_2 ($v_2 < v_1$). He immediately applies brakes that give his train a constant retardation a . What must be the critical value of x that will just avert a collision?

Solution: We first calculate the time t in which the speed of the first train moving at an initial speed v_1 and having a constant retardation a is slowed down to a value v_2 , the speed of the second train. We have

$$v_2 = v_1 - at \quad \text{or} \quad t = \frac{v_1 - v_2}{a}$$

We next calculate the distances moved by the two trains in this time. A collision would be just averted if the first train covers a distance that is just x more than the distance moved by the second train in this time. We have,

$$\text{Distance moved by the first train in time } t \left(= \frac{v_1 - v_2}{a} \right)$$

$$= v_1 t - \frac{1}{2} at^2$$

$$= v_1 \left(\frac{v_1 - v_2}{a} \right) - \frac{1}{2} a \left(\frac{v_1 - v_2}{a} \right)^2$$

$$= \left(\frac{v_1 - v_2}{a} \right) \left[v_1 - \frac{v_1 - v_2}{2} \right]$$

$$= \left(\frac{v_1 - v_2}{a} \right) \left(\frac{v_1 + v_2}{2} \right) = \frac{v_1^2 - v_2^2}{2a}$$

$$\text{Distance moved by the second train in time } t \left(= \frac{v_1 - v_2}{a} \right)$$

$$= v_2 t$$

$$= v_2 \left(\frac{v_1 - v_2}{a} \right)$$

Thus a collision is just averted if

$$x = \frac{v_1^2 - v_2^2}{2a} - \frac{v_2 (v_1 - v_2)}{a}$$

$$= \left(\frac{v_1 - v_2}{2a} \right) [v_1 + v_2 - 2v_2]$$

$$= \left(\frac{v_1 - v_2}{2a} \right) (v_1 - v_2)$$

$$= \frac{(v_1 - v_2)^2}{2a}$$

This is the required critical separation.

Example 10: A car, starting from rest, first moves with an acceleration of 5 ms^{-2} for some time and then, after moving with a uniform speed for some time, starts decelerating at the same rate to come to rest in a total time of 25 s. If the average velocity of the car over the whole journey is 20 ms^{-1} , for how long does it move with a uniform speed?

Solution: It is clear that the times taken during the first and the third parts of the journey must be the same. Let each of these times be t_1 . Also let t_2 be the time for which the car moves with a uniform speed. We then have

$$t_1 + t_2 + t_1 = 25 \text{ s} \quad \text{or} \quad t_2 = 25 - 2t_1$$

The speed v at the end of the first part of the journey is given by

$$v = 0 + 5 \times t_1 = 5t_1$$

The distance moved during this part is

$$\begin{aligned} \frac{1}{2} \times 5 \times t_1^2 \\ = 2.5 t_1^2 \end{aligned}$$

This is also the distance moved during the last part of the journey. The distance moved during the second part when the car is moving with a uniform speed v for a time t_2 is $v t_2$. This equals

$$(5 t_1) t_2 \quad \text{or} \quad 5 t_1 t_2$$

The total distance moved is thus

$$2.5 t_1^2 + 5 t_1 t_2 + 2.5 t_1^2$$

$$\text{or} \quad 5 t_1 (t_1 + t_2)$$

The average speed over the whole journey is, therefore, $\frac{5 t_1 (t_1 + t_2)}{25}$

$$\text{Hence} \quad \frac{t_1 (t_1 + t_2)}{5} = 20$$

$$\text{or} \quad t_1^2 + t_1 (25 - 2t_1) - 100 = 0$$

$$\text{or} \quad -t_1^2 + 25 t_1 - 100 = 0$$

$$\text{or} \quad (t_1 - 20) (t_1 - 5) = 0$$

$$\therefore \quad t_1 = 20 \text{ s} \quad \text{or} \quad t_1 = 5 \text{ s}$$

The solution $t = 20$ s is unacceptable as that would make the total time of the journey exceed 25 s. Hence

$$t_1 = 5 \text{ s}$$

It follows then that

$$\begin{aligned} t_2 &= (25 - 2 \times 5) \text{ s} \\ &= 15 \text{ s} \end{aligned}$$

Thus the car moves with a uniform speed for 15 second.

Example 11: Two mirrors, mounted vertically, are made to move towards each other with a speed $v \text{ ms}^{-1}$ each. A 'particle' that can bounce back between the two mirrors starts from one mirror when the mirrors are d metres apart. On reaching the second mirror, it bounces back and so on. If the 'particle' keeps on travelling at a constant speed of $3v \text{ ms}^{-1}$, how many 'trips' can it make before the mirrors run into each other? What total distance does it cover?

Solution: The initial positions of the two mirrors are at M_1 and M_2 that are a distance ' d ' apart. Since the speed of the particle is 3 times that of the mirrors, the particle will reach the second mirror at 'A' which is $3/4$ th the original distance of separation, i.e. $3/4 d$. In this time the mirror M_1 will have also reached B such that $M_1 B = \frac{1}{4} d$.

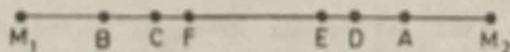


Fig. 2.6

During the second trip, the particle will move from A to B ($AB = d/2$) and will reach M_1 at C where $AC = 3/4 AB = 3/4 d/2 = 3/8 d$. In this time, the mirror M (originally at A) will have moved to D such that $AD = 1/4 AB = 1/8 d$.

For the third trip, the distance of separation of the mirrors is CD ($CD = d/2 - 1/8 d = 1/8 d = d/4$). The particle now starts from C towards D and will reach the second mirror after covering a distance $3/4 CD = 3/16 d$. In this time the first mirror will have moved to F such that $CF = 1/4 CD = 1/16 d$.

For the fourth trip, the distance of separation of the mirrors will be EF where $EF = d/4 - 1/16 d = d/8$. Since this process can continue indefinitely, the particle can make an infinite number of trips before the mirrors run into each other.

The distances moved by the particle in the successive trips are $3/4 d$, $3/8 d$, $3/16 d$, ... and so on. The total distance covered is

$$\begin{aligned}
 &= \frac{3}{4}d + \frac{3}{8}d + \frac{3}{16}d + \dots \infty \\
 &= \frac{3}{4}d \left[1 + \frac{1}{2} + \frac{1}{4} + \dots \infty \right] \\
 &= \frac{3}{4}d \frac{1}{1 - \frac{1}{2}} \\
 &= \frac{3}{4}d \times 2 = \frac{3}{2}d \\
 &= 1.5d
 \end{aligned}$$

Example 12 Two particles are initially located at points A and B a distance d apart. They start moving at time $t = 0$ such that the velocity $\vec{u}$ of B is always along the horizontal direction and the velocity $\vec{v}$ of A is continually aimed at B . At $t = 0$, $\vec{u} \perp \vec{v}$. When will the two particles meet?

Solution Let the particles meet at some time T . The conditions of the two particles at $t = 0$ and $t = T$ are as shown in Fig. 2.7 (a) and (b).

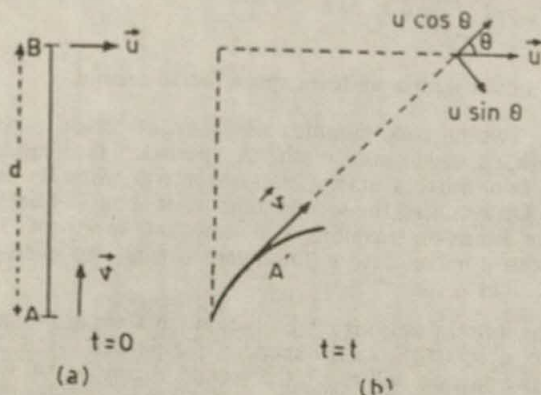


Fig. 2.7

Resolving $\vec{u}$ into two components $u \cos \theta$ and $u \sin \theta$ along and perpendicular to the direction of $\vec{v}$, we find that the relative velocity of approach of A towards B is $(v - u \cos \theta)$. The particles A and B will meet after a time T if

$$\int_0^T (v - u \cos \theta) dt = d$$

and
$$\int_0^T v \cos \theta dt = u T$$

The first of these two equations gives

$$v T - u \int_0^T \cos \theta dt = d$$

while the second equation gives

$$\int_0^T \cos \theta dt = \frac{u T}{v}$$

Substituting this value of $\int_0^T \cos \theta dt$, we get

$$v T - u \frac{u T}{v} = d$$

or
$$(v^2 - u^2) T = v d$$

or
$$T = \frac{v d}{(v^2 - u^2)}$$

This is the time after which the particles will meet.

Example 13: A particle located at $x = 0$ at time $t = 0$ starts moving along the positive x direction with a velocity v that varies as $v = \alpha \sqrt{x}$. How do the velocity and acceleration of the particle vary with time? What is the average velocity of the particle over the first s metres of its path?

Solution: We know that the instantaneous velocity of the particle is given by

$$v = \frac{dx}{dt}$$

Since $v = \alpha \sqrt{x}$, we have

$$\frac{dx}{dt} = \alpha \sqrt{x} \quad \text{or} \quad \frac{dx}{\sqrt{x}} = \alpha dt$$

Integrating from $t = 0$ ($x = 0$) to $t = t$ ($x = x$), we have

$$\int_0^x x^{-1/2} dx = \alpha \int_0^t dt$$

$$\therefore \left[\frac{x^{1/2}}{1/2} \right]_0^x = \alpha t$$

or
$$x = \frac{\alpha^2 t^2}{4}$$

The time dependence of the velocity is obtained by differentiating both sides of this relation w.r.t. time t . Thus

$$v = \frac{dx}{dt} = \frac{\alpha^2}{4} 2t = \frac{\alpha^2}{2} t$$

The velocity x of the particle is thus increasing in direct proportion to time.

Similarly the time dependence of acceleration is obtained by differentiating both sides of this relation w.r.t. ' t '. Thus

$$a = \frac{dv}{dt} = \frac{\alpha^2}{2}$$

The particle is thus moving with a constant acceleration.

To find the average velocity over the first s metre, we assume that the time taken to cover this distance is τ . Using

$$x = \alpha^2 t^2 / 4$$

we get

$$s = \frac{\alpha^2 \tau^2}{4} \text{ or } \tau = \frac{2\sqrt{s}}{\alpha}$$

The average velocity v_{av} ($= s/\tau$) is, therefore

$$v_{av} = \left(\frac{\alpha}{2} \sqrt{s} \right)$$

Example 14: A particle having a velocity $v = v_0$ at $t = 0$ is decelerated at a rate

$$|a| = \alpha \sqrt{v}$$

where α is a positive constant. After what time and after covering what distance will the particle be brought to rest?

Solution: Remembering that $a = \frac{dv}{dt}$ we have

$$a = \frac{dv}{dt} = -\alpha \sqrt{v}$$

$$\therefore v^{-1/2} dv = -\alpha dt$$

Let $v = v$ at $t = t$. Also since $v = v_0$ at $t = 0$, we have

$$\int_{v_0}^v v^{-1/2} dv = -\alpha \int_0^t dt$$

$$\text{or } 2(v^{1/2} - v_0^{1/2}) = -\alpha t$$

Let the particle come to rest at $t = \tau$. Then at $t = \tau$, $v = 0$.

$$\therefore -2v_0^{1/2} = -\alpha \tau$$

$$\text{or } \tau = \frac{2\sqrt{v_0}}{\alpha}$$

This is the time taken by the particle to come to rest. To find the distance s covered in this time, we note that

$$2(v^{1/2} - v_0^{1/2}) = -\alpha t$$

$$\text{or } v^{1/2} = v_0^{1/2} - \frac{1}{2}\alpha t$$

But $v = \frac{ds}{dt}$. We thus have

$$v = \frac{ds}{dt} = (v_0^{1/2} - \frac{1}{2}\alpha t)^2$$

Noting that at $t = 0$, $s = 0$ and at $t = \tau$, $s = s$, we have

$$\int_{s=0}^{s=s} ds = \int_0^{\tau} (v_0^{1/2} - \frac{1}{2} \alpha t)^2 dt$$

Thus
$$s = v_0 \tau + \frac{1}{4} \frac{\alpha^2 \tau^3}{3} - \frac{\alpha \sqrt{v_0}}{2} \tau^2$$

But
$$\tau = 2\sqrt{v_0}/\alpha$$

$$\begin{aligned} s &= \frac{2 v_0^{3/2}}{\alpha} + \frac{1}{12} \alpha^2 \frac{8 v_0^{3/2}}{\alpha^3} - \frac{\alpha \sqrt{v_0}}{2} \frac{4 v_0}{\alpha^2} \\ &= \frac{2 v_0^{3/2}}{\alpha} + \frac{2 v_0^{3/2}}{3 \alpha} - \frac{2 v_0^{3/2}}{\alpha} \\ &= 2/3 (v_0^{3/2}/\alpha) \end{aligned}$$

Thus the particle comes to rest after a time $2v_0^{1/2}/\alpha$ after covering a distance $2/3 (v_0^{3/2}/\alpha)$.

Example 15: Two particles A and B start from the same high point O at $t = 0$ in opposite directions with horizontal velocities of 3 m s^{-1} and 4 m s^{-1} respectively. If they experience the same uniform vertical acceleration of 9.8 m s^{-2} (due to earth's gravitational field), how far apart are the particles when their velocity vectors become mutually perpendicular?

Solution: Let the velocity vectors of the two particles become perpendicular at time $t = t$. The paths of the particles are parabolic under the combined influence of their constant horizontal and uniformly accelerated vertical velocities.

From Fig. 2.8(b) it follows that

$$\alpha_1 + \alpha_2 = \frac{\pi}{2}, \text{ so that}$$

$$\sin(\alpha_1 + \alpha_2) = \sin \frac{\pi}{2} = 1$$

or
$$\sin \alpha_1 \cos \alpha_2 + \cos \alpha_1 \sin \alpha_2 = 1$$

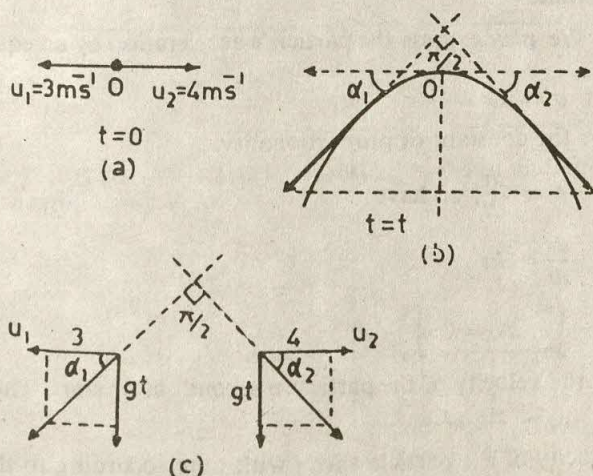


Fig. 2.8

But from Fig. 2.8(c), we have

$$\sin \alpha_1 = \frac{gt}{\sqrt{g^2 t^2 + 9}}, \cos \alpha_1 = \frac{3}{\sqrt{g^2 t^2 + 9}}$$

$$\sin \alpha_2 = \frac{gt}{\sqrt{g^2 t^2 + 16}}, \cos \alpha_2 = \frac{4}{\sqrt{g^2 t^2 + 16}}$$

We thus have

$$\frac{4gt}{\sqrt{g^2 t^2 + 9} \sqrt{g^2 t^2 + 16}} + \frac{3gt}{\sqrt{g^2 t^2 + 16} \sqrt{g^2 t^2 + 9}} = 1$$

or

$$4gt + 3gt = \sqrt{g^2 t^2 + 9} \sqrt{g^2 t^2 + 16}$$

or

$$49g^2 t^2 = (g^2 t^2 + 9)(g^2 t^2 + 16)$$

Calling

$$x = g^2 t^2, \text{ we have}$$

$$(x + 9)(x + 16) - 49x = 0$$

or

$$x^2 + 25x + 144 - 49x = 0$$

or

$$x^2 - 24x + 144 = 0$$

or

$$(x - 12)^2 = 0$$

or

$$x = g^2 t^2 = 12$$

$$\therefore t = \frac{\sqrt{12}}{9.8} \text{ s} \approx 0.353 \text{ s}$$

The velocity of the two particles, therefore, become perpendicular 0.353 s after they start. The horizontal separation between the two particles at this instant is

$$s = (3 + 4)t = 7 \times 0.353 \text{ m}$$

$$= 2.474 \text{ m}$$

Example 16: A particle, starting from rest, undergoes an acceleration that increases linearly with time. Estimate (i) the particle's velocity n second after start and (ii) the distance moved by the particle in these n second. Draw graphs showing the variation of acceleration, velocity and displacement with time.

Solution: We may express the particle's acceleration by an equation of the form

$$a = k t$$

where k is the constant of proportionality.

Since $a = \frac{dv}{dt}$, we have

$$\frac{dv}{dt} = kt$$

$$\therefore \int_0^{v_n} dv = k \int_0^n t dt$$

where v_n is the velocity of the particle n second after start. Thus

$$v_n = \frac{1}{2} k n^2$$

and the velocity of the particle varies with time according to the relation

$$v = \frac{1}{2} k t^2$$

Now $v = \frac{ds}{dt}$ so that

$$ds = \frac{1}{2} k t^2 dt$$

If s_n is the distance moved by the particle in n seconds, we have

$$\int_0^{s_n} ds = \frac{1}{2} k \int_0^n t^2 dt$$

$$\therefore s_n = \frac{1}{6} k n^3$$

Thus the variation of the acceleration, velocity and displacement are governed by the relations

$$a = kt, v = \frac{1}{2} kt^2, s = \frac{1}{6} kt^3$$

Figure 2.9 shows the required graphs showing the variation of acceleration, velocity and displacement with time where, for simplicity, we have taken $k = 1 \text{ m s}^{-2}$.

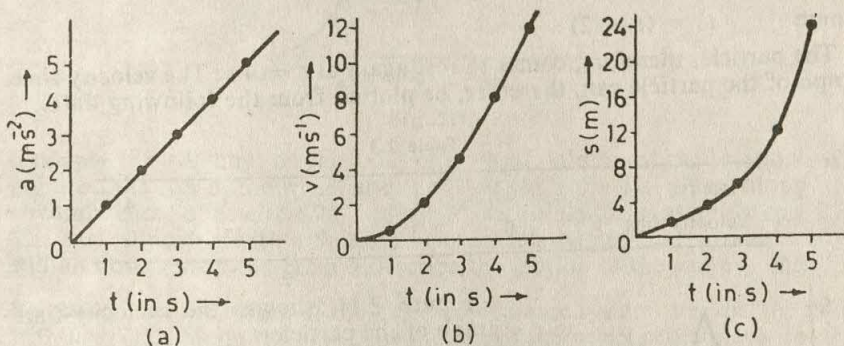


Fig. 2.9

The necessary data is given in the following table:

Table 2.2

t (s)	0	1	2	3	4	5
a (m s^{-2})	0	1	2	3	4	5
v (m s^{-1})	0	0.5	2	4.5	8	12.5
s (m)	0	0.167	1.33	4.5	10.67	20.83

Example 17: A particle starts from rest at $t=0$ and undergoes an acceleration as shown in Fig. 2.10.

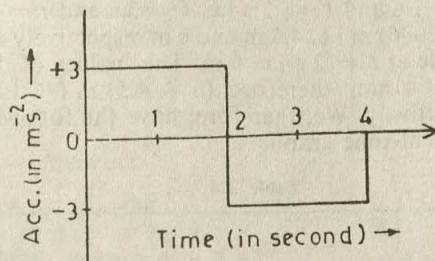


Fig. 2.10

(a) Draw a neat sketch showing the velocity of the particle as a function of time during the interval from 0 to 4 seconds indicating each second on the abscissa.

(b) Draw a neat sketch showing the displacement of the particle as a function of time during the same interval.

In both the above cases, explain the various steps.

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Solution: (a) It is clear from Fig. 2.10 that the acceleration of the particle from $t = 0$ to $t = 2$ s is $+3 \text{ m s}^{-2}$. Therefore, the velocity of the particle in this interval is given by

$$v = 0 + 3 \times t = 3t; 0 \leq t \leq 2$$

At $t = 2$ s, the velocity of the particle is 6 m s^{-1} and the acceleration changes sign. From $t = 2$ s to $t = 4$ s, the particle, starting with an 'initial' velocity of 6 m s^{-1} , decelerates at -3 m s^{-2} . The velocity of the particle at any instant in this interval is given by

$$v' = 6 - 3t' \quad 0 \leq t' \leq 2$$

where $t' = (t - 2) \text{ s}$

The particle, therefore, comes to rest again at $t = 4$ s. The velocity-time graph of the particle can, therefore, be plotted from the following data.

Table 2.3

Time (s)	0	1	2	3	4
Velocity (m s^{-1})	0	3	6	3	0

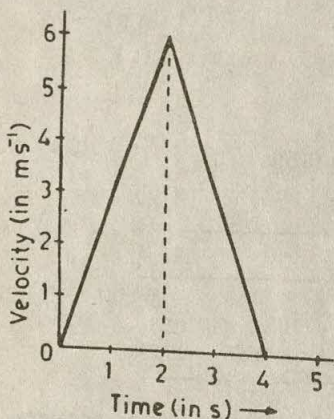


Fig. 2.11

Figure 2.11 shows the velocity-time graph of the particle.

(b) The displacement of the particle from $t = 0$ to $t = 2$ s is given by

$$s = \frac{1}{2} \times 3 \times t^2 = 1.5 t^2$$

Its values at $t = 0$, $t = 1$ s and $t = 2$ s are, therefore, 0, 1.5 m and 6 m respectively.

From $t = 2$ s to $t = 4$ s or $t' = 0$ to $t' = 2$ s, the displacement of the particle is given by

$$\begin{aligned} s' &= 6t' - \frac{1}{2} \times 3 \times t'^2 \\ &= 6t' - 1.5t'^2 \end{aligned}$$

Its values at $t' = 1$ s and $t' = 2$ s i.e. $t = 3$ s and $t = 4$ s are, therefore (6 - 1.5) m and (12 - 6) m i.e. 4.5 m and 6 m respectively above the displacement of the particle at $t = 2$ s i.e. 6 m. The values of the displacement at $t = 3$ s and $t = 4$ s are, therefore, (6 + 4.5) m (= 10.5 m) and (6 + 6) m (= 12 m) respectively. We, therefore, have the following data for plotting the displacement-time graph.

Table 2.4

Time (s)	0	1	2	3	4
Displacement (m)	0	1.5	6	10.5	12

The displacement-time graph is, therefore, parabolic as shown in Fig. 2.12.

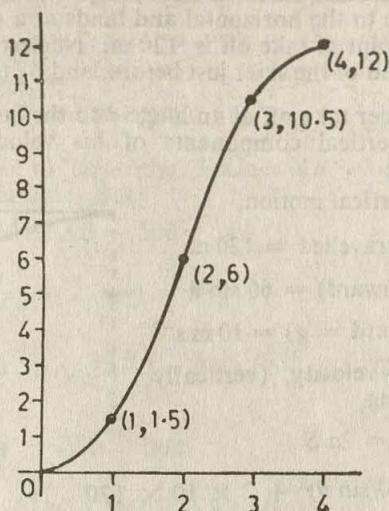


Fig. 2.12

Example 18: A 'bug' on the axle of a wheel 'observes' the motion of a point on the rim of the wheel and 'finds' it to take its place along the circumference of a circle of radius R with a uniform angular speed ω . The axle is moving with a uniform speed v relative to the ground. How will an observer on the ground describe the motion of the same point?

Solution: Let the origin of the bug's co-ordinate system be on the axle with his z -axis along the axle. The motion of the wheel takes place in the x - y plane of this co-ordinate system. If at $t = 0$, $x = R$ and $y = 0$, the corresponding values of x and y at an instant of time t will be

$$x = R \cos \omega t,$$

$$y = R \sin \omega t$$

We next look at the same motion from a co-ordinate system S' fixed relative to the ground and assume that the 'two' origins were coincident at $t = 0$ (see Fig. 2.13). Since S is moving relative to S' with a uniform speed v , along the x -axis, we have

$$x' = x + vt$$

$$= R \cos \omega t + vt$$

and

$$y' = y = R \sin \omega t$$

The graph of y' against x' , when plotted, has the form shown in Fig. 2.13 (b). The curve is known as a *cycloid*.

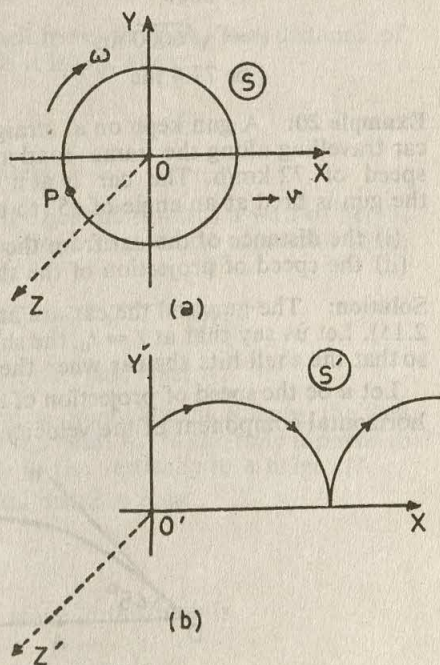


Fig. 2.13

Example 19: A skier takes off from a ski jump with a velocity of 60 m s^{-1} at an unknown angle to the horizontal and lands at a point whose vertical distance below the point of take off is 120 m . Neglecting the friction of air, calculate the speed of the skier just before landing (Take $g = 10 \text{ m s}^{-2}$.)

Solution: Let the skier take off at an angle θ to the horizontal (Fig. 2.14). The horizontal and vertical components of his velocity are $60 \cos \theta$ and $60 \sin \theta$ respectively.

Considering his vertical motion,

Downward distance travelled = 120 m

Initial velocity (downward) = $60 \sin \theta$

Acceleration (downward = g) = 10 m s^{-2}

Let v' be the final velocity (vertically downwards). Using

$$v'^2 - u^2 = 2aS$$

$$\begin{aligned} \text{we have } v'^2 &= (60 \sin \theta)^2 + 2 \times 10 \times 120 \\ &= 3600 \sin^2 \theta + 2400 \end{aligned}$$

The horizontal component of the velocity of the skier remains constant at its initial value (we are neglecting air friction), i.e. $60 \cos \theta$. The velocity (v) of the skier just before landing is, therefore, given by

$$\begin{aligned} v^2 &= v'^2 + (60 \cos \theta)^2 \\ &= 3600 \sin^2 \theta + 2400 + 3600 \cos^2 \theta \\ &= 3600 + 2400 \\ &= 6000 \\ v &= \sqrt{6000} \text{ ms}^{-1} \\ &\cong 77.4 \text{ ms}^{-1} \end{aligned}$$

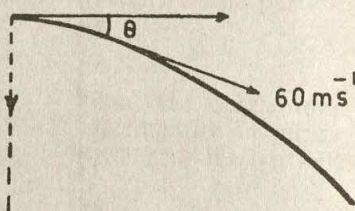


Fig. 2.14

Example 20: A gun kept on a straight horizontal road is used to hit a car travelling along the same road away from the gun with a uniform speed of 72 km/h . The car is at a distance of 500 m from the gun when the gun is fired at an angle of 45° to the horizontal. Find

- the distance of the car from the gun when the shell hit it, and
- the speed of projection of the shell from the gun. (IIT 1974)

Solution: The gun and the car are at O and A respectively at $t = 0$ (Fig. 2.15). Let us say that at $t = t_0$, the shell and the car reach B simultaneously so that the shell hits the car when the car is at a distance OB from the gun.

Let u be the speed of projection of the shell from the gun. Then the initial horizontal component of the velocity of the shell = $u \cos 45^\circ = u/\sqrt{2}$

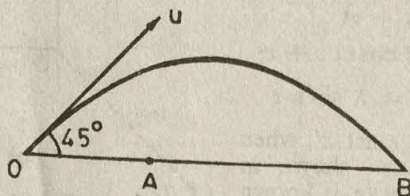


Fig. 2.15

and the initial vertical component of the velocity of the shell $= u \sin 45^\circ = u/\sqrt{2}$

Time of flight of the shell

$$= \frac{2(u/\sqrt{2})}{g} = \sqrt{2}(u/g)$$

The car takes this time to cover the distance AB while the shell covers the distance OB in this time. But

$$OB = OA + AB = 500 + AB$$

Also $OB = \frac{u}{\sqrt{2}} \sqrt{2}(u/g) = u^2/g$

and $AB = 20 \times \sqrt{2} \left(\frac{u}{g} \right) = 20 \sqrt{2} \frac{u}{g}$ ($\because 72 \text{ km/h} = 20 \text{ m s}^{-1}$)

$\therefore \frac{u^2}{g} = 500 + 20 \sqrt{2} (u/g)$

or $u^2 - 20 \sqrt{2} u - 500 \times 9.8 = 0$

or $u^2 - 20 \sqrt{2} u - 4900 = 0$

or $u = \frac{20 \sqrt{2} \pm \sqrt{400 \times 4 + 4 \times 4900}}{2} \text{ m s}^{-1}$

$$= (10 \sqrt{2} \pm \sqrt{5300}) \text{ m s}^{-1}$$

$$= 10 [\sqrt{2} \pm \sqrt{53}] \text{ m s}^{-1}$$

$$= 10 [1.414 + 7.280] \text{ m s}^{-1} = 86.94 \text{ m s}^{-1}$$

This is the speed of projection of the shell from the gun. The distance of the car from the gun when the shell hits it is OB where

$$OB = u^2/g = \frac{(86.94)^2}{9.8} \text{ m}$$

$$= 771.3 \text{ m approximately.}$$

Example 21: A stone is thrown from the ground towards a wall of height H with a speed u at an angle α to the horizontal. Show that the stone must be thrown from a point distant

$$\frac{1}{2g} [u^2 \sin^2 \alpha \pm 2u \cos \alpha \sqrt{u^2 \sin^2 \alpha - 2gH}]$$

from the foot of the wall in order that it may just clear it.

Solution: Let the stone be thrown from a point distant R from the foot of the wall. The stone will just clear the wall if, in the time it needs to cover a horizontal distance R , it is able to rise vertically to a height H .

Now, time taken to cover a horizontal distance R is

$$\frac{R}{u \cos \alpha}$$

The vertical distance moved up by the stone in this time is

$$(u \sin \alpha) \left(\frac{R}{u \cos \alpha} \right) - \frac{1}{2} g \left(\frac{R}{u \cos \alpha} \right)^2$$

Thus the stone will be just able to clear the wall if

$$H = R \tan \alpha - \frac{gR^2}{2u^2 \cos^2 \alpha}$$

or $gR^2 - (2u^2 \cos^2 \alpha \tan \alpha) R + 2H u^2 \cos^2 \alpha = 0$

or $gR^2 - (u^2 \sin 2\alpha) R + (2H u^2 \cos^2 \alpha) = 0$

or
$$R = \frac{u^2 \sin 2\alpha \pm \sqrt{u^4 \sin^2 2\alpha - 8gH u^2 \cos^2 \alpha}}{2g}$$

$$= \frac{1}{2g} \left[u^2 \sin 2\alpha \pm u \sqrt{u^2 \sin^2 2\alpha - 8gH \cos^2 \alpha} \right]$$

$$= \frac{1}{2g} \left[u^2 \sin 2\alpha \pm u \cdot 2 \cos \alpha \sqrt{u^2 \sin^2 \alpha - 2gH} \right]$$

Thus if the stone is thrown from a point, distant

$$R = \frac{1}{2g} \left\{ u^2 \sin 2\alpha \pm 2u \cos \alpha \sqrt{u^2 \sin^2 \alpha - 2gH} \right\}$$

from the foot of the wall, it will just clear it.

Example 22: A particle is projected at $t = 0$ from a point P on the ground with a speed v_0 at an angle of 45° to the horizontal. Find the magnitude and direction of the angular momentum of the particle about the point P at time $t = v_0/g$. (IIT 1984)

Solution: If the co-ordinates of point P are taken $(0, 0)$, then those of the point Q where the particle is at time $t = v_0/g$ may be taken to be (x, y) . Now see Fig. 2.16.

Initial horizontal component of velocity of the particle

$$= v_0 \cos 45^\circ$$

$$= \frac{v_0}{\sqrt{2}}$$

and

Initial vertical component of the velocity $= v_0 \sin 45^\circ = \frac{v_0}{\sqrt{2}}$

$x =$ horizontal distance covered in time $t = \left(\frac{v_0}{g} \right)$

$$= \frac{v_0}{\sqrt{2}} \cdot \frac{v_0}{g} = \frac{v_0^2}{g\sqrt{2}}$$

and

$y =$ vertical distance covered in time $t = \left(\frac{v_0}{g} \right)$

$$= \frac{v_0}{\sqrt{2}} \cdot \frac{v_0}{g} - \frac{1}{2} g \left(\frac{v_0}{g} \right)^2$$

$$= \frac{v_0^2}{g\sqrt{2}} - \frac{v_0^2}{2g} = \frac{v_0^2}{2g} (\sqrt{2} - 1)$$

Also, the horizontal velocity v_x of the particle at $Q = \frac{v_0}{\sqrt{2}}$

and the vertical velocity v_y of the particle at $Q = \frac{v_0}{\sqrt{2}} - g \left(\frac{v_0}{g} \right)$

$$= \frac{v_0}{\sqrt{2}} - v_0$$

$$= \frac{v_0}{\sqrt{2}} (1 - \sqrt{2}) = -\frac{(\sqrt{2} - 1)}{\sqrt{2}} v_0$$

Now by definition, the angular momentum of the particle at Q about P (0, 0) is

$$m (v_y x - v_x y)$$

Substituting the values we get

Angular momentum at $t = v_0/g$ is given by

$$\begin{aligned} &= m \left[\frac{v_0^2}{g \sqrt{2}} \times -(\sqrt{2} - 1) \frac{v_0}{\sqrt{2}} - \frac{v_0}{\sqrt{2}} \frac{v_0^2}{2g} (\sqrt{2} - 1) \right] \\ &= \frac{m v_0^3}{2\sqrt{2} g} [-\sqrt{2} (\sqrt{2} - 1) - (\sqrt{2} - 1)] = -\frac{m v_0^3}{2\sqrt{2} g} \end{aligned}$$

The direction of the angular momentum will be perpendicular to both the position vector $\vec{r}$ of the particle and its momentum vector $m \vec{v}$ at point Q .

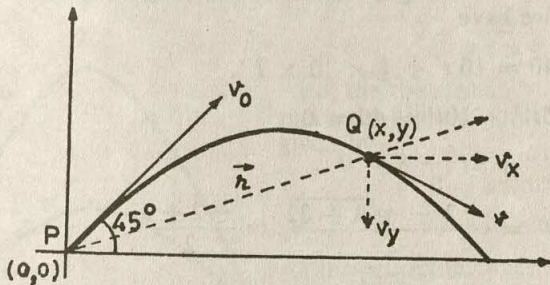


Fig. 2.16

Example 23: From the top of a tower of height 40 m, a ball is projected upwards with a speed of 20 m s^{-1} at an angle of 30° to the horizontal. When and at what distance from the foot of the tower does the ball hit the ground? What is the velocity of the ball at this instant? (Take $g = 10 \text{ m s}^{-2}$).

Solution: The initial horizontal and vertical components of the velocity of the ball are (see Fig. 2.17)

$$20 \cos 30^\circ = 10\sqrt{3} \text{ m s}^{-1} \text{ and } 20 \sin 30^\circ = 10 \text{ m s}^{-1}$$

respectively.

The ball reaches the highest point P in time t after its projection where t is given by

$$0 = 10 - 10t \quad \text{or} \quad t = 1 \text{ s}$$

The time of flight of the ball, i.e. the time taken by it to reach O , where O is the point in level with the top of the tower, is, therefore, 2 s. Also AO , the 'range' of the ball is given by

$$AO = 10\sqrt{3} \times 2 \text{ m} = 20\sqrt{3} \text{ m}$$

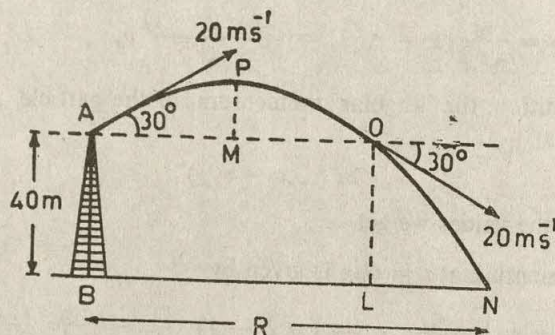


Fig. 2.17

At O the velocity of the ball is again 20 m s^{-1} directed downwards as shown. The horizontal and vertical components of its velocity are now $10\sqrt{3} \text{ m s}^{-1}$ and 10 m s^{-1} (downwards) respectively. The ball will hit the ground after travelling a vertical distance of 40 m. If the time taken for this is t' , we have

$$40 = 10 t' + \frac{1}{2} \times 10 \times t'^2$$

$$\text{or} \quad 5t'^2 + 10t' - 40 = 0$$

$$\text{or} \quad t'^2 + 2t' - 8 = 0$$

$$\text{or} \quad t' = \frac{-2 \pm \sqrt{4 + 32}}{2} = \frac{-2 \pm 6}{2}$$

$$\text{Thus} \quad t' = 2 \text{ s (leaving out the negative sign)}$$

The horizontal distance LN moved by the ball in this time is given by

$$LN = 10\sqrt{3} \times 2 \text{ m} = 20\sqrt{3} \text{ m}$$

Thus the ball hits the ground $(2 + 2) \text{ s} = 4 \text{ s}$ after it is projected. Its distance from the foot of the tower at this instant is

$$20\sqrt{3} + 20\sqrt{3} = 40\sqrt{3} \text{ m } 69.3 \text{ m nearly}$$

The horizontal component of the velocity of the ball when it hits the ground is $10\sqrt{3} \text{ m s}^{-1}$. The vertical component (downwards) at this instant is

$$10 + 10 \times 2 = 30 \text{ m s}^{-1}$$

The resultant velocity of the ball at the instant it hits the ground is, therefore,

$$\begin{aligned} & \sqrt{(10\sqrt{3})^2 + (30)^2} \text{ m s}^{-1} \\ &= \sqrt{300 + 900} \text{ m s}^{-1} \\ &= \sqrt{1200} \text{ m s}^{-1} = 34.64 \text{ m s}^{-1} \end{aligned}$$

It is inclined at an angle α to the horizontal where (see Fig. 2.18)

$$\tan \alpha = \frac{-30}{10\sqrt{3}} = -\sqrt{3}$$

$$\therefore \alpha = -60^\circ$$

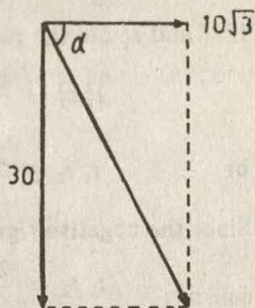


Fig. 2.18

Example 24: A particle is projected with a certain velocity so as to pass through a point at a horizontal distance R from the point of projection. Show that if t_1 and t_2 are the times taken to reach this point in the two possible ways in which the particle can be projected with the given velocity, then

$$t_1 t_2 = 2R/g$$

Solution: Let the particle be projected with an initial velocity v at an angle θ to the horizontal. The time taken by it to cover the horizontal distance R is then (see Fig. 2.19).

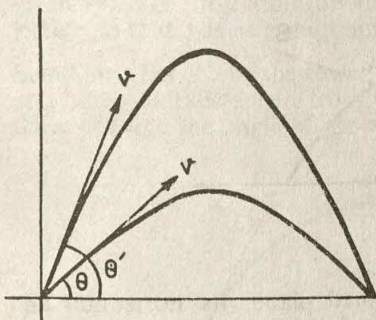


Fig. 2.19

$$t = \frac{R}{v \cos \theta} \quad (1)$$

Also, the time taken by the projectile to reach the highest point in its motion is given by

$$0 = v \sin \theta - g t' \quad (2)$$

or

$$t' = v \sin \theta / g$$

As the projectile takes the same time in coming back to the level of its point of projection, we have $t = 2t'$ so that

$$t = \frac{2v \sin \theta}{g} \quad (3)$$

From Eq. (1):

$$\cos \theta = R/vt$$

From Eq. (2):

$$\sin \theta = \frac{gt'}{v} = \frac{gt}{2v}$$

$$\therefore \left(\frac{R}{vt}\right)^2 + \left(\frac{gt}{2v}\right)^2 = 1$$

$$\text{or} \quad \frac{R^2}{v^2 t^2} + \frac{g^2 t^2}{4v^2} = 1$$

$$\text{or } 4R^2 + g^2 t^4 - 4v^2 t^2 = 0$$

$$\text{or } g^2 t^4 - 4v^2 t^2 + 4R^2 = 0$$

If t_1^2 and t_2^2 are the two values of t^2 satisfying this equation, we must have

$$t_1^2 t_2^2 = \frac{4R^2}{g^2}$$

$$\text{or } t_1 t_2 \pm \frac{2R}{g}$$

Since the negative sign is not permissible, we must have

$$t_1 t_2 = \frac{2R}{g}$$

which is the required condition.

Example 25: A projectile is launched with a speed v at an angle θ to the horizontal hits an inclined plane inclined at an angle α to the horizontal ($\alpha < \theta$) and passing through the point of launching. Obtain an expression for the range r of the projectile on this inclined plane. When does the projectile hit this plane?

Solution: Let OAB be the inclined plane making an angle α to the horizontal and let the projectile hit it at a point A where $OA = r$ (Fig. 2.20)

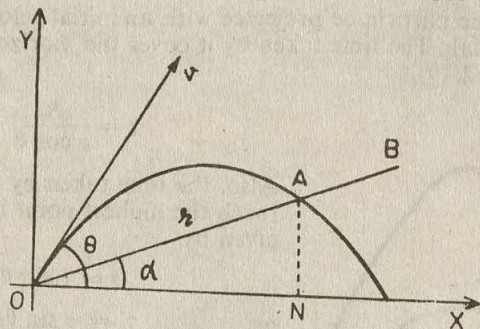


Fig. 2.20

At the instant the projectile hits the inclined plane, its horizontal and vertical displacements are $ON (= r \cos \alpha)$ and $NA (= r \sin \alpha)$ respectively.

Now the time taken to cover a horizontal distance $r \cos \alpha$ is clearly $r \cos \alpha / v \cos \theta$. The vertical distance moved in this time being $r \sin \alpha$, we have

$$r \sin \alpha = (v \sin \theta) \left(\frac{r \cos \alpha}{v \cos \theta} \right)$$

$$\text{or } r (\sin \alpha - \tan \theta \cos \alpha) = - \frac{1}{2} g \frac{\cos^2 \alpha}{v^2 \cos^2 \theta} r^2$$

$$\begin{aligned} \therefore r &= \frac{2v^2 \cos^2 \theta}{g \cos^2 \alpha} [\tan \theta \cos \alpha - \sin \alpha] \\ &= \frac{2v^2 \cos^2 \theta}{g \cos^2 \alpha} \left[\frac{\sin \theta \cos \alpha - \cos \theta \sin \alpha}{\cos \theta} \right] \\ &= \frac{2v^2 \cos \theta}{g \cos^2 \alpha} \sin (\theta - \alpha) \end{aligned}$$

Thus the range of the projectile on the inclined plane is

$$\frac{2 v^2 \cos \theta}{g \cos^2 \alpha} \sin (\theta - \alpha)$$

The time at which the projectile hits the inclined plane being $(r \cos \alpha / v \cos \theta)$, we have

$$t = \frac{r \cos \alpha}{v \cos \theta} = \frac{2 v^2 \cos \theta}{g \cos^2 \alpha} \sin (\theta - \alpha) \times \frac{\cos \alpha}{v \cos \theta}$$

or
$$t = \frac{2 v \sin (\theta - \alpha)}{g \cos \alpha}$$

A confirmation of these results is obtained by putting $\alpha = 0$ for which we get

$$r = \frac{2 v^2 \cos \theta \sin \theta}{g} = \frac{v^2 \sin 2 \theta}{g}$$

and
$$t = \frac{2 v \sin \theta}{g}$$

the usual expressions for the range and time of flight of a projectile.

Example 26: A ball is thrown from a point in level with and at a horizontal distance r from the top of a tower of height H . How must the speed and angle of projection of the ball be related to r in order that the ball may just go grazing past the top edge of the tower? At what horizontal distance x from the foot of the tower does the ball hit the ground? For a given speed of projection, obtain an equation for finding the angle of projection so that x is at a minimum.

Solution: Let AB be the tower of height H and O the point of projection at a horizontal distance r from A as shown in Fig. 2.21. Let u and θ be the speed and the angle of projection of the ball.

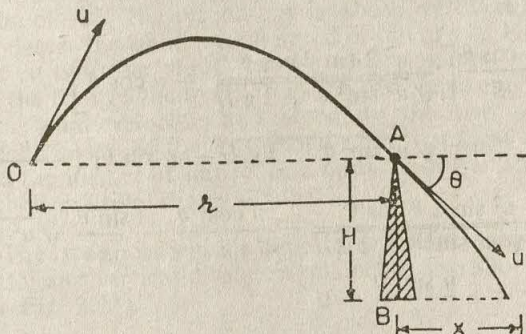


Fig. 2.21

The ball will go just grazing past the top edge of the tower if r equals the horizontal range of the projectile, i.e. if

$$r = u \cos \theta \cdot \frac{2 u \sin \theta}{g} = \frac{u^2 \sin 2 \theta}{g}$$

Thus r , u and θ must be related according to the relation

$$gr = u^2 \sin 2\theta$$

At point A , the velocity of the ball is again u at an angle θ to the horizontal. The horizontal distance x at which the ball strikes the ground from the foot of the tower is the horizontal distance covered with a horizontal velocity $u \cos \theta$ in the time the ball falls vertically through a distance H starting with an initial vertically downward velocity $u \sin \theta$ and having a vertically downward acceleration g . If t is this time, we have

$$H = (u \sin \theta) t + \frac{1}{2} g t^2$$

$$\text{or } g t^2 + 2 u \sin \theta t - 2 H = 0$$

$$\begin{aligned} \text{or } t &= \frac{-2 u \sin \theta \pm \sqrt{4 u^2 \sin^2 \theta + 8 g H}}{2 g} \\ &= \frac{-2 u \sin \theta \pm 2 \sqrt{u^2 \sin^2 \theta + 2 g H}}{2 g} \\ &= \frac{1}{g} [-u \sin \theta \pm \sqrt{u^2 \sin^2 \theta + 2 g H}] \end{aligned}$$

Taking only the positive sign, we have

$$t = \frac{1}{g} [\sqrt{u^2 \sin^2 \theta + 2 g H} - u \sin \theta]$$

Thus

$$\begin{aligned} x &= u \cos \theta t \\ &= \frac{u \cos \theta}{g} [\sqrt{u^2 \sin^2 \theta + 2 g H} - u \sin \theta] \end{aligned}$$

The angle of projection θ for which x is minimum for a given value of u is given by

$$\frac{dx}{d\theta} = 0$$

Thus,

$$\begin{aligned} \frac{u \cos \theta}{g} \left[\frac{u^2 2 \sin \theta \cos \theta}{\sqrt{u^2 \sin^2 \theta + 2 g H}} - u \cos \theta \right] \\ + [\sqrt{u^2 \sin^2 \theta + 2 g H} - u \sin \theta] \left(\frac{-u \sin \theta}{g} \right) = 0 \end{aligned}$$

$$\begin{aligned} \text{or } \frac{u^2 \sin 2\theta \cos \theta}{g \sqrt{u^2 \sin^2 \theta + 2 g H}} - \frac{u \cos^2 \theta}{g} - \frac{\sin \theta}{g} \sqrt{u^2 \sin^2 \theta + 2 g H} \\ + \frac{u \sin^2 \theta}{g} = 0 \end{aligned}$$

$$\text{or } \frac{u^2 \sin 2\theta \cos \theta}{\sqrt{u^2 \sin^2 \theta + 2 g H}} - \sin \theta \sqrt{u^2 \sin^2 \theta + 2 g H} - u \cos 2\theta = 0$$

$$\begin{aligned} \text{or } u^2 \sin 2\theta \cos \theta - \sin \theta (u^2 \sin^2 \theta + 2 g H) \\ - u \cos 2\theta \sqrt{u^2 \sin^2 \theta + 2 g H} = 0 \end{aligned}$$

The angle of projection for which x is a minimum is a solution of this equation.

Example 27: A ball rolls off the top of a stairway with a horizontal velocity $u \text{ m s}^{-1}$. If the steps are h meters high and w metre wide, show that the ball will just hit the edge of the n -th step if

$$n = \frac{2 h u^2}{g w^2}$$

Solution: The horizontal and vertical distances moved by the ball while just hitting the edge of the n -th step are clearly nw and nh respectively (Fig. 2.22).

Let the time taken by the ball for these horizontal and vertical displacements be t . Then,

$$nw = ut$$

$$\text{and } nh = 0 + \frac{1}{2} g t^2$$

Eliminating t between these two equations, we get

$$nh = \frac{1}{2} g \left(\frac{nw}{u} \right)^2$$

$$\text{or } nh = \frac{1}{2} g \frac{n^2 w^2}{u^2}$$

$$\text{or } n = \frac{2 h u^2}{g w^2}$$

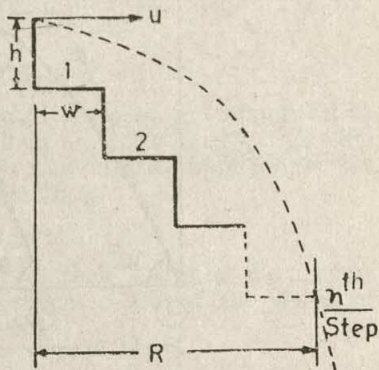


Fig. 2.22

Thus the ball just grazes past the edge of the n -th step if

$$n = \frac{2 h u^2}{g w^2}$$

Example 28: Two boys simultaneously aim their guns at a bird sitting on a tower. The first boy releases his 'shot' with a speed of 100 m s^{-1} at an angle of projection of 30° . The second boy is 'ahead' of the first by a distance of 50 m and releases his 'shot' with a speed of 80 m s^{-1} . How must he aim his gun so that both the 'shots' hit the bird simultaneously? What is the distance of the foot of the tower from the two boys and the height of the tower? With what velocities and when do the two shots hit the bird? Assuming the masses of the two shots to be identical and the energy of each shot at the instant of impact as a measure of the 'share' of each boy, determine their relative 'shares'.

Solution: The two shots will hit the bird simultaneously at some time t if the horizontal and vertical displacements of the two shots at time t are as follows (see Fig. 2.23).

First shot: Horizontal displacement $= x_1$

Vertical displacement $= h$

Second shot: Horizontal displacement $= x_2$

Vertical displacement $= h$

Here $x_1 = x_2 + 50$ and h is the height of the tower at which the bird is perched.

Now, for the first shot,

$$h = (100 \sin 30^\circ) t - \frac{1}{2} \times 9.8 \times t^2$$

And for the second shot,

$$h = (80 \sin \theta) t - \frac{1}{2} \times 9.8 \times t^2$$

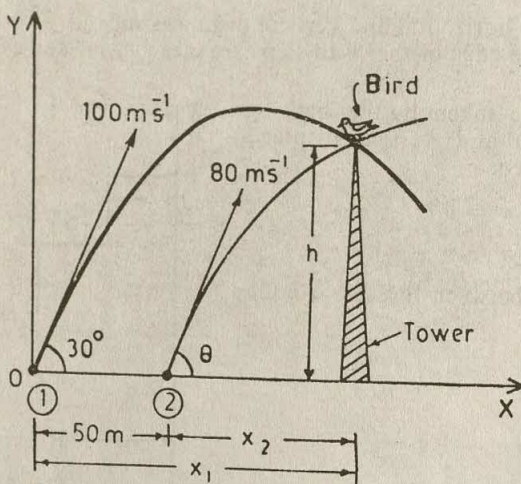


Fig. 2.23

We therefore have

$$(100 \sin 30^\circ) t - 4.9 t^2 = (80 \sin \theta) t - 4.9 t^2$$

$$\text{or} \quad \sin \theta = \frac{100 \sin 30^\circ}{80} = \frac{50}{80} = 0.625$$

$$\therefore \quad \theta = \sin^{-1} 0.625 = 38^\circ 68'$$

Thus, the second boy must aim his 'shot' at an angle of $38^\circ 68'$ to the horizontal.

$$\text{Now} \quad x_1 = (100 \cos 30^\circ) t \quad \text{and} \quad x_2 = 80 \cos (38^\circ 68') t$$

$$\therefore \quad (100 \cos 30^\circ) t = 50 + 80 (\cos 38^\circ 68') t$$

$$\text{or} \quad t \left[100 \frac{\sqrt{3}}{2} - 80 \times 0.7806 \right] = 50$$

$$\text{or} \quad t [86.6025 - 62.4499] = 50$$

$$\text{or} \quad t = \frac{50}{24.1526} \text{ s} = 2.070 \text{ s}$$

Thus the two 'shots' hit the bird together 2.07 s after firing. The distances of the foot of the tower from the two boys are

$$x_1 = 100 \cos 30^\circ \times t = 86.6025 \times 2.070 \text{ m} \\ = 179.27 \text{ m}$$

$$x_2 = (80 \cos 38^\circ 68') t = 62.4499 \times 2.070 \text{ m} \\ = 129.27 \text{ m}$$

respectively.

The height h of the tower on which the bird is perched is given by

$$\begin{aligned} h &= 100 \sin 30^\circ t - \frac{1}{2} \times 9.8 \times t^2 \\ &= 50 \times 2.070 - 4.9 \times (2.070)^2 \\ &= (103.5 - 20.996) \text{ m} \\ &= 82.504 \text{ m} \end{aligned}$$

Now, the horizontal and vertical components of the velocities of the first shot at $t = 2.07 \text{ s}$ are $100 \cos 30^\circ$ and $(100 \sin 30^\circ) - 9.8 \times 2.07$ respectively i.e. 86.6025 m s^{-1} and 29.72 m s^{-1} . The resultant velocity of the first shot at the instant of hitting the bird is

$$\sqrt{(86.6025)^2 + (29.72)^2} = 91.56 \text{ m s}^{-1}$$

For the second shot, the horizontal and vertical velocity components at $t = 2.07 \text{ s}$ are $80 \cos 38.68^\circ$ and $80 \sin 38.68^\circ - 9.8 \times 2.07$ i.e. 62.4499 m s^{-1} and 29.72 m s^{-1} respectively. The resultant velocity of the second shot at the instant of hitting the bird is, therefore,

$$\sqrt{(62.4499)^2 + (29.72)^2} = 69.16 \text{ m s}^{-1}$$

Since the masses of the two shots are identical, their energies are proportional to $(91.56)^2$ and $(69.16)^2$. This ratio being 1.75 nearly, the shares of the boys are in the ratio 1.75 : 1 or 7 : 4 nearly.

2.5 TRUE—FALSE STATEMENTS WITH REASONS

Given below are some statements. We have to decide whether they are true or false and give reasons for our choice.

- (1) The distances moved by a freely falling body (starting from rest) during the 1st, 2nd, 3rd, ... n th second of its motion are proportional to $1 : 4 : 9 : \dots : n^2$.
- (2) Two balls of different masses are thrown vertically upward with the same speed. They will then pass through their point of projection in the downward direction with the same speed. (neglect air resistance). (IIT 1983)
- (3) An arrow is fired vertically upwards with a certain velocity u and ' n ' second later another arrow is fired vertically upwards with the same velocity. The two arrows will then meet $n/2$ second after the first arrow has reached the highest point.
- (4) A metal ball is allowed to fall (from rest) freely on a perfectly elastic plate from a height of h metre. Assuming the vertically downward direction to be positive, the variation of its velocity with time is represented by Fig. 2.24.
- (5) A car covers the first half of its distance between two places at a speed of 40 km/hour and the second half at 60 km/hour. The average speed of the car is then 50 km/hour.

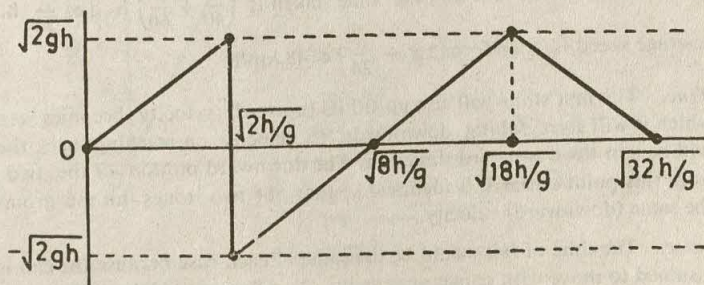


Fig. 2.24

- (6) From the top of a tower, two stones are thrown—one straight up with an initial speed u and the second straight down with the same speed u . The two stones will then have the same speed when they hit the ground. (Air-resistance is assumed negligible).
- (7) A man in a closed elevator cab with no indicator can find out whether the cab is stationary, moving upwards at constant velocity or moving downwards at constant velocity by timing the fall of an object. He would find the time of fall to be *less* when the cab is moving upwards at constant velocity and *more* when the cab is moving downwards at constant velocity compared to the time of fall in a stationary cab.
- (8) For a projectile projected at an angle $\tan^{-1} 2$, the maximum height and horizontal range are equal.
- (9) For a constant initial speed and angle of projection of a projectile, the change, dR in its horizontal range due to a change dg , in the value of the acceleration due to gravity is governed by the relation

$$\frac{dR}{R} = - \frac{dg}{g}$$

- (10) The path of one projectile as seen from another projectile is also a parabolic curve.

ANSWERS

1. *False.* The given ratio series is for the total distance covered in 1, 2, 3, ... n seconds. The ratio series for the distances covered in 1st, 2nd, 3rd, ... n th second of motion is 1 : 3 : 5 : ... : $(2n - 1)$.
2. *True.* The maximum height attained and the downward velocity attained at the point of projection are independent of the mass of the ball.
3. *True.* If $t (= u/g)$ is the time taken by the first arrow to reach the highest point in its motion, its height $n/2$ second after that (i.e. at a time $t + n/2$) will be the same as that of the second arrow $n/2$ second before, i.e. at a time $(t - n/2)$ it reaches the highest point in its motion.
4. *True.* The (downward) velocity of the freely falling ball increases proportionally to time till it hits the plate (at a time $T = \sqrt{2h/g}$, at which instant its velocity gets reversed (i.e. upward) and has the same magnitude ($\sqrt{2gh}$). The velocity of the ball becomes zero at a time $2T (= \sqrt{8h/g})$ after which the ball again starts falling downwards and again hits the plate at a time $3T (= \sqrt{18h/g})$ and so on.
5. *False.* The average speed of the car is (total distance/total time). Assuming the total distance to be $2x$ km, the time taken is $(\frac{x}{40} + \frac{x}{60})$ h i.e. $\frac{x}{24}$ h. The average speed is, therefore $2x \div \frac{x}{24}$ i.e. 48 km/h.
6. *True.* The first stone will rise up till its (upward) velocity becomes zero after which it will start falling downwards. Its velocity on reaching back the tower will be u in the downward direction. The downward motion of the two stones from this point onwards is identical so that the two stones hit the ground with the same (downward) velocity.
7. *False.* The time of fall would be the same in each case because the cab is being assumed to move with constant velocity. A difference of the type given would arise only if the cab is accelerated.

8. *False.* The horizontal range $\left(= \frac{v_0^2 \sin 2\theta}{g}\right)$ and maximum height $\left(= \frac{v_0^2 \sin^2 \theta}{2g}\right)$ are equal if

$$\sin 2\theta = \frac{1}{2} \sin^2 \theta \quad \text{or if} \quad 4 \cos \theta = \sin \theta$$

$$\text{or if} \quad \tan \theta = 4 \quad \text{i.e.} \quad \theta = \tan^{-1} 4.$$

9. *True.* The horizontal range of a projectile being

$$R = \frac{v_0^2 \sin 2\theta}{g}$$

we have

$$\frac{dR}{dg} = -\frac{v_0^2 \sin 2\theta}{g^2} \quad (v_0, \theta \text{ being constant}).$$

$$\therefore dR = -\frac{v_0^2 \sin 2\theta}{g^2} dg = -\frac{R}{g} dg$$

so that

$$\frac{dR}{R} = -\frac{dg}{g}$$

10. *False.* The x and y co-ordinates of a projectile being

$$x = v \cos \theta t; \quad y = v \sin \theta t - \frac{1}{2} g t^2$$

the 'co-ordinates' of one projectile as seen from another projectile are

$$X = (v_1 \cos \theta_1 - v_2 \cos \theta_2) t \quad \text{and} \quad Y = (v_1 \sin \theta_1 - v_2 \sin \theta_2) t$$

Hence the equation of the path of one projectile as seen from another is

$$\frac{Y}{X} = \frac{v_1 \sin \theta_1 - v_2 \sin \theta_2}{v_1 \cos \theta_1 - v_2 \cos \theta_2} = m \text{ (say)}$$

Thus the path of one projectile as seen from another is always a straight line, not a parabolic curve.

2.6 MULTIPLE CHOICE QUESTIONS

Select the best alternative in each of the following.

1. For the displacement-time graph shown in Fig. 2.25, the ratio of the magnitudes of the (constant) speeds during the first two seconds and the next four second is

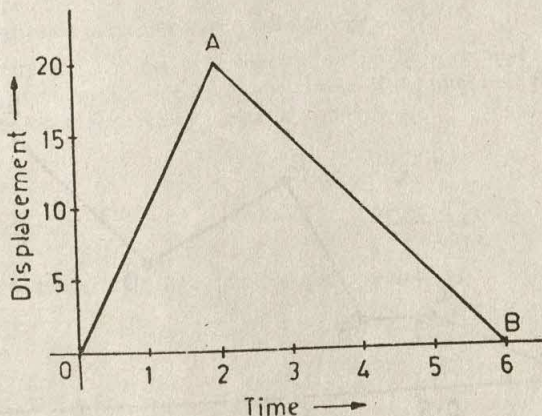


Fig. 2.25

- (a) 1 : 1
(b) 2 : 1

- (c) 1 : 2
(d) 3 : 2

2. For the velocity-time graph shown in Fig. 2.26, the total distance covered by the particle in the last two second of its motion is what fraction of the total distance covered by it in all the seven second?

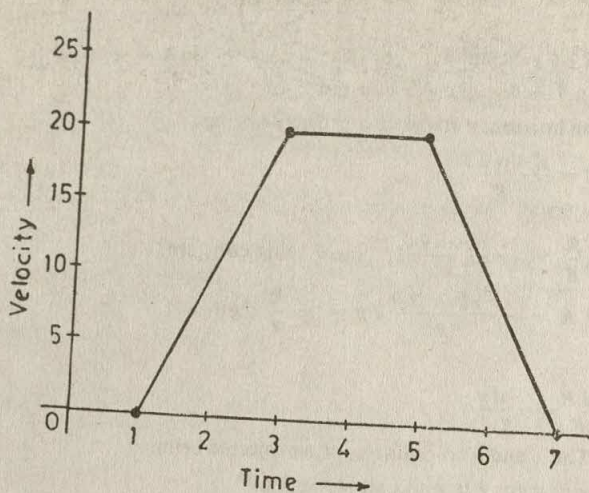


Fig. 2.26

- (a) $1/2$ (b) $2/3$ (c) $1/4$ (d) $1/3$
3. A stone falls freely from rest and the total distance covered by it in the *last* second of its motion equals the distance covered by it in the first three seconds of its motion. The stone remains in the air for
(a) 6 s (b) 5 s (c) 7 s (d) 4 s
4. The height from which the stone fell and the velocity with which it hits the ground in Q. 3 are (take $g = 10 \text{ m s}^{-2}$)
(a) 180 m, 60 m s^{-1} (b) 125 m, 50 m s^{-1}
(c) 245 m, 70 m s^{-1} (d) 80 m, 40 m s^{-1}
5. The variation in the speed of a car during its two hour journey is shown in the graph of Fig. 2.27. The magnitude of the maximum acceleration of the car occupies an interval of

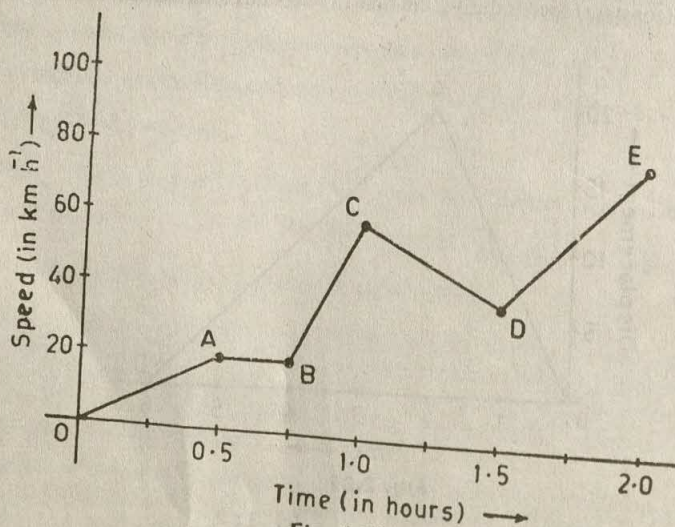


Fig. 2.27

- (a) OA (b) BC (c) CD (d) DE

6. A parachutist drops freely from an aeroplane for 10 s before the parachute opens out. Then he descends with a net retardation of 2.5 m s^{-2} . If he bails out of the plane at a height of 2495 m and $g = 10 \text{ m s}^{-2}$, his velocity on reaching the ground will be
- (a) 2.5 m s^{-1} (b) 7.5 m s^{-1}
 (c) 5 m s^{-1} (d) 10 m s^{-1}
7. The *hodograph* for the motion of a particle is defined as follows. From a fixed point O , we draw a straight line OP to represent in magnitude and direction, the instantaneous velocity of a moving particle. The *locus* of the point P gives us the *hodograph* for the motion of the particle (see Fig. 2.28)

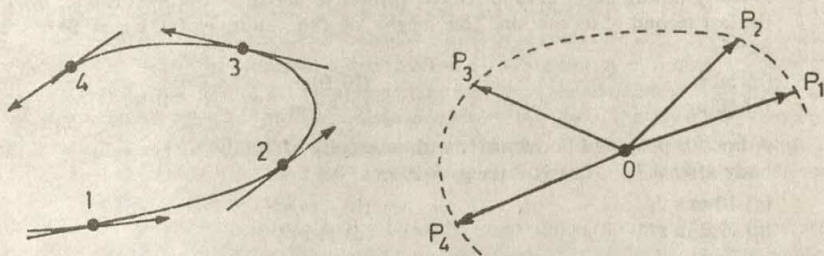


**Fig. 2.28 (i) The path of the moving particle
 (ii) The hodograph of its motion**

For a particle moving with a constant speed along a straight line AB , the hodograph will be

- (a) a straight line parallel to the line AB
 (b) a straight line inclined at 45° to the line AB
 (c) a straight line perpendicular to the line AB
 (d) just a point
8. For a particle moving with a uniform acceleration along a straight line PQ , the hodograph is
- (a) a circle
 (b) a straight line parallel to the line PQ
 (c) just a point
 (d) a straight line perpendicular to the line PQ
9. A car accelerates from rest at a constant rate α for some time after which it decelerates at a constant rate β to come to rest. If the total time elapsed is t , the maximum velocity acquired by the car is given by

- (a) $\frac{\alpha \beta}{\alpha + \beta} t$ (b) $\frac{\alpha + \beta}{\alpha \beta} t$
 (c) $\frac{\alpha^2 + \beta^2}{\alpha \beta} t$ (d) $\frac{\alpha^2 - \beta^2}{\alpha \beta} t$

10. The distance travelled by the car in Q. 9 above is given by

- (a) $\frac{1}{2} \left(\frac{\alpha \beta}{\alpha + \beta} \right) t^2$ (b) $\frac{1}{2} \left(\frac{\alpha + \beta}{\alpha \beta} \right) t^2$
 (c) $\frac{1}{2} \left(\frac{\alpha^2 + \beta^2}{\alpha \beta} \right) t^2$ (d) $\frac{1}{2} \left(\frac{\alpha^2 - \beta^2}{\alpha \beta} \right) t^2$

11. The acceleration of a particle, starting from rest, varies with time according to the relation

$$a = k t + c$$

The velocity v of the particle after a time t will be

- (a) $k t^2 + c t$ (b) $\frac{1}{2} k t^2 + c t$
 (c) $\frac{1}{2} (k t^2 + c t)$ (d) $k t^2 + \frac{1}{2} c t$

12. The acceleration of a particle, starting from rest, varies with time according to the relation

$$a = -s \omega^2 \sin \omega t$$

The displacement of this particle at a time t will be

- (a) $-\frac{1}{2} (s \omega^2 \sin \omega t) t^2$ (b) $s \omega \sin \omega t$
 (c) $s \omega \cos \omega t$ (d) $s \sin \omega t$

13. A freely falling particle, falling from a tower of height h covers a distance $h/2$ in the last second of its motion. The height of the tower is (take $g = 10 \text{ m s}^{-2}$) nearly

- (a) 58 m (b) 50 m
 (c) 60 m (d) 55 m

14. A body is projected horizontally with a velocity of 4 m s^{-1} . The velocity of the body after 0.7 s is nearly (take $g = 10 \text{ m s}^{-2}$)

- (a) 10 m s^{-1} (b) 8 m s^{-1}
 (c) 19.2 m s^{-1} (d) 11 m s^{-1}

15. At the top of the trajectory of a projectile, the directions of its velocity and accelerations are

- (a) parallel to each other
 (b) anti-parallel to each other
 (c) inclined to each other at an angle of 45°
 (d) perpendicular to each other

16. A boy aims a gun at a bird from a point, at a horizontal distance of 100 m. If the gun can impart a velocity of 500 m s^{-1} to the bullet, at what height above the bird must he aim his gun in order to hit it? (Take $g = 10 \text{ m s}^{-2}$)

- (a) 20 cm (b) 10 cm
 (c) 50 cm (d) 100 cm

17. From the top of a tower of height 40 m a ball is projected upwards with a speed of 20 m/s at an angle of elevation of 30° . Then the ratio of the total time taken by the ball to hit the ground to its time of flight (time taken to come back to the same elevation) is (Take $g = 10 \text{ m/s}^2$)

- (a) 2 : 1 (b) 3 : 1
 (c) 3 : 2 (d) 4 : 1

18. The horizontal displacement, from the foot of the tower, of the ball in the above case is nearly

- (a) 50 m (b) 70 m
 (c) 60 m (d) 80 m

19. A cannon on a level plain is aimed at an angle α above the horizontal and a shell is fired with a muzzle velocity v_0 towards a vertical cliff a distance R away. Then the height from the bottom at which the shell strikes the side walls of the cliff is

- (a) $R \sin \alpha - \frac{1}{2} \frac{g R^2}{v_0^2 \sin^2 \alpha}$ (b) $R \cos \alpha - \frac{1}{2} \frac{g R^2}{v_0^2 \cos^2 \alpha}$
 (c) $R \tan \alpha - \frac{1}{2} \frac{g R^2}{v_0^2 \cos^2 \alpha}$ (d) $R \tan \alpha - \frac{1}{2} \frac{g R^2}{v_0^2 \sin^2 \alpha}$

20. It is possible to project a particle with a given velocity in two possible ways so as to make it pass through a point at a distance r from the point of projection.

The product of the times taken to reach this point in the two possible ways is then proportional to

- (a) $1/r$
(c) r^2

- (b) r
(d) $1/r^2$

2.7 NUMERICAL EXERCISES

1. A steel ball is held by an electromagnet and when the current is switched off, it starts an electric timer which reads to the nearest 0.001 s. The ball lands on a plate 50 cm below its starting point and closes a switch, which stops the timer. The time of fall is recorded as 0.320 s. What value does it give for 'g'? How far does the ball actually fall in the first 0.05 s from rest?
2. The maximum permissible reaction time of drivers of a certain type of vehicles is 1 s. In one such test, a driver driving a vehicle at a constant speed of 20 m/s suddenly 'saw' a stationary car that was 95 m ahead of him. The driver applied the brakes after his reaction time Δt and this gave the car a retardation of 2.5 m/s. Assuming that the reaction time of the driver was good enough to just avert a collision, find his reaction time and thus decide whether the driver will pass the test or not?
3. A small ball is dropped vertically from the zero mark of a vertical scale. A 'snapshot' of the falling ball taken and shows that while the shutter is open, the ball falls from the 4.9 m to the 5.1 m mark. How long is the shutter open?
4. A helicopter ascending with a uniform vertical velocity of 5 m/s was used to drop foodpackets for people marooned in a flooded colony. If the packets reach the ground in 10 s, find the height of the helicopter when the packets hit the ground.
5. A body travels 2 m in the first two second and 2.2 m in the next four second. What will be the velocity at the end of the seventh second from the start?
6. If a , b , c be the distances moved by a particle travelling with a uniform acceleration during the x th, y th, z th second of its motion respectively, show that

$$a(y - z) + b(z - x) + c(x - y) = 0$$
7. A car is accelerating uniformly between two checkpoints that are 30 m apart. The time taken between the checkpoints is 4 s and the car's speed at the first checkpoint is 5 m/s. Find the acceleration and the speed of the car at the second checkpoint.
8. A balloon rising at 13 m s^{-1} releases a bag, at a height of 300 m above the ground. Find the maximum height attained by the bag and its position and velocity 5 s after being released. When does the bag hit the ground?
9. A particle traversed half the distance with a velocity V , the remaining part of the distance was covered with a velocity V' for half the time and with a velocity V'' for the other half of the time. Find the mean velocity of the particle averaged over the whole of its motion.
10. A circus artiste maintains four balls in motion making each in turn rise to height of 5 m from his hand. With what velocity does he project them and where will the other three balls be at the instant when the fourth one is just leaving his hand? (Take $g = 10 \text{ m s}^{-2}$).
11. A plastic ball is dropped from a height of 1 m and rebounds several times from the floor. If 1.3 s elapse from the moment it is dropped to the second impact on the floor, what is the coefficient of restitution (i.e. the ratio of the upward velocity v with which the ball rises after the first impact to the downward velocity v with which it struck the floor at the instant of its first impact)? (IIT 1977)
12. Three particles are located at the vertices of an equilateral triangle of side a . They all start moving simultaneously with a constant speed v but move in such a way that the first particle is continually headed for the second, the second for the third and the third for the first. Where and when will the particles converge?

13. A point moves rectilinearly in one direction. The graph in Fig. 2.29 shows the distance covered by the point as a function of time t . From the graph, find (a) the maximum velocity, (b) the instant of time t at which the instantaneous velocity is equal to the mean velocity averaged over the first t_0 seconds.

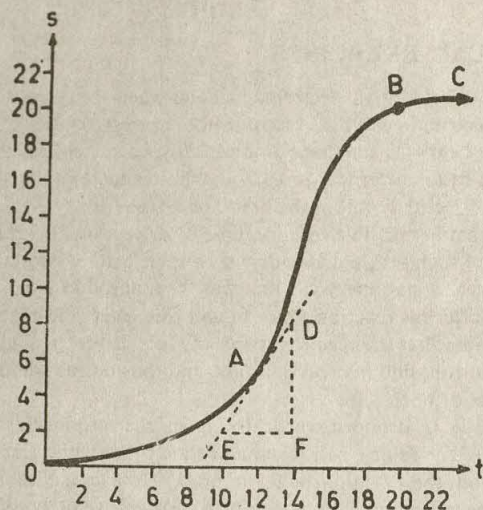


Fig. 2.29

14. A steel ball is dropped from a height of 78 m on a perfectly elastic surface. Taking the speed of the ball at $t = 0$ as zero, draw a graph representing the velocity of the ball as a function of time from $t = 0$ to $t = 8$ s. What is the velocity of the ball at $t = 1$ s and $t = 7$ s?
15. The displacement of a particle varies with time according to the relation

$$x = -15t^2 + 20t + 30$$

Find the position, velocity and acceleration of the particle at $t = 0$. When and where will the velocity of the particle become zero? Can we call the motion of the particle as one with a uniform acceleration?

16. Assuming that a ball released from rest from a considerable height feels a retardation due to air resistance proportional to its instantaneous velocity, verify that the downward vertical distance moved by it in a time t is given by

$$h = k(e^{-ct} - 1) + \frac{g}{c}t$$

where c is the proportionality constant for the retardation due to air resistance and $k = g/c^2$. When will the velocity of the ball become constant? What is the value of this constant velocity?

17. If the displacement of a particle varies with time according to the relation

$$x = \frac{u}{b}(1 - e^{-bt})$$

obtain the relations for the velocity and acceleration of the particle as a function of time and express them graphically. What are the displacement, velocity and acceleration of the particle at $t = 0$ and $t = \infty$?

18. An observer in a train moving with a uniform velocity finds that a car moving parallel to the train has a speed of 10 km/h in the direction of motion of the train. An object falls from the car and the observer in the train notices that the car has moved on for one minute, turned back, and moved with a speed of 10 km/h and picked up the object two minutes after turning. Find (a) the velocity of the train relative to the ground and (b) the velocity of the car during its forward and reverse journeys? Assume that the object comes to rest immediately on fall from the point of view of the observer on the ground.

19. A ball leaves a bat at an angle of 37° above the horizontal at a speed of 50 m s^{-1} . How high does it go? How much forward is it at its highest point? If the ball is caught at a horizontal distance of 175 m from the point where it was struck, find how far above the level at which the ball was struck, was it caught? Take $g = 9.8 \text{ m s}^{-2}$
20. An electron moves horizontally between two 'vertical' plates in an oscilloscope tube at a speed of $2 \times 10^7 \text{ m s}^{-1}$ as it enters the space between the plates. An electric field due to charge on the plates causes it to accelerate with a constant acceleration of $2.5 \times 10^{15} \text{ m s}^{-2}$ in a direction at right angles to the initial velocity of the electron. What deflection occurs from the straight line path while the electron moves forward 2.4 cm? By what angle is its direction changed?
21. A bullet is fired at an angle of 45° with the horizontal and with a velocity of 900 m s^{-1} . How high will it rise and what will be its horizontal range? What can you say about the horizontal range of the bullet if it is fired at an angle of (i) 33° and (ii) 57° ?
22. An aeroplane flying horizontally with a speed of 490 m s^{-1} , releases a bomb when at a height of 1960 m. When will the bomb hit the ground and what will be its velocity then? If the bomb was aimed at a target at a horizontal distance of 10 km from the point vertically below the point of its release, by how much distance will the bomb miss the target?
23. A projectile is fired with a muzzle velocity of 400 m s^{-1} at an angle of elevation of 15° above the horizontal. At what height will it strike a vertical cliff, distant 5000 m horizontally from the cliff? What will be its velocity when it strikes the cliff?
24. Find the angle of elevation of a gun which fires a shell with a muzzle velocity of 366.5 m s^{-1} at a target on the same level but 1527 m distant. How high will the shell rise?
25. A water hose on the ground shoots a stream of water at an angle of 40° to the horizontal with a speed of 20 m s^{-1} . Find the height at which the water stream will strike a wall 8 m away.
26. A ball has to be thrown horizontally from a window 152.4 m above the ground in one tall building so as to enter a window 15.24 m above the ground in a second tall building that is at a horizontal distance of 60.96 m from the first building. What is the speed with which the ball must be thrown?
27. An observer on the ground finds that an object is falling vertically with a constant acceleration g . How is the motion of the object described by an observer in the train moving along a straight track with a constant velocity v ? Assume that the 'origins' of the two observers are coincident at $t = 0$.
28. A projectile is fired upward with a velocity v_0 at an angle θ . It strikes a roof

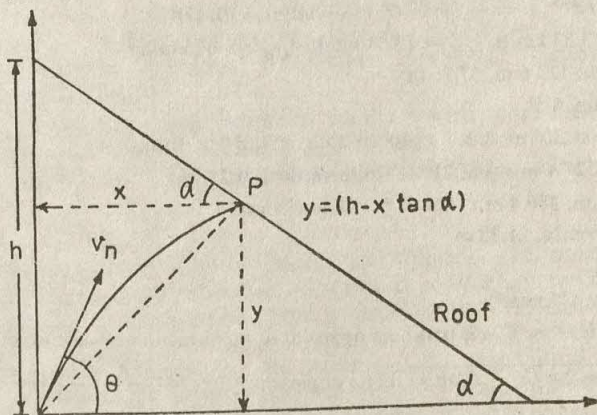


Fig. 2.30

inclined at an angle α to the negative direction of the x -axis (see Fig. 2.30). With what velocity does the projectile strike the roof in a minimum time?

29. A bomber, if flying level at a speed of 72 m s^{-1} at an elevation of $h = 100 \text{ m}$. When directly over the point O (located at $x = 0$), bomb B is released and strikes a truck which is moving along a level road (the x -axis) with a constant velocity v_2 . At the instant the bomb is released the truck is at a distance $x_0 = 125 \text{ m}$ from O . Find the value of v_2 and the time of flight of B .

ANSWERS

MULTIPLE CHOICE QUESTIONS

- | | | | |
|---------|---------|---------|---------|
| 1. (b) | 2. (c) | 3. (b) | 4. (b) |
| 5. (b) | 6. (c) | 7. (d) | 8. (b) |
| 9. (a) | 10. (a) | 11. (b) | 12. (d) |
| 13. (a) | 14. (b) | 15. (d) | 16. (a) |
| 17. (a) | 18. (b) | 19. (c) | 20. (b) |

NUMERICAL EXERCISES

- 9.765 m s^{-2} , 1.22 cm
- 0.75 s , yes
- 0.01 s
- 490 m
- 0.1 m s^{-1}
- _____
- 1.25 m s^{-2} , 10 m s^{-1}
- 308.6 m , 242 m , 36 m s^{-1} downward, 9.3 s
- $2 V(V' + V'')/(V' + V'' + 2V)$
- 10 m s^{-1} , $3.75 \uparrow$, 5 m , $3.75 \downarrow$
- 0.94
- $\text{CG}, t \approx (\pi a/3 \sqrt{3} v)$
- $v_{\max} = 1.44 \text{ m s}^{-1}$, 14.8 s after the start
- $v_1 = 9.8 \text{ m s}^{-1}$, $v_7 = 8.232 \text{ m s}^{-1}$
- $(30 \text{ m}, 20 \text{ m s}^{-1}, -30 \text{ m s}^{-2})$; $(0.67 \text{ s}, 36.67 \text{ m})$; yes
- For $t \rightarrow \infty$, g/c
- $v = u e^{-bt}$, $a = -bv$; $0 (O, u, -bu)$; $(u/b, 0, O)$
- $v_T = 3.33 \text{ km/h}$, $v_{CF} = 13.33 \text{ km/h}$, $v_{CB} = 6.67 \text{ km/h}$
- 46.2 m , 122.6 m , 37.76 m
- 1.8 mm , 8.5°
- $2.065 \times 10^4 \text{ m}$, $8.265 \times 10^4 \text{ m}$ both $= 7.551 \times 10^4 \text{ m}$
- 20 s , 527.4 m s^{-1} at $21^\circ 48'$ with vertical, 0.2 km
- 516.2 m , 386.4 m s^{-1} at 4° below horizontal
- 3.2° nearly, 21.33 m
- 5.4 m
- 11.6 m s^{-1} nearly
- $y' = h - gx^2/2v^2$, parabola opposite to the direction of motion of the train.
- $\sqrt{v_0^2 - 2g(h - x \tan \alpha)}$ at an angle $\tan^{-1} \left\{ \frac{v_0^2 \sin^2 \theta - 2g(h - x \tan \alpha)}{v_0 \cos \theta} \right\}$
- 44.33 m s^{-1} , 4.52 s

Particle Dynamics

3.1 REVIEW OF BASIC CONCEPTS

3.1.1 Newton's Laws of Motion The wellknown three laws of motion, as stated by Newton, are

First Law A particle remains at rest or moves in a straight line with constant speed unless it is compelled to change that state by an external unbalanced force.

Second Law The rate of change of momentum of a particle is directly proportional to the force acting on it and takes place in the direction of the applied force.

Third Law When two particles interact with each other, the force by the first particle on the second (action) is equal in magnitude but opposite in direction to the force exerted by the second particle on the first (reaction).

The three laws, taken together provide us with a complete insight into the meaning of force, real forces, to be precise. Whereas the first law tells us what real forces do (they produce changes in velocity or acceleration), the second law tells us that (quantitatively and dimensionally), 'force' is given by the rate of change of momentum. The third law tells us that, by nature, force is an interaction and that forces exist in pairs.

3.1.2 Inertial Frames We know that we need a 'frame of reference' to describe and understand the motion of a particle. The simplest 'frames of reference' used are known as the *inertial frames*. A frame of reference is known as an *inertial frame* if, within it, all accelerations of any particle are caused by the action of 'real forces' on that particle. When we talk about accelerations produced by 'fictitious' or 'pseudo' forces, the frame of reference is a non-inertial one.

For most practical situations, wherein we deal with particles moving with small velocities over small distances and for small time intervals, we are quite justified in treating the earth as an inertial frame. This is what we usually do.

3.1.3 Applying the Laws of Motion While considering the motion of a system made up of a number of bodies, we find it convenient to consider each body separately for applying the laws of motion. We take into account all the forces acting on the body and equate the *net* force acting on the body to its mass times acceleration. By doing so for each body, we get the 'equations of motion' of all the bodies. Each of these equations is the 'free body equation of motion' of the body. Solving all these equations, we can get the values of the unknowns in the problem.

The above technique of solving a problem is known as the 'free body technique'. If we represent all the forces acting on a body on a diagram, we get the *free body diagram* of the body. Such diagrams are very useful in visualizing the forces acting on each of the bodies of the system.

3.1.4 Momentum The linear momentum of a body is a special characteristic associated with it when it is in a state of motion. It is determined by the product of the mass and velocity of the body

$$\text{Linear momentum} = \text{mass} \times \text{velocity}$$

Like velocity, momentum is a vector quantity. It is measured in units of kg m s^{-1} in the SI system.

3.1.5 Law of Conservation of Linear Momentum According to this law, when no external force is applied to the system, its total linear momentum remains constant. In other words, in the absence of external forces the total linear momentum of a system is a conserved quantity.

It is important to understand the words 'external' and 'system' for a given problem. Thus, for a system of two interacting particles, the forces exerted by one particle on the other are *not* external to the system.

The law of conservation of linear momentum along with the general form of the law of conservation of energy are the two most general laws of physics.

3.1.6 Friction The opposition to any relative motion between two surface in contact is referred to as *friction*. It arises because of the 'intermeshing' of the surface irregularities of the two surfaces in contact. We say that because of this intermeshing at the points of actual contact, the two surfaces may be considered as having formed a 'cold weld'. The external forces tending to produce relative motion between the two surfaces has to untangle these 'intermeshings' or to break apart these 'cold welds'. It, therefore, experiences an opposition and this manifests itself as the force of friction.

It is clear that this opposition to the external force will always be present irrespective of the direction in which the external force may be applied. We, therefore, say that the force of friction always oppose the relative motion between the two surfaces in contact. Equivalently, we say that the direction of the force of friction is always opposite to the direction in which the motion is intended.

3.1.7 Effect of Polishing or Smoothness of the Surfaces When the surfaces in contact are progressively smoothened, we first experience a decrease in the force of friction. This is because of the reduction in the 'intermeshing' of the surface irregularities or a reduction in the 'intermeshing' of the surface irregularities or a reduction in the number of 'cold welds' between the surfaces. However, if the surfaces are so smooth that their surface irregularities are less than the range of intermolecular attraction ($\sim 10^{-8}$ m), there is a direct interaction between the molecules of the two surfaces. At this stage, the two surfaces will be held together very strongly and it will be very difficult to produce a relative motion between them. Thus, continued polishing or smoothening of the two surfaces in contact after first decreasing friction between them, will later on, cause a very sharp increase in the force of friction.

3.1.8 Static and Dynamic (Kinetic) Friction The frictional forces between two surfaces in contact (i) before and (ii) after a relative motion between

them has started, are referred to as *static* and *dynamic friction* respectively. Static friction is always a little more than dynamic friction.

3.1.9 Limiting Friction Before a relative motion actually starts, the force of static friction between the two surfaces is a *self-adjusting* force. Its magnitude is equal to that of the external force. The force of friction (under given external conditions) can, however, increase only upto a limit and when the external force is increased beyond this limit, the relative motion between the surfaces ensues. This maximum value of the force of friction between the two surfaces (under given external conditions) is known as their limiting friction.

3.1.10 Laws of Friction The limiting friction between two surfaces in contact.

- (i) does not depend upon their (macroscopic) area of apparent contact
- (ii) acts in the plane of motion and opposite to the direction of motion
- (iii) is proportional to the normal force keeping the two surfaces together
- (iv) depends on the (microscopic) area of actual contact.

3.1.11 Coefficient of Friction The coefficient of friction (μ) between two surfaces is the ratio of their limiting frictional force to the normal force between them, i.e.

$$\mu = \frac{\text{limiting frictional force}}{\text{normal reaction}} = \frac{F}{R}$$

When there is no relative motion $\mu = \mu_s$, the coefficient of static friction. When there is a relative motion $\mu = \mu_k$, the coefficient of kinetic friction.

3.1.12 Angle of Friction It is the angle which the resultant of the force of limiting friction ($\vec{F}$) and the normal reaction ($\vec{R}$) makes with the direction of the normal reaction. If θ is the angle of friction, we have (see Fig. 3.1).

$$\tan \theta = \frac{F}{R} = \mu$$

$$\therefore \theta = \tan^{-1} \mu$$

3.1.13 Angle of Repose When a body is kept in contact with an inclined plane surface, it just starts sliding down the plane for a certain angle of inclination of the surface with the horizontal. This angle (α) is called the angle of repose of the inclined surface with respect to the body in contact with it.

Referring to Fig. 3.2, we have

$$mg \sin \alpha = F = \text{force of limiting friction}$$

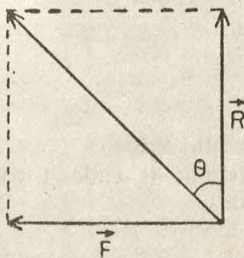


Fig. 3.1

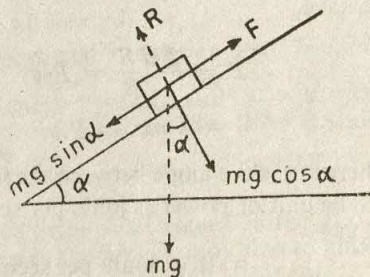


Fig. 3.2

and $mg \cos \alpha = R = \text{force of normal reaction}$

Therefore, $\tan \alpha = \frac{F}{R} = \mu = \tan \theta$

Thus, $\alpha = \theta = \tan^{-1} \mu$

3.1.14 Work Work is said to be done by a force if, as a result of the application of the force, its point of application is displaced along some direction. It is measured by the product of the force and the distance moved in the direction of the force, i.e.

$$W = \vec{F} \cdot \vec{S}$$

In general, the force F and the displacement S of its point of application may not be in the same direction. Then

$$W = \vec{F} \cdot \vec{S} = FS \cos \theta$$

where θ is the angle between vectors F and S .

The SI unit of work is called the joule (J). One joule of work is said to be done if, as a result of the application of one newton of force, its point of application moves through one metre in the direction of the force. Thus

$$\begin{aligned} 1 \text{ joule} &= 1 \text{ newton} \times 1 \text{ metre} = 1 \text{ newton-metre} \\ &= 1 \text{ N-m} \end{aligned}$$

3.1.15 Energy The energy of a body is its capacity for doing work. Anything which is able to do work is said to possess energy. Energy is measured in the same units as those of work, namely, joule.

The energy possessed by a body by virtue of its position or condition is known as its *potential energy*.

The energy possessed by a body by virtue of its motion is known as its *kinetic energy*.

3.1.16 Power It is the rate of doing work, i.e. the work done per unit time.

$$P = \frac{dW}{dt}$$

Since $W = \vec{F} \cdot \vec{S}$, we have

$$P = \frac{d}{dt} (\vec{F} \cdot \vec{S}) = \vec{F} \cdot \frac{d\vec{S}}{dt} + \vec{S} \cdot \frac{d\vec{F}}{dt}$$

When we are concerned with the rate of doing work by a constant force, we have,

$$\begin{aligned} P &= \vec{F} \cdot \frac{d\vec{S}}{dt} = \vec{F} \cdot \vec{v} \\ &= Fv \cos \alpha \end{aligned}$$

where α is the angle between the force $\vec{F}$ and the velocity $\vec{v}$.

The unit of power is joule per second. This unit is called the watt (W). Thus

$$\text{watt} = \text{joule per second}$$

$$\text{or } W = \text{J s}^{-1}$$

3.1.17 Horse Power It is a popular unit of power and is equal to 746 W.

i.e. $1 \text{ H P} = 746$

3.1.18 Electric energy is usually measured in units of kilowatt hours (kWh)

$$\begin{aligned} 1 \text{ kWh} &= 1000 \text{ W} \times (60 \times 60 \text{ s}) \\ &= 1000 \text{ J s}^{-1} \times 3600 \text{ s} \\ &= 3.6 \times 10^6 \text{ J} = 3.6 \text{ mega joule} \end{aligned}$$

3.2 SI UNITS

Table 3.1

Physical Quantity	SI Unit
Momentum	kilogram metre per second (kg m s^{-1})
Force	newton (N)
Coefficient of friction	No units
Work	joule (J) or newton metre (N.m)
Energy	joule (J)
Power	watt (W) or joule per second (J s^{-1})

3.3 FORMULAE

$$1. F = \frac{dp}{dt} = \frac{d}{dt}(mv) = m \frac{dv}{dt} + v \frac{dm}{dt}$$

where F is the force and p is the momentum

$$2. F = m \frac{dv}{dt} = ma \text{ (for constant } m); a = \text{acceleration,}$$

$$3. p = m v$$

$$4. \text{ In the absence of external forces } \sum m_i v_i = \text{constant}$$

5. $\mu = F/R$ where μ = coefficient of friction, F = limiting frictional force, and R = normal reaction.

$$6. \theta = \tan^{-1} \mu, \text{ where } \theta \text{ is the angle of friction.}$$

$$7. \alpha = \theta, \text{ where } \alpha \text{ is the angle of repose and } \theta \text{ is the angle of friction.}$$

8. $W = \vec{F} \cdot \vec{S}$, where W = work done, $\vec{F}$ = force and $\vec{S}$ = displacement of the point of application of the force. Equivalently, $W = FS \cos \theta$, where θ is the angle between $\vec{F}$ and $\vec{S}$. For $\theta = 0$ (i.e. S parallel to F); $W = FS$.

9. Potential energy of a mass m raised through a height h in the gravitational field of the earth = mgh where g is the acceleration due to gravity.

$$10. \text{ Kinetic energy} = \frac{1}{2} m v^2, \text{ where } m = \text{mass, and } v = \text{its velocity.}$$

11. $P = \frac{dW}{dt}$, where $P = \text{Power}$. In general,

$$\begin{aligned}
 P &= \vec{F} \cdot \frac{d\vec{S}}{dt} + \vec{S} \cdot \frac{d\vec{F}}{dt} \\
 &= \left. \begin{aligned} &\vec{F} \cdot \frac{d\vec{S}}{dt} \\ &\vec{F} \cdot \vec{v} \end{aligned} \right\} \text{, if } F \text{ is a constant force.}
 \end{aligned}$$

3.4 NUMERICAL EXAMPLES

Example 1: An object weighing 70 kg is kept in a lift. Find its weight as recorded by a spring balance when the lift (a) moves upwards with a uniform velocity of 5 m s^{-1} , (b) moves upwards with a uniform acceleration of 2.2 m s^{-2} , (c) moves downwards with a uniform acceleration of 2.8 m s^{-2} and (d) falls freely under gravity.

Solution: (a) When the lift moves upwards with a uniform velocity of 5 m s^{-1} (acceleration is zero), the reaction R or the pressure on the base is

$$R = mg = 70 \times 9.8 \text{ N} = 686 \text{ N}$$

(b) When the lift is moving upwards with a uniform acceleration of 2.2 m s^{-2} , the reaction R' or the pressure on the base increases and is given by

$$\begin{aligned}
 R' &= m(g + a) \\
 &= 70(9.8 + 2.2) \text{ N} = 840 \text{ N}
 \end{aligned}$$

(c) When the lift descends with a uniform acceleration of 2.8 m s^{-2} , the reaction R'' is given by

$$R'' = m(g - a) = 70(9.8 - 2.8) \text{ N} = 490 \text{ N}$$

(d) When the lift falls freely under gravity, the reaction R''' is given by

$$R''' = m(g - g) = 0$$

i.e. the object appears to have become weightless.

Example 2: The bodies with masses 10 kg and 12 kg are connected by a light inextensible string passing over a smooth fixed pulley. Find (a) the velocity at the end of 3 s (b) the distance covered in 3 s (c) if, at the end of 3 s the string is cut, find the distance moved by the bodies in the next 6 s.

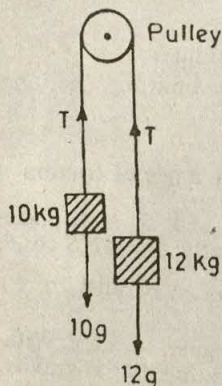


Fig. 3.3

Solution: Let T be the tension in the string and a the common acceleration of the two masses (see Fig. 3.3)

Writing the equations of motion of the two masses, we have

$$12 \times g - T = 12 \times a$$

$$\text{and} \quad T - 10 \times g = 10 \times a$$

Adding these equations, we get

$$a = \frac{g}{11} = \frac{9.8}{11} = 0.89 \text{ m s}^{-2}$$

(a) Since the bodies start from rest, their velocity at the end of 3 s is

$$v_3 = 0 + 0.89 \times 3 = 2.67 \text{ m s}^{-1}$$

(b) The distance moved by each body in 3 s is

$$s_3 = 0 + 0.89 \times (3)^2 = 4.005 \text{ m}$$

(c) At the end of 3 s, the string is cut. The bodies now fall freely under gravity. We now have

For the 10 kg mass

$$u = 2.67 \text{ m s}^{-1} \quad \text{upwards}$$

$$a = 9.8 \text{ m s}^{-2} \quad \text{downwards}$$

$$t = 6 \text{ s}$$

$$s = -2.67 \times 6 + \frac{1}{2} \times 9.8 \times (6)^2 \\ = 160.38 \text{ m}$$

For the 12 kg mass

$$u = 2.67 \text{ m s}^{-1} \quad \text{downwards}$$

$$a = 9.8 \text{ m s}^{-2} \quad \text{downwards}$$

$$t = 6 \text{ s}$$

$$\therefore s = (2.67 \times 6 + \frac{1}{2} \times 9.8 \times 6^2) \text{ m} \\ = 192.4 \text{ m}$$

Example 3: A cricket ball of mass 150 g is moving with a velocity of 12 m s^{-1} and is hit by a bat so that the ball is turned back with a velocity of 20 m s^{-1} . The force of the blow acts for 0.01 s. Find the average force exerted on the ball by the bat. (IIT 1974)

Solution: The impulse of the force exerted by the bat is given by the change in the momentum of the ball. Now

$$\text{Initial momentum of the ball} = \frac{150}{1000} \times 12 \text{ kg m s}^{-1} \\ = 1.8 \text{ kg m s}^{-1}$$

$$\text{Final momentum of the ball} = -\frac{150}{1000} \times 20 \text{ kg m s}^{-1} \\ = -3.0 \text{ kg m s}^{-1}$$

$$\text{Change in momentum of the ball} = [1.8 - (-3.0)] \text{ kg m s}^{-1} \\ = 4.8 \text{ kg m s}^{-1}$$

This equals the impulse of the force exerted by the bat. Since

$$\text{Impulse} = \text{force} \times \text{time}$$

we have

$$\text{Average force exerted} = \frac{\text{impulse}}{\text{time}} \\ = \frac{4.8 \text{ kg m s}^{-1}}{0.01 \text{ s}} = 480 \text{ kg m s}^{-2} \\ = 480 \text{ N}$$

Example 4: Two particles of equal masses m and m are connected by a light string of length $2l$ as shown in Fig. 3.4. A constant force F is applied continuously always along the perpendicular bisector of the straight line joining the two particles at the midpoint of the string. What is the acceleration of approach between the two particles at the instant the distance between them has become $2x$?

Solution: The situation corresponding to the instant when the distance between the two particles has become $2x$ is shown in Fig. 3.4(b). If T is the tension in each part of the string, we have

$$2T \sin \theta = F \quad \text{or} \quad T = \frac{F}{2} \sin \theta$$

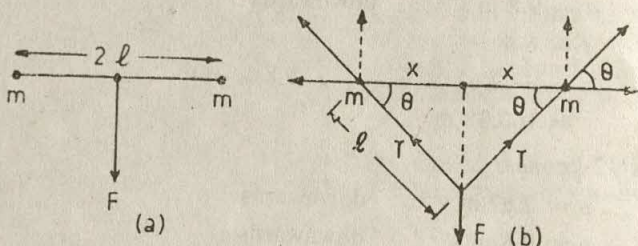


Fig. 3.4 (a) and (b)

The horizontal force acting on each mass is $T \cos \theta$, Therefore, the acceleration of each mass is

$$a = \frac{T \cos \theta}{m} = \frac{F}{2m} \cot \theta$$

From Fig. 3.4 (b), we have

$$\cot \theta = \frac{x}{\sqrt{l^2 - x^2}}$$

$$a = \frac{F}{2m} \frac{x}{\sqrt{l^2 - x^2}}$$

Since the two particles are approaching each other with this acceleration, their acceleration of approach is

$$\begin{aligned} 2 \times a &= 2 \times \frac{F}{2m} \frac{x}{\sqrt{l^2 - x^2}} \\ &= \frac{F}{m} \frac{x}{\sqrt{l^2 - x^2}} \end{aligned}$$

Example 5: The blocks connected by an inextensible string passing over a light frictionless pulley are resting on two smooth inclined planes as shown

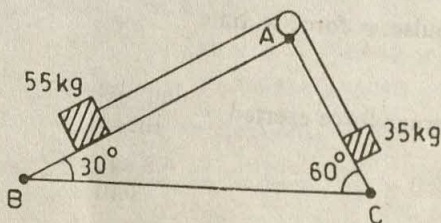


Fig. 3.5

in Fig. 3.5. Determine the acceleration of the blocks and the tension in the strings. Assume string to be massless.

Solution: The forces acting on each of the two blocks are shown in Fig. 3.6(a) and (b) respectively. The components of the weights of each of the two blocks perpendicular to their respective inclined planes are balanced by the normal reactions N and N' respectively at these planes.

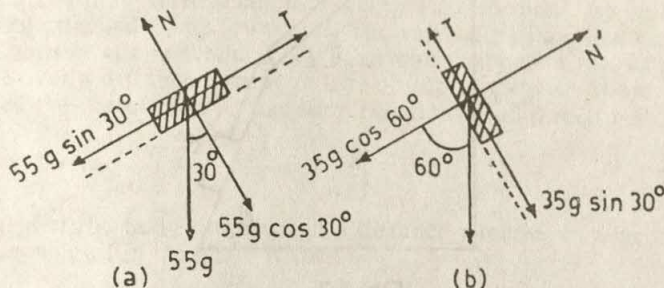


Fig. 3.6 (a) and (b)

The tension in the string and the acceleration of both the blocks are the same, as the blocks are connected by a light and inextensible string passing over a frictionless pulley. Writing the equation of motion for the two masses, we have

$$55g \sin 30^\circ - T = 55 \times a$$

and $T - 35g \sin 60^\circ = 35 \times a$

Solving, we get

$$g(55 \sin 30 - 35 \sin 60^\circ) = (55 + 35) a$$

Therefore

$$\begin{aligned} a &= \frac{9.8(55 \times 0.5 - 35 \times 0.866)}{90} \\ &= \frac{9.8(27.5 - 30.311)}{90} \\ &= \frac{-9.8 \times 2.811}{90} = -0.306 \text{ m s}^{-2} \end{aligned}$$

The negative sign shows that, contrary to our original assumption, it is the 35 kg mass that moves downwards. Thus the 35 kg mass descends and the 55 kg mass ascends from their respective planes.

Tension T is given by

$$\begin{aligned} T &= 55g \sin 30^\circ - 55 \times (-0.306) \\ &= 55(9.8 \times 0.5 + 0.306) \\ &= 55 \times 5.206 \\ &= 286.33 \text{ N} \end{aligned}$$

Example 6: Two cubes of masses m_1 and m_2 lie on two frictionless slopes of a block A which rests on a horizontal table. The cubes are connected by a string which passes over a pulley as shown in Fig. 3.7. To what horizontal acceleration f should the whole system (i.e. the block and

cubes) be subjected so that the cubes do not slide down the plane? What is the tension in the string in this situation?

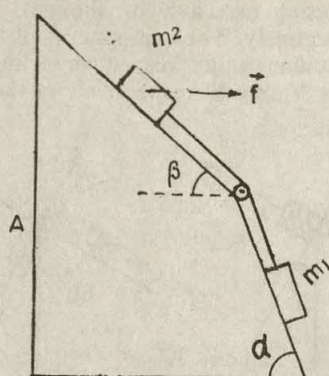


Fig. 3.7

Solution: Let T be the tension in the connecting string. The forces on the two masses are shown in Fig. 3.8 (a) and (b) respectively.

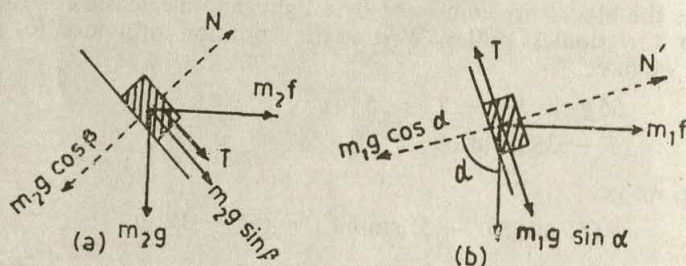


Fig. 3.8 (a) and (b)

In order that the cubes do not slide down the plane, we must have

$$T + m_2 g \sin \beta = m_2 f \cos \beta \quad (i)$$

and

$$m_1 g \sin \alpha - T = m_1 f \cos \alpha \quad (ii)$$

These two equations can be solved to obtain f and T . Adding (i) and (ii) we get

$$g (m_2 \sin \beta + m_1 \sin \alpha) = f (m_2 \cos \beta + m_1 \cos \alpha)$$

Thus

$$f = \frac{m_1 \sin \alpha + m_2 \sin \beta}{m_1 \cos \alpha + m_2 \cos \beta} g \quad (iii)$$

Substituting this value of f in, say Eq. (i) we have

$$\begin{aligned} T &= m_2 f \cos \beta - m_2 g \sin \beta \\ &= m_2 g \left[\frac{m_1 \sin \alpha + m_2 \sin \beta}{m_1 \cos \alpha + m_2 \cos \beta} \cos \beta - \sin \beta \right] \end{aligned}$$

$$\begin{aligned}
 &= \frac{m_2 g}{m_1 \cos \alpha + m_2 \cos \beta} [m_1 \sin \alpha \cos \beta \\
 &\quad - m_1 \cos \alpha \sin \beta] \\
 &= \frac{m_1 m_2 g \sin (\alpha - \beta)}{m_1 \cos \alpha + m_2 \cos \beta}
 \end{aligned}$$

Example 7 A light vertical chain is being used to haul up an object of mass M kg attached to its lower end. The vertical pull applied has a magnitude F newton at $t = 0$ and it decreases uniformly at a rate of f newton per metre over a distance h through the object is raised. Show that the velocity of the object after it has been raised through h metre is given by

$$v = \left\{ \frac{2h}{m} \left(F - Mg - f \frac{h}{2} \right) \right\}^{1/2}$$

Solution: Let the body rise through a distance y metre in t second. The instantaneous pull in the chain is then

$$P = (F - fy) \quad \text{newton}$$

If a is the upward acceleration of the object at this instant, we have, from Newton's second law,

$$P - Mg = Ma$$

$$a = \frac{F - fy - Mg}{M}$$

But
$$a = \frac{dv}{dt} = \frac{dv}{dy} \frac{dy}{dt} = v \frac{dv}{dy} = \frac{1}{2} \frac{d}{dy} (v^2)$$

$$\therefore \frac{d}{dy} (v^2) = \frac{2}{M} (F - fy - Mg)$$

or
$$d(v^2) = \frac{2}{M} (F - fy - Mg) dy$$

If V is the velocity of the body when it has risen h metre, we have, on integration between the limits $y = 0$ to $y = h$,

$$\int_0^V d(v^2) = \frac{2}{M} \int_0^h (F - fy - Mg) dy$$

$$\begin{aligned}
 V^2 &= \left[\frac{2}{M} (F - Mg) y \right]_0^h - \left[\frac{2fy^2}{M} \right]_0^h \\
 &= \frac{2h}{M} \left(F - Mg - \frac{fh}{2} \right)
 \end{aligned}$$

$$\therefore V = \left\{ \frac{2h}{M} \left(F - Mg - \frac{fh}{2} \right) \right\}^{1/2}$$

Example 8: An aeroplane requires for take off a speed of 80 km per hour; the run on the ground being 100 m. The mass of the plane is 10,000 kg and the coefficient of friction between the plane and the ground is 0.2. Assume that the plane accelerates uniformly during the take off. What is the minimum force required by the engine of the plane for take off? (IIT 1977)

Solution: The engine needs force to (i) accelerate the plane from rest to a speed of 80 km/hour ($= 200/9 \text{ m s}^{-1}$) over a distance of 100 m and (ii) to

overcome the force of friction ($= \mu R = \mu mg$). The total force needed is the sum of the forces needed in (i) and (ii).

To calculate the force needed for accelerating the plane, we note that, since $v^2 - u^2 = 2as$, we have

$$\left(\frac{200}{9}\right)^2 - 0^2 = 2 \times a \times 100$$

$$\therefore a = \frac{200 \times 200}{9 \times 9 \times 200} \text{ m s}^{-2}$$

$$= 2.469 \text{ m s}^{-2}$$

The force needed for providing this acceleration is

$$\begin{aligned} \text{Force} &= \text{mass} \times \text{acceleration} \\ &= 10,000 \times 2.469 \text{ N} \\ &= 24690 \text{ N} \end{aligned}$$

The force needed to overcome the force of friction is

$$\begin{aligned} \mu mg &= 0.2 \times 10,000 \times 9.8 \text{ N} \\ &= 19600 \text{ N} \end{aligned}$$

Hence the total force needed $= (24690 + 19600) \text{ N}$

$$= 44290 \text{ N} = 4.429 \times 10^4 \text{ N}$$

Example 9: An object of weight W is resting on an inclined plane inclined at an angle θ to the horizontal. Show that the ratio of the minimum horizontal force needed to (a) just push the object up the plane and (b) just prevent it from sliding down is

$$\left(\frac{\sin \theta + \mu \cos \theta}{\sin \theta - \mu \cos \theta}\right) \left(\frac{\cos \theta + \mu \sin \theta}{\cos \theta - \mu \sin \theta}\right)$$

where μ is the coefficient of static friction.

Solution: Let P_1 and P_2 be the minimum horizontal forces needed to (a) just push the object up the plane and (b) just prevent it from sliding down respectively. The various forces keeping the object in equilibrium in the two cases are shown in Fig. 3.9 (a) and (b) respectively.

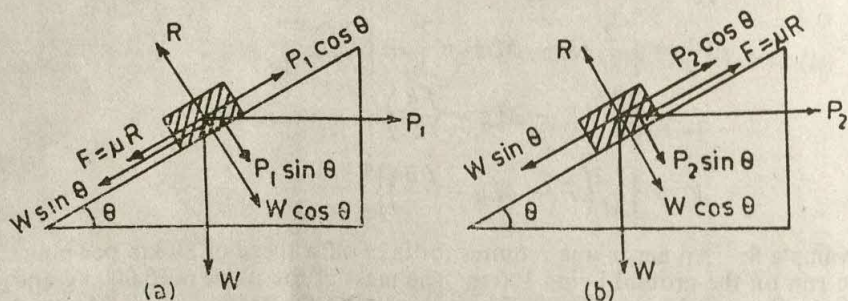


Fig. 3.9 (a) and (b)

For case (a) we have

$$R = W \cos \theta + P_1 \sin \theta$$

and
$$P_1 \cos \theta = W \sin \theta + \mu (W \cos \theta + P_1 \sin \theta)$$

$$P_1 = \frac{W (\sin \theta + \mu \cos \theta)}{(\cos \theta - \mu \sin \theta)} \quad (i)$$

For case (b) we have

$$R = W \cos \theta + P_2 \sin \theta$$

and
$$P_2 \cos \theta + \mu (W \cos \theta + P_2 \sin \theta) = W \sin \theta$$

$$\therefore P_2 = \frac{W (\sin \theta - \mu \cos \theta)}{(\cos \theta + \mu \sin \theta)} \quad (ii)$$

From Eqs (i) and (ii) we have

$$\frac{P_1}{P_2} = \frac{(\sin \theta + \mu \cos \theta)}{(\sin \theta - \mu \cos \theta)} \times \frac{(\cos \theta + \mu \sin \theta)}{(\cos \theta - \mu \sin \theta)}$$

Example 10: An object is first sliding down an inclined plane of inclination θ with a constant velocity v . It is next projected up the plane with an initial velocity u . Calculate the distance up to which it will rise before coming to rest. What will happen to the object after it has come to rest?

Solution: Let m be the mass of the object. The forces acting on it when it is (a) moving down, and (b) moving up the 'incline' are as shown in Figs. 3.10(a) and (b) respectively.

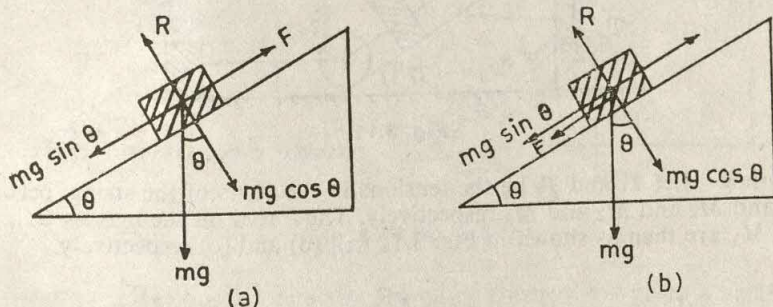


Fig. 3.10 (a) and (b)

Case (a): Since the object is sliding down with a constant velocity, the net force acting on it must be zero. Thus

$$F = m g \sin \theta$$

But
$$F = \mu_k R = \mu_k m g \cos \theta$$

where, μ_k is the coefficient of friction (kinetic). Therefore,

$$\mu_k m g \cos \theta = m g \sin \theta$$

or
$$\mu_k = \tan \theta$$

Case (b): Since the body is projected upwards with an initial velocity u , it experiences a downward acceleration of magnitude

$$(m g \sin \theta + \mu_k m g \cos \theta)/m \quad \text{or} \quad (g \sin \theta + \mu_k g \cos \theta)$$

But,
$$\mu_k = \tan \theta, \text{ therefore,}$$

$$\begin{aligned} \text{Downward acceleration} &= g \sin \theta + \tan \theta \cdot g \cos \theta \\ &= 2g \sin \theta \end{aligned}$$

Let x be the distance moved up the plane before the object comes to rest. We then have

$$0 - u^2 = 2 \times (-2g \sin \theta) x$$

$$x = u^2 / 4g \sin \theta$$

Once the object comes to rest, the frictional force to be overcome is the force of static friction. The object can, therefore, slide down the plane if its inclination is $\tan^{-1} \mu_s$. Since $\mu_s > \mu_k$ and the inclination of the given plane is $\tan^{-1} \mu_k$, the object will stay at rest and will not slide down again.

Example 11: Masses M_1 , M_2 and M_3 are connected by strings of negligible mass which pass over massless and frictionless pulleys P_1 and P_2 as shown in Fig. 3.11. The masses move such that the portion of the string between P_1 and P_2 is parallel to the inclined plane and the portion of the string between P_2 and M_3 is horizontal. The masses M_2 and M_3 are 4 kg each and the co-efficient of kinetic friction between the surfaces is 0.25. If the mass M_1 moves downwards with a uniform velocity, find

(i) the mass M_1 and (ii) the tension in the horizontal portion of the string.
(Take $g = 9.8 \text{ m s}^{-2}$ and $\sin 37^\circ = 3/5$) (IIT 1981)

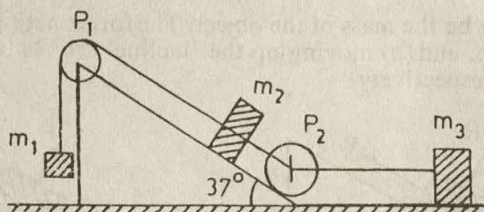


Fig. 3.11

Solution: Let T_1 and T_2 be the tensions in the parts of the string between M_1 and M_2 and M_2 and M_3 respectively. The forces on the masses M_1 , M_2 and M_3 are then as shown in Fig. 3.12 (a), (b) and (c) respectively.

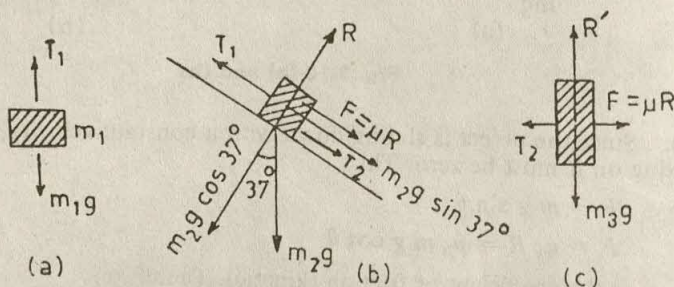


Fig. 3.12 (a), (b) and (c)

For mass M_1 , we have

$$T_1 = M_1 g \quad (i)$$

For mass M_2 , we have (since $\sin 37^\circ = 3/5$ and $\cos 37^\circ = 4/5$)

$$\begin{aligned} T_1 &= T_2 + M_2 g \sin 37^\circ + \mu M_2 g \cos 37^\circ \\ &= T_2 + 4 \times 9.8 \times 3/5 + 0.25 \times 4 \times 9.8 \times 4/5 \\ &= T_2 + 23.52 + 7.84 \\ &= T_2 + 31.36 \end{aligned} \quad (ii)$$

For mass M_3 , we have

$$\begin{aligned} T_2 = F &= \mu R = \mu M_3 g \\ &= 0.25 \times 4 \times 9.8 \\ &= 9.8 \text{ N} \end{aligned} \quad (\text{iii})$$

Hence $T_1 = (9.8 + 31.36) = 41.16 \text{ N}$

Using this value of T_1 in Eq. (i), we get

$$M_1 = T_1/g = \frac{41.16}{9.8} \text{ kg} = 4.2 \text{ kg}$$

Thus the mass M_1 is 4.2 kg and the tension (T_2) in the horizontal portion of the string is 9.8 N.

Example 12: A right angled triangular block A has a body B of mass m placed on it as shown in Fig. 3.13 (a). Block A is given a horizontal acceleration a directed towards the left. What is the maximum value of this acceleration so that B is stationary relative to A ? The coefficient of friction μ is such that $\mu < \cot \theta$.

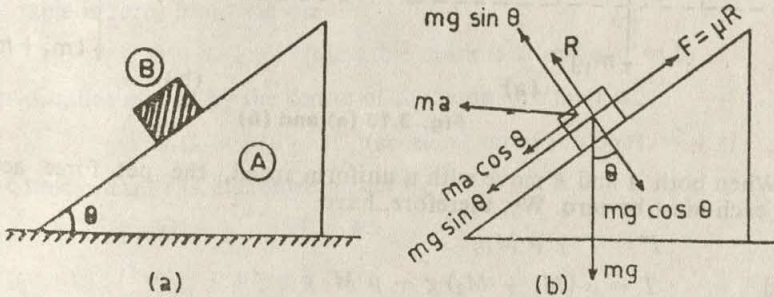


Fig. 3.13 (a) and (b)

Solution: The various forces acting on block B when A is given a horizontal acceleration directed to the left are as shown in Fig. 3.13 (b). Since the block has to remain at rest relative to A , we must have

$$m a \sin \theta + R = m g \cos \theta$$

$$\text{or } R = m (g \cos \theta - a \sin \theta)$$

$$\begin{aligned} \text{and } F &= \mu R = \mu m (g \cos \theta - a \sin \theta) \\ &= m (a \cos \theta + g \sin \theta) \end{aligned}$$

$$\text{Therefore, } \mu g \cos \theta - g \sin \theta = a (\cos \theta + \mu \sin \theta)$$

$$a = \frac{g (\mu \cos \theta - \sin \theta)}{(\cos \theta + \mu \sin \theta)}$$

$$\text{or } a = \frac{g (\mu \cot \theta - 1)}{(\cot \theta + \mu)}$$

Example 13: Two blocks A and B of masses M_1 and M_2 respectively are placed on each other and their combination rests on a fixed horizontal surface C . A massless string passing over a smooth pulley as shown in Fig. 3.14 is used to connect A and B . Assuming the co-efficient of sliding friction is

tion between all the surfaces to be μ , show that both A and B will move with a uniform speed if A is dragged with a force $\mu (3 M_1 + M_2) g$ to the left.

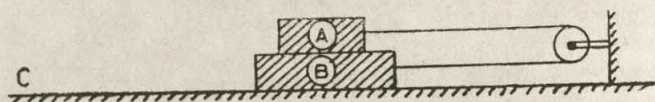


Fig. 3.14

Solution: When A is dragged to the left with, say, a force P , the block B starts moving to the right. The forces acting on A and B are, therefore, as shown in Fig. 3.15 (a) and (b) respectively. Here T is the tension in the connecting string.

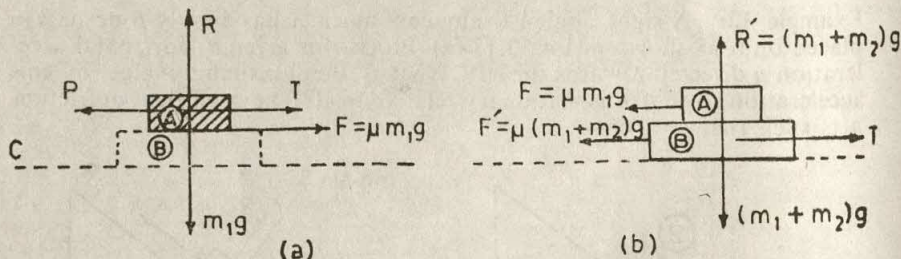


Fig. 3.15 (a) and (b)

When both A and B move with a uniform speed, the net force acting on each must be zero. We, therefore, have

$$P = T + \mu M_1 g$$

and

$$\begin{aligned} T &= \mu (M_1 + M_2) g + \mu M_1 g \\ &= \mu (2 M_1 + M_2) g \end{aligned}$$

Hence,

$$\begin{aligned} P &= \mu (2 M_1 + M_2) g + \mu M_1 g \\ &= \mu (3 M_1 + M_2) g \end{aligned}$$

Example 14: A book rests on a table cloth spread over a table as shown in Fig. 3.16. The centre of the book is at a distance of b from the edge of the table at time $t = 0$; the table cloth is suddenly pulled at this instant with an acceleration a . Show that the cloth will slip from under the book if $a > \mu g$, where μ is the coefficient of sliding friction between the table cloth and the book. Assuming this condition to be satisfied, calculate (i)

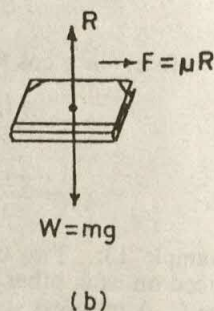
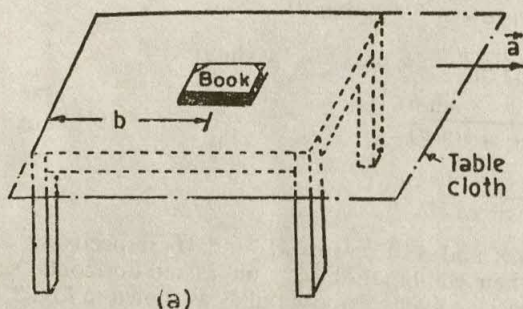


Fig. 3.16

time instant, (ii) the velocity, and (iii) the distance of the centre of the book from the edge of the table when the edge of the table cloth passes over the centre of the book.

Solution: The forward pulling of the table cloth will tend to push the book backwards. The force of friction on the book will, therefore, be in the forward direction and will have a magnitude

$$F = \mu R = \mu m g$$

where m is the mass of the book. The acceleration of the book will, therefore, be

$$\frac{F}{m} = \frac{\mu m g}{m} = \mu g$$

Thus if, $\mu g < a$, the acceleration of the table cloth, the cloth will slip from under the book.

At the instant (say $t = t$), the edge of the table cloth comes under the centre of the book, the distance of this table cloth edge and the centre of the book from the edge of the table must be the same. Now the distance moved by the edge of the table cloth (the initial distance from the edge of the table is zero) in a time t is

$$x = 0 + \frac{1}{2} a t^2 \text{ (the table cloth is at rest at } t = 0 \text{)}$$

The distance moved by the centre of the book in time t is

$$x' = 0 + \frac{1}{2} (\mu g) t^2 \text{ (acceleration of the book } = \mu g \text{)}$$

The time instant t is, therefore, given by

$$\frac{1}{2} a t^2 = \frac{1}{2} \mu g t^2 + b$$

$$\text{or } t^2 (a - \mu g) = 2b$$

$$\text{or } t = \sqrt{\frac{2b}{(a - \mu g)}}$$

The velocity of the book and the distance of its centre from the edge of the table at this instant are given by

$$\begin{aligned} v &= 0 + \mu g \sqrt{\frac{2b}{(a - \mu g)}} \\ &= \sqrt{\frac{2b \mu^2 g^2}{(a - \mu g)}} \end{aligned}$$

$$\text{and } s = b + \frac{1}{2} \mu g \left(\frac{2b}{(a - \mu g)} \right)$$

$$= b \left\{ 1 + \frac{\mu g}{(a - \mu g)} \right\}$$

$$= \frac{ab}{(a - \mu g)}$$

Example 15: A 0.5 kg block slides from point A (Fig. 3.17) on a horizontal track with an initial speed of 3 m s^{-1} towards a weightless spring of length 1 m and having a force constant 2 N m^{-1} . The part AB of the track is frictionless and the part BC has coefficients of static and kinetic friction

as 0.22 and 0.20 respectively. If the distances AB and BD are 2 m and 2.14 m respectively, find the total distance through which the block moves before it comes to rest completely. (Take $g = 10 \text{ m s}^{-2}$). (IIT 1983)

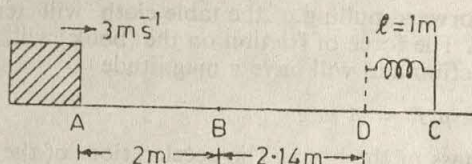


Fig. 3.17

Solution: The block possesses $KE (= \frac{1}{2}mv^2 = \frac{1}{2} \times 0.5 \times 3^2 \text{ J} = 2.25 \text{ J})$, a part of which is used in doing work against the force of (kinetic) friction over the part BD of the track. The remaining energy of the block is used up in compressing the spring through a distance, say x . The block will come to rest at this stage if the restoring force developed in the compressed spring is *not* sufficient to make the block move against the force of (static) friction: otherwise, the block will be pushed backwards and will start oscillating before coming to rest.

To find out what happens to the block, we first calculate x . The work done against the force of (kinetic) friction over the part BD of the track is

$$W = (\mu mg) \times BD = 0.20 \times 0.5 \times 10 \times 2.14 \text{ J} \\ = 2.14 \text{ J}$$

The energy left with the block when it meets the spring is, therefore, equal to $(2.25 - 2.14) \text{ J} = 0.11 \text{ J}$. It uses this energy to compress the spring through a distance x . Now the work needed to compress a spring of force constant K through a distance x is given by

$$W' = \int_0^x K \times dx = \frac{1}{2} K x^2$$

While compressing the spring, the block again does work against the force of (kinetic) friction over a distance x . We, therefore, have

$$0.11 = 0.20 \times 0.5 \times 10 \times x + \frac{1}{2} K x^2 \\ = x + \frac{1}{2} \times 2 x^2 = x + x^2$$

Therefore,

$$x^2 + x - 0.11 = 0$$

$$\text{or } x = \frac{-1 \pm \sqrt{1 + 0.44}}{2} = \frac{-1 \pm 1.2}{2} = 0.1 \text{ m}$$

where the negative value of x has been neglected.

Thus the spring gets compressed through a distance of 0.1 m. The restoring force developed due to this compression is

$$f = K \times (0.1) \text{ N} = 2 \times 0.1 \text{ N} = 0.2 \text{ N}$$

The force of (static) friction, on the other hand, is

$$\mu_s mg = 0.22 \times 0.5 \times 10 \text{ N} = 1.1 \text{ N}$$

Thus the restoring force developed in the compressed spring is *not* sufficient to make the block move and, thereby, cause its oscillations. The

block, therefore, comes to rest after compressing the spring through a distance of 0.1 m.

Hence the total distance moved by the block before coming to rest is

$$\begin{aligned} d &= AB + BD + x \\ &= (2.0 + 2.14 + 0.1) \text{ m} \\ &= 4.24 \text{ m} \end{aligned}$$

Example 16: A string, with one end fixed on a rigid wall, passing over a fixed frictionless pulley at a distance of 2 m from the wall, has a point mass $M = 2 \text{ kg}$ attached to it at a distance of 1 m from the wall, as shown in Fig. 3.18. A mass $m = 0.5 \text{ kg}$ attached at the free end is held at rest so that the string is horizontal between the wall and pulley and vertical beyond the pulley. What will be the speed with which the mass M will hit the wall when the mass m is released? (IIT 1985)

Solution When mass m is released, it will start rising up and the mass M will move along the curved path as shown in Fig. 3.19.

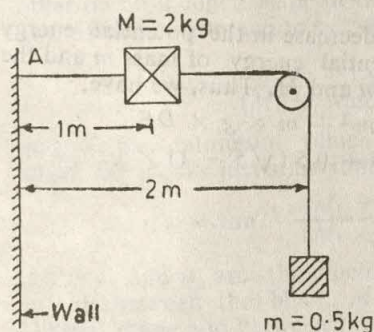


Fig. 3.18

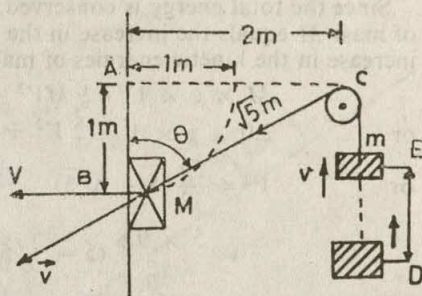


Fig. 3.19

The figure shows the instant the mass M hitting the wall. At this instant the mass M is at B which is 1 m below the fixed point A on the wall. The mass m has risen through a height DE which is given by

$$\begin{aligned} DE &= CB + BA - AC \\ &= (\sqrt{5} + 1 - 2) = (\sqrt{5} - 1) \text{ m} \end{aligned}$$

It is clear that the lighter mass m gains potential energy while the heavier mass M loses potential energy. Thus,

$$\begin{aligned} \text{Decrease in potential energy of } M &= M \times g \times 1 \\ &= 2.0 \times g \times 1 = 2g \end{aligned}$$

where $g = 9.8 \text{ m s}^{-2}$

$$\begin{aligned} \text{Increase in potential energy of } m &= m \times g \times DE \\ &= 0.5 \times g \times (\sqrt{5} - 1) \end{aligned}$$

Since both the masses are moving, their kinetic energies increase. Let v be the velocity with which the mass m rises and V be the velocity with which the mass M strikes the wall. It is clear from Fig. 3.19 that v and V are related as

$$r = V \cos \theta$$

$$= \frac{2}{\sqrt{5}} V$$

$$(\text{since } \cos \theta = 2/\sqrt{5})$$

Thus, we have,

Increase in kinetic energy of

$$M = \frac{1}{2} M V^2$$

$$= \frac{1}{2} \times 2.0 \times 4/5 V^2$$

$$= 4/5 V^2$$

Increase in kinetic energy of

$$m = \frac{1}{2} m v^2$$

$$= \frac{1}{2} \times 0.5 \times v^2$$

$$= v^2/4$$

$$= V^2/5$$

$$(\text{since } v = 2 V/\sqrt{5})$$

Since the total energy is conserved, the decrease in the potential energy of mass M equals the increase in the potential energy of mass m and the increase in the kinetic energies of masses m and M . Thus, we have,

$$M \times g \times 1 = \frac{1}{2} M V^2 + \frac{1}{2} m v^2 + m \times g \times DE$$

$$\text{or } 2.0 \times g \times 1 = 2 V^2 + V^2/5 + 0.5 (\sqrt{5} - 1) \times g$$

$$\text{or } V^2 = \frac{5g}{12} (5 - \sqrt{5})$$

$$= \frac{5 \times 9.8}{12} (5 - \sqrt{5})$$

$$= 3.36 \text{ ms}^{-1} (\text{approx.})$$

Example 17: A weight is moved from the bottom to the top of an inclined plane. Show that a smaller force will be required to drag it along the plane than to lift it if the co-efficient of friction μ is such that

$$\mu < \tan (\pi/4 - \theta/2)$$

where θ is the inclination of the plane to the horizontal.

Solution: Let P be the force needed to drag the body along the inclined plane (see Fig. 3.20).

Since the force of friction, in this case, will act downwards, we have

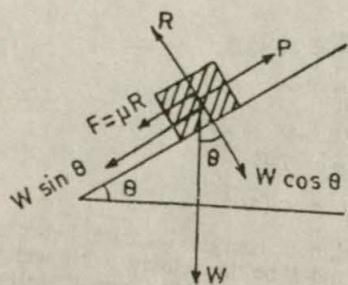


Fig. 3.20

$$P = W \sin \theta + F$$

$$= W \sin \theta + \mu W \cos \theta$$

$$= W (\sin \theta + \mu \cos \theta)$$

The force P' needed to lift the body up is, of course, just W . Thus,

$$P < P' \text{ if}$$

$$W (\sin \theta + \mu \cos \theta) < W$$

$$\text{or } (\sin \theta + \mu \cos \theta) < 1$$

$$\text{or } \mu < \frac{1 - \sin \theta}{\cos \theta}$$

Now
$$\frac{1 - \sin \theta}{\cos \theta} = \frac{\sin^2 \theta/2 + \cos^2 \theta/2 - 2 \sin \theta/2 \cos \theta/2}{\cos^2 \theta/2 - \sin^2 \theta/2}$$

$$= \frac{(\cos \theta/2 - \sin \theta/2)^2}{(\cos \theta/2 - \sin \theta/2)(\cos \theta/2 + \sin \theta/2)}$$

$$= \frac{\cos \theta/2 - \sin \theta/2}{\cos \theta/2 + \sin \theta/2}$$

Thus $P < P'$

if
$$\mu < \frac{\cos \theta/2 - \sin \theta/2}{\cos \theta/2 + \sin \theta/2} = \left(\frac{1 - \tan \theta/2}{1 + \tan \theta/2} \right)$$

$$= \tan \left(\frac{\pi}{4} - \frac{\theta}{2} \right)$$

Example 18: Two blocks of masses M_1 and M_2 are kept touching each other on an inclined plane of inclination α with the horizontal. Show that (i) the force of interaction between the blocks is

$$\frac{(\mu_1 - \mu_2) M_1 M_2 g \cos \alpha}{(M_1 + M_2)}$$

and (ii) the minimum values of α at which the blocks just start sliding is

$$\alpha = \tan^{-1} \frac{\mu_1 M_1 + \mu_2 M_2}{(M_1 + M_2)}$$

where μ_1 and μ_2 are the coefficients of friction between the block M_1 and the inclined plane and between the block M_2 and the inclined plane respectively. (See Fig. 3.21)

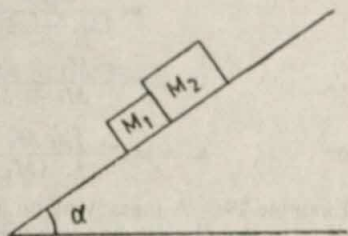


Fig. 3.21

Solution: We note that, whereas the block M_2 will press the block M_1 , downwards, the block M_1 will try to 'hold' the block M_2 from sliding downwards. Let P be the value of this force of interaction between the two blocks. The various forces acting on the two blocks are shown in Fig. 3.22 (a) and (b) respectively.

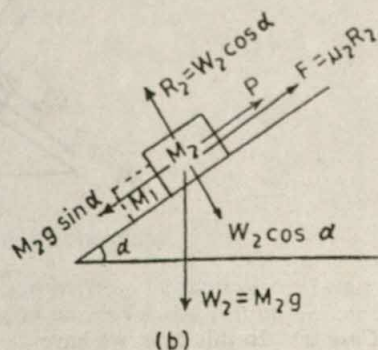
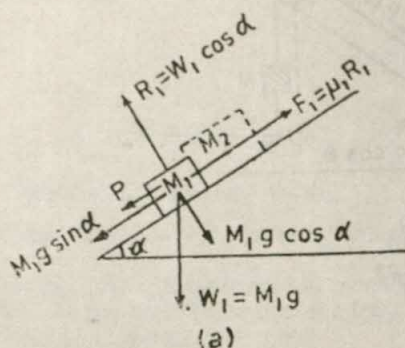


Fig. 3.22 (a) and (b)

Since the blocks are just in equilibrium, we have

$$M_1 g \sin \alpha + P = F_1 = \mu_1 M_1 g \cos \alpha \quad (i)$$

and $M_2 g \sin \alpha - P = F_2 = \mu_2 M_2 g \cos \alpha \quad (ii)$

Multiplying Eq. (i) by M_2 and Eq. (ii) by M_1 , we get

$$(M_1 + M_2) P = (\mu_1 - \mu_2) M_1 M_2 g \cos \alpha$$

$$P = \frac{(\mu_1 - \mu_2) M_1 M_2 g \cos \alpha}{(M_1 + M_2)} \quad (iii)$$

Using this value of P in Eq. (i) we get

$$M_1 g \sin \alpha = \mu_1 M_1 g \cos \alpha - \frac{(\mu_1 - \mu_2) M_1 M_2 g \cos \alpha}{(M_1 + M_2)}$$

or $\sin \alpha = \mu_1 \cos \alpha - \frac{(\mu_1 - \mu_2) M_2 \cos \alpha}{M_1 + M_2}$

$$= \frac{\cos \alpha}{(M_1 + M_2)} [\mu_1 (M_1 + M_2) - (\mu_1 - \mu_2) M_2]$$

$$= \frac{\cos \alpha}{(M_1 + M_2)} (\mu_1 M_1 + \mu_2 M_2)$$

$\therefore \tan \alpha = \frac{\mu_1 M_1 + \mu_2 M_2}{M_1 + M_2}$

or $\alpha = \tan^{-1} \left\{ \frac{\mu_1 M_1 + \mu_2 M_2}{(M_1 + M_2)} \right\}$

Example 19: A mass lying on an inclined plane of inclination θ is connected to an equal mass hanging vertically through a massless string passing over a pulley. Find the limits of θ , in terms of the angle of friction λ so that motion may take place (i) up the inclined plane and (ii) down the inclined plane.

Solution: The various forces acting on the two masses are shown in Fig. 3.23. The force of friction F acts down the plane in case (i) and up the plane in case (ii).

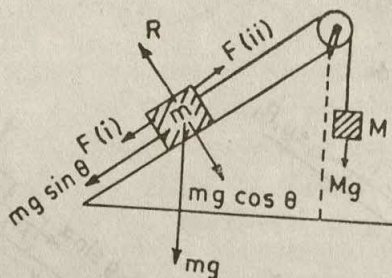


Fig. 3.23

Case (i) In this case, we have

$$T = m g \text{ for the mass hanging vertically}$$

$$\begin{aligned}\text{and } T &= m g \sin \theta + F \\ &= m g \sin \theta + \mu m g \cos \theta\end{aligned}$$

where μ , the coefficient of friction, equals $\tan \lambda$.

$$m g = m g \sin \theta + (\tan \lambda) m g \cos \theta$$

$$\text{or } \sin \theta + (\tan \lambda) \cos \theta = 1$$

$$\text{or } R \sin (\theta + \alpha) = 1$$

where $R \cos \alpha = 1$ and $R \sin \alpha = \tan \lambda$. Therefore,

$$\begin{aligned}\sin (\theta + \alpha) &= \frac{1}{R} = \frac{1}{\sqrt{1 + \tan^2 \lambda}} \\ &= \cos \lambda \\ &= \sin \left(\frac{\pi}{2} - \lambda \right)\end{aligned}$$

But,

$$\tan \alpha = \frac{\tan \lambda}{1} = \tan \lambda \text{ so that } \alpha = \lambda$$

Thus, when $\theta = \frac{\pi}{2} - 2\lambda$, the body will move up the plane.

Case (ii) In this case, we have

$$T = m g \text{ and } T + F = m g \sin \theta$$

$$m g + \mu m g \cos \theta = m g \sin \theta$$

$$\text{or } 1 + \tan \lambda \cos \theta = \sin \theta$$

$$\text{or } 1 = \sin \theta - \tan \lambda \cos \theta$$

Again putting $1 = r \cos \beta$ and $\tan \lambda = r \sin \beta$, we have

$$r \sin (\theta - \beta) = 1$$

$$\begin{aligned}\text{or } \sin (\theta - \beta) &= \frac{1}{r} = \frac{1}{\sqrt{1 + \tan^2 \lambda}} = \cos \lambda \\ &= \sin \left(\frac{\pi}{2} - \lambda \right)\end{aligned}$$

$$\therefore \theta - \beta = \frac{\pi}{2} - \lambda$$

Since $\beta = \lambda$, we have $\theta = \pi/2$

Thus when $\theta = \pi/2$, the body will move down the plane.

Example 20: A uniform rod is made to lean between a rough vertical wall and the ground. Show that the least angle at which the rod can be leaned without slipping is given by

$$\theta = \tan^{-1} \left(\frac{1 - \mu_1 \mu_2}{2 \mu_2} \right)$$

where μ_1 and μ_2 stand for the coefficient of friction between (i) the rod and the wall and (ii) the rod and the ground.

Solution: Figure 3.24 shows the various forces acting on the rod AB when it is leaning between the wall and the ground without slipping.

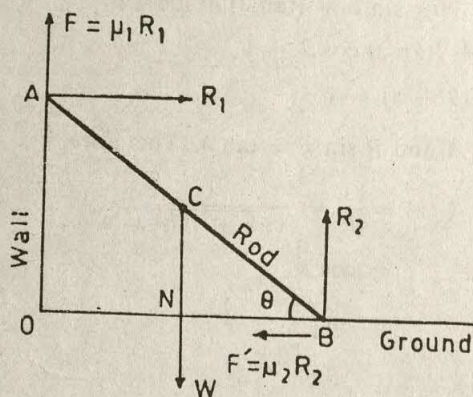


Fig. 3.24

Since the rod is in equilibrium, the net force as well as the net torque on it must each be zero. Considering the forces acting on the rod: we have

$$R_1 + (-F') = 0 \quad \text{or} \quad R_1 = \mu_2 R_2$$

and $R_2 + F + (-W) = 0 \quad \text{or} \quad R_2 + \mu_1 R_1 = W$

Considering next the moments of all forces about A , we have

$$R_2 \times OB = W \times ON + \mu_2 R_2 \times OA$$

or $R_2 \times AB \cos \theta = W \times \frac{AB \cos \theta}{2} + \mu_2 R_2 \times AB \sin \theta$

or $\cos \theta (R_2 - W/2) = \sin \theta (\mu_2 R_2)$

or $\tan \theta = \frac{(R_2 - W/2)}{\mu_2 R_2}$

Now $W = R_2 + \mu_1 R_1 = R_2 + \mu_1 (\mu_2 R_2)$
 $= R_2 (1 + \mu_1 \mu_2)$

$\therefore R_2 - W/2 = R_2 - \frac{R_2}{2} (1 + \mu_1 \mu_2)$

$$= R_2 \left(\frac{1}{2} - \frac{\mu_1 \mu_2}{2} \right) = \frac{R_2}{2} (1 - \mu_1 \mu_2)$$

$\therefore \tan \theta = \frac{R_2 (1 - \mu_1 \mu_2)}{2 \mu_2 R_2} = \frac{(1 - \mu_1 \mu_2)}{2 \mu_2}$

$\therefore \theta = \tan^{-1} \left\{ \frac{1 - \mu_1 \mu_2}{2 \mu_2} \right\}$

Example 21: A small mass slides down an inclined plane of inclination θ with the horizontal. The coefficient of friction $\mu = \mu_0 x$, where x is the

distance through which the mass slides down and μ_0 is a constant. Find the distance covered by the mass before it stops. What is its maximum speed over this distance?

Solution: We consider the mass at the instant when it has slid down a distance x along the inclined plane (see Fig. 3.25).

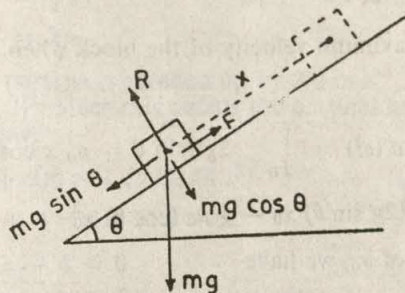


Fig. 3.25

The net downward force at this instant is

$$\begin{aligned} m g \sin \theta - F &= m g \sin \theta - \mu R \\ &= m g \sin \theta - (\mu_0 x) m g \cos \theta \\ &= m g (\sin \theta - \mu_0 x \cos \theta) \end{aligned}$$

The instantaneous acceleration a of the mass is, therefore, given by

$$\begin{aligned} a &= \frac{m g (\sin \theta - \mu_0 x \cos \theta)}{m} \\ &= g (\sin \theta - \mu_0 x \cos \theta) \end{aligned}$$

But $a = \frac{dv}{dt} = \frac{dv}{dx} \frac{dx}{dt} = \frac{dv}{dx} v = \frac{1}{2} \frac{d}{dx} (v^2)$

$$\therefore \frac{1}{2} \frac{d}{dx} (v^2) = g (\sin \theta - \mu_0 x \cos \theta)$$

or $d(v^2) = 2 g (\sin \theta - \mu_0 x \cos \theta) dx$

Let S be the distance moved by the mass before it stops again. The

$$\int_0^0 d(v^2) = \int_0^S 2 g (\sin \theta - \mu_0 x \cos \theta) dx$$

or $0 = 2 g \sin \theta \left| x \right|_0^S - 2 g \mu_0 \cos \theta \left| \frac{x^2}{2} \right|_0^S$

or $2 g \sin \theta S - 2 g \mu_0 (\cos \theta) \frac{S^2}{2} = 0$

or since $S \neq 0$, $2 \sin \theta = (\mu_0 \cos \theta) S$

or $S = \frac{2}{\mu_0} \tan \theta$

Thus the block again comes to rest after covering a distance

$$S = \frac{2}{\mu_0} \tan \theta$$

The maximum velocity is attained at the instant the instantaneous acceleration becomes zero. This happens for a value x_0 of x for which

$$g (\sin \theta - \mu_0 x_0 \cos \theta) = 0$$

$$\therefore x_0 = \frac{1}{\mu_0} \frac{\sin \theta}{\cos \theta} = \frac{\tan \theta}{\mu_0}$$

Let $v_{\max}$ be the maximum velocity of the block when $x = x_0$. We then have

$$\int_0^{v_{\max}} d(v^2) = \int_0^{x_0} 2g (\sin \theta - \mu_0 x \cos \theta) dx$$

$$\text{or } v_{\max}^2 = (2g \sin \theta) x_0 - g \mu_0 (\cos \theta) x_0^2$$

Putting the value of x_0 , we have

$$v_{\max}^2 = 2g \sin \theta \frac{\sin \theta}{\mu_0 \cos \theta} - g \mu_0 \cos \theta \frac{\sin^2 \theta}{\mu_0^2 \cos^2 \theta}$$

$$= g \frac{\sin^2 \theta}{\mu_0 \cos \theta} = \frac{g}{\mu_0} \sin \theta \tan \theta$$

$$\therefore v_{\max} = \sqrt{\left(\frac{g}{\mu_0} \sin \theta \tan \theta \right)}$$

Example 22: An engine is hauling a train of mass M kg on a level track at a constant speed of v ms⁻¹. If the resistance due to friction is f N kg⁻¹, what power is the engine producing? What extra power must the engine develop to maintain the speed up a gradient of 1 in l ?

Solution: The engine has to do work *only* against friction on the level track as the train is moving at a constant speed. The force needed to overcome friction is Mf newton. Since the distance moved in one second is v metres, the work done per second or the power of the engine is

$$P = Mfv \text{ watt}$$

When the train is being hauled up an incline of 1 in l ($\sin \theta = 1/l$), the engine has to do extra work against the downward component ($= Mg \sin \theta = Mg/l$) of the weight of the train. The work per second is

$$\frac{Mg}{l} \times v = \frac{Mvg}{l}$$

The extra power needed is, therefore, Mvg/l . The total power developed by the engine now is

$$Mfv + Mvg/l$$

$$\text{or } \frac{Mv}{l} (fl + vg)$$

Example 23: A particle of mass m has half the kinetic energy of another particle of mass $m/2$. If the speed of the heavier particle is increased by 2.0 m s⁻¹, its new kinetic energy equals the original kinetic energy of the lighter particle. Calculate the original speeds of the two particles.

Solution: Let v and v' be the original speeds of the heavier and the lighter particles respectively. We then have

$$\frac{1}{2} m v^2 = \frac{1}{2} \left\{ \frac{1}{2} \left(\frac{m}{2} \right) v'^2 \right\}$$

$$\therefore v^2 = \frac{1}{4} v'^2$$

$$\text{or } v' = 2 v$$

When the heavier particle is speeded up by 2.0 m s^{-1} , its kinetic energy becomes $\frac{1}{2} m (v + 2)^2$. Since this equals the original kinetic energy of the lighter particle, we have

$$\frac{1}{2} m (v + 2)^2 = \frac{1}{2} (m/2) (4 v^2)$$

$$v^2 + 4 + 4 v = 2 v^2$$

$$\text{or } v^2 - 4 v - 4 = 0$$

$$v = \frac{4 \pm \sqrt{16 + 16}}{2} = \frac{4 \pm 2\sqrt{8}}{2}$$

$$3.24 \text{ m s}^{-1}$$

$$v' = 2 v = 6.48 \text{ m s}^{-1}$$

Thus the original speeds of the heavier and the lighter particles are 3.24 m s^{-1} and 6.48 m s^{-1} respectively.

Example 24: Two identical cylindrical vessels with their bases at the same level, each contain a liquid of density ρ . The height of the liquid in one vessel is h_1 and that in the other vessel is h_2 . The area of either base is A . What is the work done by gravity in equalizing the levels when the vessels are interconnected? (IIT 1981)

Solution: The work done by gravity equals the change in potential energy of the system after the vessels are interconnected. We may regard the liquid in each vessel as equivalent to a point mass kept at their respective centres of gravity. Remembering that the mass of the liquid is given by $(A h \rho)$ and that the PE of a mass at a height h in earth's gravity is $m g h$, we have

$$\begin{aligned} \text{Total PE at start} &= (A h_1 \rho) g \frac{h_1}{2} + (A h_2 \rho) g \frac{h_2}{2} \\ &= \frac{A \rho g}{2} (h_1^2 + h_2^2) \end{aligned}$$

After the vessels are connected, the height of liquid in each vessel is $(h_1 + h_2)/2$. Hence

$$\begin{aligned} \text{PE after connection} &= 2 \left\{ A \rho \left(\frac{h_1 + h_2}{2} \right) g \left(\frac{h_1 + h_2}{4} \right) \right\} \\ &= \frac{A \rho g}{4} (h_1 + h_2)^2 \end{aligned}$$

$$\text{Change in PE} = \frac{A \rho g}{4} \{ (h_1 + h_2)^2 - 2 (h_1^2 + h_2^2) \}$$

$$= - \frac{A \rho g}{4} (h_1 - h_2)^2$$

$$= - A \rho g \left(\frac{h_1 - h_2}{2} \right)^2$$

This must be equal to the work done 'by' gravity on the liquid. Thus the work done 'by' gravity is

$$A \rho g \left(\frac{h_1 - h_2}{2} \right)^2$$

Example 25: A gun weighing 5 kg fires a bullet of mass 10^{-2} kg with a speed of $5 \times 10^2 \text{ m s}^{-1}$. Obtain the kinetic energy of the bullet and the gun. Also obtain the ratio of the distance the gun moves backward (while the bullet is in the barrel) to the distance moved by the bullet.

Solution: $K_B =$ kinetic energy of bullet $= \frac{1}{2} m_b \times v_b^2$

Given

$$m_b = 10^{-2} \text{ kg}, \quad v_b = 5 \times 10^2 \text{ m s}^{-1}$$

$$\begin{aligned} K_B &= \frac{1}{2} \times 10^{-2} \times (5 \times 10^2)^2 \\ &= 12.5 \times 10^2 \text{ J} \end{aligned}$$

When the bullet is fired, the gun recoils with a speed v_g . From the law of conservation of linear momentum, we have

$$0 = m_b v_b + m_g v_g$$

$$v_g = - m_b v_b / m_g$$

$$= - \frac{10^{-2} \times 5 \times 10^2}{5} \text{ m s}^{-1}$$

$$= - 1 \text{ m s}^{-1}$$

$$K_g = \text{kinetic energy of the gun}$$

$$= \frac{1}{2} m_g v_g^2$$

$$= \frac{1}{2} \times 5 \times (1)^2 \text{ J}$$

$$= 2.5 \text{ J}$$

Considering the bullet and the gun to constitute a single system, the centre of mass of the system is at rest and the linear momentum is conserved. Therefore,

$$0 = m_g v_g + m_b v_b$$

$$= m_g \frac{\Delta x_g}{\Delta t} + m_b \frac{\Delta x_b}{\Delta t}$$

where Δx_g and Δx_b stand for the distance moved by the gun and the bullet respectively. Hence

$$\left| \frac{\Delta x_g}{\Delta x_b} \right| = \frac{m_b}{m_g}$$

$$= \frac{10^{-2}}{5} = 2 \times 10^{-3}$$

Example 26: A hammer of mass M drops from a height h . It strikes a nail placed vertically on the ground and drives it into the ground through a distance d . Calculate (i) the average resistance offered by the ground, assuming that the hammer and nail remain stuck together after impact, (ii) the time for which the nail is in motion and (iii) the loss in kinetic energy in impact.

Solution: The hammer of mass M falls freely under gravity through a distance h . Let V be the speed acquired by the hammer when it strikes the nail. Obviously

$$V = \sqrt{2gh} \quad (i)$$

On impact, the hammer and nail are stuck together. Let V' be the speed of the combination after impact. The law of conservation of linear momentum gives

$$MV = (M + m)V' \quad (ii)$$

where m is the mass of the nail.

Let F_G be the average resistance (or resistive force) exerted by the ground. The net upward force F on the hammer-nail combination is

$$F = F_G - (M + m)g$$

The combination moves through a distance d against the net upward force. Obviously, the work done against the force F in a distance d must be equal to the kinetic energy the combination had just after impact. Hence

$$\begin{aligned} \frac{1}{2}(M + m)V'^2 &= F \times d \\ &= [F_G - (M + m)g] \times d \end{aligned} \quad (iii)$$

Using Eqs (i) and (ii), we can rewrite Eq. (iii) as

$$\frac{1}{2}(M + m) \left[\frac{MV}{M + m} \right]^2 = [F_G - (M + m)g] \times d$$

$$\text{or} \quad \frac{1}{2} \frac{M^2}{(M + m)} \times 2gh = [F_G - (M + m)g] \times d$$

$$\text{or} \quad F_G = \frac{M^2 gh}{(M + m)d} + (M + m)g \quad (iv)$$

(ii) Let the hammer-nail combination be moving for a time Δt , before coming to rest. If ΔP is the change in momentum, then from Newton's second law, we have

$$\frac{\Delta P}{\Delta t} = F_G - (M + m)g$$

$$\begin{aligned} \text{Also} \quad \Delta P &= (M + m)V' - 0 \\ &= (M + m) \frac{MV}{(M + m)} \\ &= MV \end{aligned}$$

$$\begin{aligned} \therefore \Delta t &= \frac{\Delta P}{F_G - (M + m)g} \\ &= \frac{MV}{\left\{ \frac{M^2}{(M + m)} \frac{gh}{d} + (M + m)g \right\} - (M + m)g} \\ &= \frac{(M + m)d}{M^2 gh} MV = \frac{(M + m)d}{M gh} \sqrt{2gh} \\ &= \frac{M + m}{M} d \sqrt{\frac{2}{gh}} \end{aligned}$$

(iii) The kinetic energy of the hammer-nail combination just before the impact is $\frac{1}{2} MV^2$ and after impact, it is $\frac{1}{2} (M + m) V'^2$. Therefore,

$$\begin{aligned}\text{loss in kinetic energy} &= \frac{1}{2} MV^2 - \frac{1}{2} (M + m) V'^2 \\ &= \frac{1}{2} MV^2 - \frac{1}{2} \frac{M^2 V^2}{(M + m)} \\ &= \frac{1}{2} \frac{M m}{(M + m)} V^2\end{aligned}$$

Example 27: Two monkeys are connected to each other by a massless, inextensible string passing over a pulley as shown in Fig. 3.26. The system is let go from rest and the heavier monkey starts descending. At the end of 5 s, suddenly another monkey of mass M joins the ascending one. Calculate (i) the resultant speed after $t = 5$ s and (ii) the loss of kinetic energy of the descending monkey when the other monkey gets attached to the ascending one.

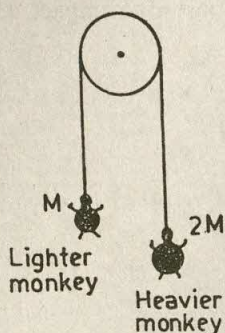


Fig. 3.26

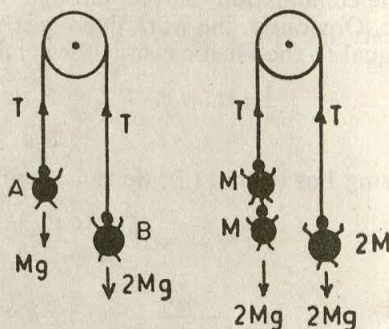
(a) $t < 5$ s(b) $t > 5$ s

Fig. 3.27

Solution: (i) Figure 3.27 (a) shows the situation before 5 s. Let T be the tension in the string and a the common acceleration of A and B . Obviously,

$$2Ma = 2Mg - T$$

$$\text{and} \quad T - Mg = Ma$$

$$\text{Therefore} \quad 3Ma = Mg$$

$$\text{or} \quad a = \frac{g}{3}$$

If V is the speed acquired by A in the downward direction (or the speed acquired by B in the upward direction) at $t = 5$ s, then

$$V = 0 + a \times 5$$

$$= \frac{5}{3} g$$

(i)

When another monkey of mass M is attached to B , the situation is shown in Fig. 3.27 (c). Now the resultant force on the system is zero and, therefore, the acceleration is reduced to zero. Also, the system has a momentum of $3MV$ before the monkey joins B . Let V' be the speed of both the monkeys when the new one has joined B . Obviously, now the momentum is $4MV'$. From the law of conservation of momentum we have

$$3MV = 4MV'$$

$$V' = \frac{3}{4} V$$

$$\text{or } V' = \frac{5}{4} g \quad (\text{ii})$$

(ii) The kinetic energy of A before the new monkey joins, is $\frac{1}{2} (2M) V^2$. Its kinetic energy after the new monkey joins B is $\frac{1}{2} (2M) V'^2$. Therefore,

$$\begin{aligned} \text{Loss of kinetic energy} &= \frac{1}{2} (2M) \left(\frac{5}{3} g \right)^2 - \frac{1}{2} (2M) \left(\frac{5}{4} g \right)^2 \\ &= 1.215 M g^2 J \end{aligned}$$

Example 28: A heavy mass M resting on the ground is attached to a smaller mass m via a massless, inextensible string passing over a pulley as shown in Fig. 3.28. The string connected to M is loose. The smaller mass m falls freely through a height h and the string becomes taut. Obtain the time from this instant onwards when the heavier mass again makes contact with the ground. Also obtain the loss in kinetic energy when mass M is jerked into motion.

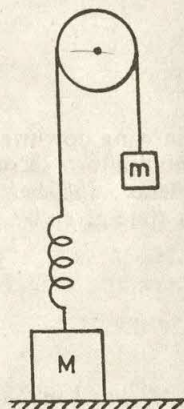


Fig. 3.28

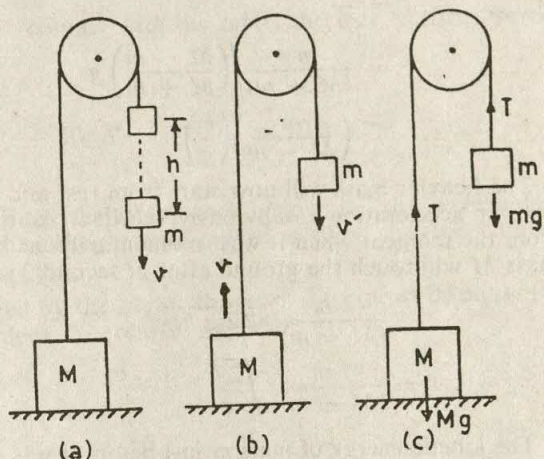


Fig. 3.29 (a) (b) and (c)

Solution: Figure 3.29 (a) shows the situation when mass m has descended through a height h and the string is just about to become taut. At this moment the speed acquired by mass m is v . This gain in the kinetic energy of m is equal to the loss in its potential energy. Therefore,

$$m g h = m v^2$$

$$\text{or } v = \sqrt{2 g h} \quad (1)$$

Figure 3.29 (b) shows the situation just after the string has become taut. Now, mass M is also joined to mass m . Let v' be the common velocity of both m and M at this moment. From the law of conservation of linear momentum, we have

$$(M + m) v' = m v$$

$$\text{or } v' = \frac{m}{M + m} v \quad (2)$$

Figure 3.29 (c) shows the situation when the heavier mass M has just left the ground. The forces acting are also shown in the diagram. If a is the acceleration of M in the downward direction and that of m in the upward direction, the equations of motion of the two masses are

$$M a = M g - T$$

and $m a = T - m g$

where T is the tension in the string. Hence,

$$a (M + m) = (M - m) g$$

or $a = \frac{M - m}{M + m} g$ (3)

At this moment, mass M has a speed v in the upward direction and an acceleration a in the downward direction. It will rise up to some height above the ground and come to rest. Let mass M come to rest momentarily after time t second. Then

$$0 = v' - a t$$

or $t = \frac{v}{a}$

$$= \frac{m v}{(M + m)} \left/ \left(\frac{M - m}{M + m} \right) g \right.$$

$$= \left(\frac{m}{M - m} \right) \left(\frac{v}{g} \right)$$

The heavier mass will now start from rest and begin moving downwards with an acceleration a . Obviously, it will strike the ground after t second from the moment when it was momentarily at rest. Hence, the heavier mass M will touch the ground after $2t$ second, i.e. after time given by

$$\frac{2 m}{(M - m)} \sqrt{2 g h / g}$$

$$= \frac{2 m}{(M - m)} \sqrt{\frac{2 h}{g}}$$

The kinetic energy of mass m just before it was to jerk M , was $\frac{1}{2} m v^2$ and that after it had jerked up the mass M was $\frac{1}{2} (M + m) v'^2$. Therefore, loss in kinetic energy = $\frac{1}{2} m v^2 - \frac{1}{2} (M + m) v'^2$

$$= \frac{1}{2} m v^2 \left(\frac{M}{M + m} \right)$$

Example 29: A force of 100 N acts, in the direction as shown in Fig. 3.30, on a block of mass 10 kg resting on a smooth horizontal table. Calculate the speed acquired by the block after it has moved a distance of 10 m. If the table is not smooth and the coefficient of friction between the block and table is 0.4, what would be the speed acquired by the block under the same conditions?

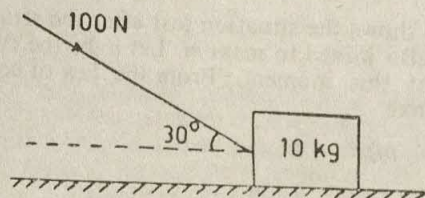


Fig. 3.30

Solution: The various forces acting are shown in Fig. 3.31.

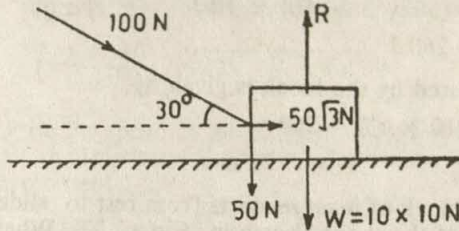


Fig. 3.31

The force of 100 N has (i) a horizontal component of $100 \cos 30^\circ = 100 \frac{\sqrt{3}}{2}\text{ N} = 50\sqrt{3}\text{ N}$ and (ii) a vertical component $= 100 \sin 30^\circ = 50\text{ N}$.

Since the block is always in contact with the table, the net vertical force must be zero, i.e.

$$\begin{aligned} R &= W + 50 \\ &= (10 \times 10 + 50)\text{ N} \quad (\because g = 10\text{ m s}^{-2}) \\ &= 150\text{ N} \end{aligned}$$

When the block moves along the table, work is done by the horizontal component of the force. Since the distance moved is 10 m , the work done is $= 50\sqrt{3} \times 10 = 500\sqrt{3}\text{ J}$.

If v is the speed acquired by the block, the work done must be equal to the kinetic energy of the block. Therefore, we have

$$\begin{aligned} 500\sqrt{3} &= \frac{1}{2} \times 10 \times v^2 \\ \text{or} \quad v^2 &= 100\sqrt{3} \\ v &= 13.17\text{ m s}^{-1} \end{aligned}$$

When the table is not smooth, the various forces acting are as shown in Fig. 3.32. As before

$$R = 150\text{ N}$$

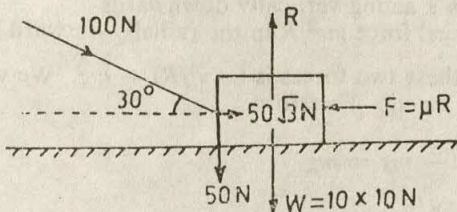


Fig. 3.32

The frictional force F_f is

$$\begin{aligned} F_f &= \mu R = 0.4 \times 150\text{ N} \\ &= 60\text{ N} \end{aligned}$$

Now, when the block moves through a distance of 10 m, the net work done is given by

$$\begin{aligned} W &= (50 \sqrt{3} - 60) \times 10 \text{ J} \\ &= 260 \text{ J} \end{aligned}$$

The speed v' acquired by the block is given by

$$\begin{aligned} \frac{1}{2} \times 10 \times v'^2 &= 260 \\ v' &= 7.22 \text{ m s}^{-1} \end{aligned}$$

Example 30: A block of mass m starts from rest to slide along a smooth frictionless track of the shape shown in Fig. 3.33. What should be the height h so that when the mass reaches point A on the track, it pushes the track with a force equal to its weight?

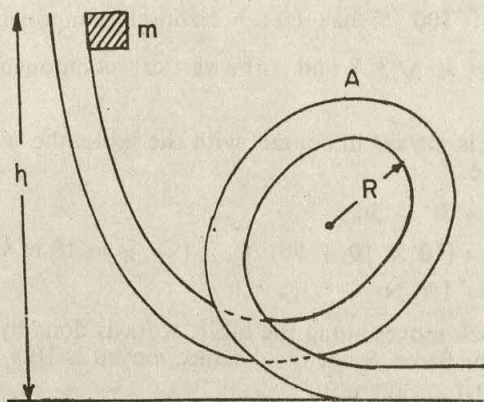


Fig. 3.33

Solution: When the block starts from the top of the track, it has a potential energy equal to mgh . As it reaches point A , it has both kinetic as well as potential energies. If v is the speed of the block at A , its kinetic energy is $\frac{1}{2}mv^2$. Its potential energy at A is $mg(2R)$. Obviously the law of conservation of energy demands that

$$mgh = mg(2R) + \frac{1}{2}mv^2 \quad (1)$$

The forces acting on the block at A are

- (i) its weight mg acting vertically downwards
- (ii) the centrifugal force mv^2/R in the radially outward direction.

The resultant of these two forces is $(mv^2/R) - mg$. We want this resultant force to be equal to mg . Therefore

$$\frac{mv^2}{R} - mg = mg$$

$$\text{or} \quad v^2 = 2Rg \quad (2)$$

Combining Eqs (1) and (2), we get

$$\begin{aligned} mgh &= mg(2R) + mgR \\ &= 3mgR \\ h &= 3R \end{aligned}$$

Example 31: Figure 3.34 shows a ballistic pendulum. A bullet of mass m and speed v striking and getting embedded into a block of mass M . After impact, the pendulum swings through an angle θ . Show that

$$v = \left(\frac{m + M}{m} \right) \sqrt{2gl(1 - \cos \theta)}$$

Solution: Let V be the speed acquired by the combined masses m and M immediately after the impact of the bullet. According to the law of conservation of linear momentum

$$mv = (M + m)V$$

$$\text{or} \quad V = \frac{m}{(M + m)} v \quad (1)$$

The combined mass $(M + m)$, therefore, acquires kinetic energy equal to $\frac{1}{2}(M + m)V^2$. The pendulum swings through an angle θ and is momentarily at rest. In the process of swinging, the centre of gravity of the system rises through a distance h as shown in Fig. 3.35.

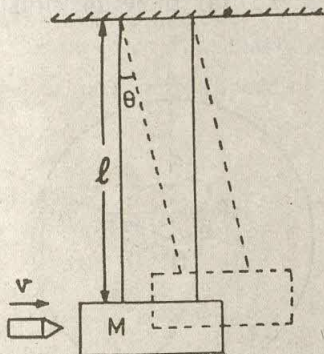


Fig. 3.34

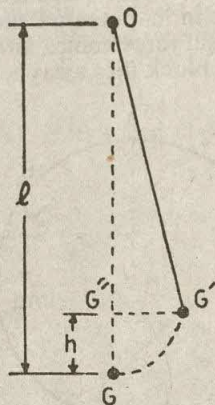


Fig. 3.35

Obviously, in this new position, the whole of kinetic energy is converted into potential energy. Therefore,

$$\frac{1}{2}(M + m)V^2 = (M + m)gh$$

$$\text{or} \quad V = \sqrt{2gh}$$

From the diagram, it is clear that

$$\begin{aligned} h &= l - OG' \\ &= l - l \cos \theta = l(1 - \cos \theta) \end{aligned} \quad (2)$$

Combining Eqs (1) and (2) we have

$$\begin{aligned} v &= \frac{M + m}{m} V \\ &= \frac{(M + m)}{m} \sqrt{2gl(1 - \cos \theta)} \end{aligned}$$

Example 32: A small block slides from rest from the top of a frictionless sphere of radius R (see Fig. 3.36). How far below the top does the block lose contact with the sphere? The sphere is at rest all through.

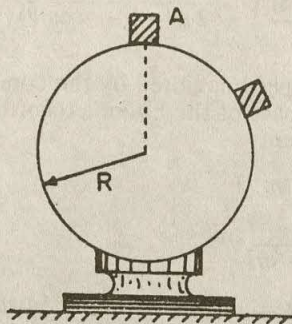


Fig. 3.36

Solution: As the block slides down the sphere, it acquires a velocity due to a loss in its potential energy. As the block moves with a finite speed, a centrifugal force comes into play. Figure 3.37 (a) shows the situation before the block flies away.

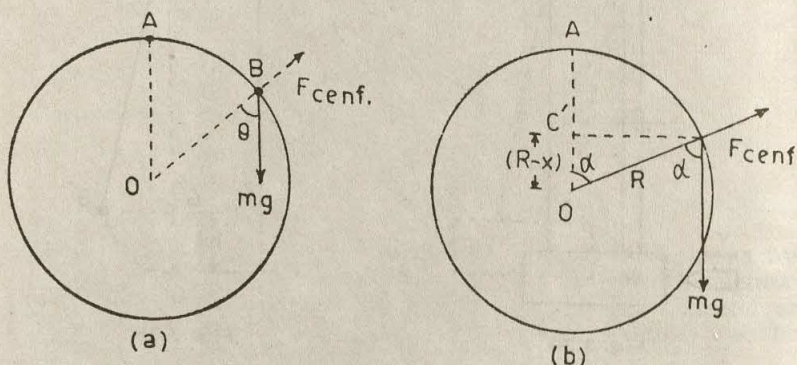


Fig. 3.37 (a) and (b)

The forces acting are

- (i) $W = mg$ in the vertically downward direction
- (ii) F_{cent} ; the centrifugal force directed radially outwards
- (iii) The reaction force N of the sphere on the block. This is along the radius at the point of interest.

The component of weight along the radial direction is $mg \cos \theta$. So long as $mg \cos \theta > F_{cent}$, N is finite and the block is in contact with the sphere. When the block is about to fly off $mg \cos \alpha = F_{cent}$ and $N = 0$ as shown in Fig. 3.37 (b). It is clear from this diagram that

$$mg \cos \alpha = \frac{mv^2}{R} \quad (1)$$

Also v is the speed acquired by the mass as it falls through a vertical distance x . Hence

$$v^2 = 2gx \quad (2)$$

From the diagram, it is also obvious that

$$\cos \alpha = \frac{(R - x)}{R} \quad (3)$$

From Eqs (1) and (2) we get

$$m g \frac{(R - x)}{R} = m \frac{2 g x}{R}$$

or $R - x = 2 x$

$$x = \frac{R}{3}$$

Example 33: AB is a quarter of a smooth circular track of radius 4 m (see Fig. 3.38). A particle P of mass 5 kg moves along the track from A to B under the action of the following forces:

- (1) A force F_1 directed always towards point B ; its magnitude is constant and equals 4 N.
- (2) A force F_2 that is directed along the instantaneous tangent to the circular track; its magnitude is $(20 - 10s)$ newton where s is the distance travelled in metre.
- (3) A horizontal force F_3 of magnitude 25 N.

If the particle starts with a speed of 10 m s^{-1} , what is its speed at point B ?

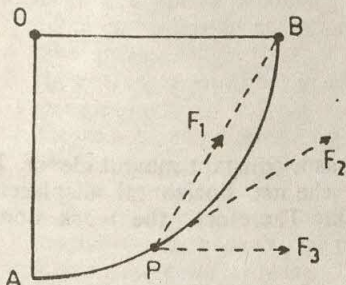


Fig. 3.38

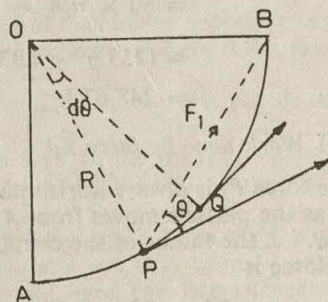


Fig. 3.39

Solution: We shall first calculate the work done by each individual force as the particle moves from A to B .

(1) *Work done by force F_1*

Let the particle be at point P at some instant of time $t = t$. It moves to a position Q in a small time interval dt . The direction of force on the particle at P is $\vec{PB}$ (see Fig. 3.39). The small amount of work done in time dt is

$$\begin{aligned} dW_1 &= \vec{F}_1 \cdot d\vec{s} \\ &= F_1 \cos \theta ds \\ &= 4 R \cos \theta d\theta \quad (1) \quad (ds = R d\theta) \end{aligned}$$

The total work done as the particle moves from A to B is given by the integral of Eq. (1) with in limits $\theta = 0$ to $\theta = \pi/4$.

$$\begin{aligned} W_1 &= 4 \times 4 \int_0^{\pi/4} \cos \theta \, d\theta \\ &= 16 \left[\sin \theta \right]_0^{\pi/4} \\ &= 8 \sqrt{2} \text{ J} = 11.32 \text{ J} \end{aligned} \quad (2)$$

(2) *Work done by force F_2*

The force F_2 is always along the direction of the instantaneous tangent which is also the direction of the instantaneous displacement. Therefore, the small amount of work done as the particle moves from P to Q is

$$\begin{aligned} dW_2 &= F_2 \times ds \\ &= (20 - 10s) \, ds \end{aligned}$$

But $s = R\theta$ and $ds = R \, d\theta$

$$dW_2 = (20 - 4\theta) \times 4 \, d\theta$$

Hence,

$$\begin{aligned} W_2 &= \int_0^{\pi/4} (20 - 4\theta) \times 4 \, d\theta \\ &= 80 \times \pi/4 - \frac{16}{2} (\pi/4)^2 \\ &= (253.6 - 4.93) \text{ J} \\ &= 248.67 \text{ J} \end{aligned}$$

(3) *Work done by force F_3*

Since force F_3 is always horizontal and has a constant magnitude of 25 N and as the particle moves from A to B , the net horizontal displacement is OB , i.e. the radius of the circular track. Therefore, the work done by this force is

$$W_3 = 25 \times 4 \text{ J} = 100 \text{ J}$$

Work done against weight In addition to the above three forces, there is another force, namely, the weight of the particle equal to $5 \times 9.8 \text{ N} = 49 \text{ N}$ acting vertically downwards. As the particle moves from A and B , the net vertical displacement produced is again equal to the radius of the track. Since the force due to the weight is always constant, in magnitude and direction, the work done by the force of gravity is

$$W_g = -49 \times 4 \text{ J} = -196 \text{ J}$$

The negative sign is indicative of the fact that the forces of the weight and displacement are opposite directions.

The total work done in the presence of all the forces is

$$\begin{aligned} W &= W_1 + W_2 + W_3 + W_4 \\ &= (11.32 + 248.67 + 100 - 196) \text{ J} \\ &= 163.99 \text{ J} \end{aligned}$$

Let v_A and v_B be the velocity of the particle at A and B respectively. From the work-energy theorem, we have

$$\frac{1}{2} m v_B^2 - \frac{1}{2} m v_A^2 = W$$

$$\text{or} \quad \frac{1}{2} \times 4 v_B^2 - \frac{1}{2} \times 4 \times 10^2 = 163.95$$

$$\text{or} \quad v_B^2 = 181.797$$

$$v_B = 13.42 \text{ m s}^{-1}$$

3.5 TRUE-FALSE STATEMENTS WITH REASONS

We are given some statements below. We have to decide whether they are true or false giving, in brief, the reasons in support of our answers.

1. Newton's first law tells us that in the absence of nearby objects—the cause of 'real' forces—it is possible to find a family of reference frames in which the particle has no acceleration.
2. A physical quantity can be regarded as a vector if we can associate a magnitude as well as a direction with it.
3. It is possible to sometimes have a single isolated force.
4. A spring is attached to a 'stand' at one end and has a block, at rest, suspended from its other end. In general, such a spring will exert different forces on the stand and the block.
5. We will need different forces to give the same acceleration, say a , to a given body in (a) gravity-free space and (b) a horizontal frictionless surface on the earth.
6. An object kept on a smooth inclined plane of 1 in l can be kept stationary relative to the 'incline' by giving a horizontal acceleration of $g/\sqrt{l^2 - 1}$ to the inclined plane.
7. The total energy of a body in motion is equal to the work it can do in being brought to rest.
8. The work done by a force $F(x)$ in displacing a particle from a position x_1 to a position x_2 is equal to the area enclosed by the $F(x)$ versus x curve; the x -axis and the two ordinates at $x = x_1$ and $x = x_2$.
9. A body is moving down a long inclined plane of 1 in 2 . The coefficient of friction between the body and the plane varies as $\mu = 0.49 x$, where x is the distance moved down the plane. The body will then have its maximum velocity when it reaches the bottom of the plane.
10. An insect crawling up a fixed hemispherical bowl of radius R can only crawl upto a height of 68% of the radius of the bowl if the coefficient of friction is $1/3$.
11. If two surfaces are progressively polished or smoothened, the frictional force between them will continue to decrease indefinitely.
12. An open knife-edge of mass m is dropped from a height h on a wooden floor. If the blade penetrates S cm into the wood, the average resistance offered by the wood to the blade is mgh/S .
13. When a body is moved along a straight line by a machine delivering constant power, the distance moved by it in time t is proportional to $t^{3/2}$.
14. A uniform chain of length L and mass M is lying on a smooth table and one-third of its length is hanging vertically down over the edge of the table. The work required to pull the hanging part of the chain on to the table is $MgL/18$.
15. The time period of a simple pendulum in a lift accelerating upwards with an acceleration $g/8$ is $2\sqrt{2/3}$ times its value in a stationary lift.

ANSWERS

1. *True* The first law tells us that a body persists in its state of rest or of uniform motion in a straight line, unless it is compelled to change that state by forces impressed on it. Since the acceleration of a body is dependent on the reference frame used to measure it, we can regard the first law as a statement about reference frames.
2. *False* A physical quantity is regarded as a vector, if in addition to having a magnitude and direction, it also obeys the laws of vector addition.
3. *False* Forces in nature always occur in pairs—action and reaction. Any single force is only a 'one-sided view' of a mutual interaction between a body and its surroundings or between two bodies.
4. *True* In general, the weight of the spring has to be taken into account. It is only when the spring is assumed to be 'weightless' that it may be regarded as transmitting a force from one end to the other without change.
5. *False* The force needed in both the cases is ma , where m , the mass of the body is the same in both the situations.
6. *True* As shown in Fig. 3.40, a horizontal acceleration a to the inclined plane has a component $a \cos \theta$ down the plane. If this equals $g \sin \theta$, the acceleration of the object down the 'incline', the object will appear stationary relative to the incline. Thus

$$a \cos \theta = g \sin \theta$$

$$\text{or } a = g \tan \theta$$

$$= g/\sqrt{l^2 - 1}$$

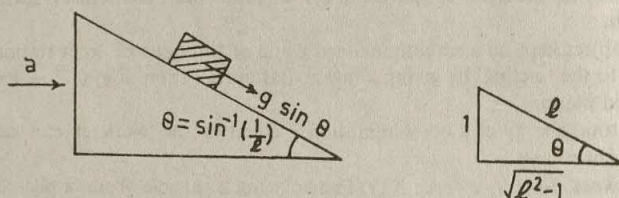


Fig. 3.40

7. *False* It is not the total energy but only the kinetic energy that is 'used up' when the body is brought to rest. We must, therefore, say that the *kinetic* energy of a body in motion is equal to the work it can do in being brought to rest.
8. *True* For a variable force $F(x)$, the work done is given by

$$W = \int_{x_1}^{x_2} F(x) dx$$

and this equals the area enclosed by the $F(x)$ versus x curve, the x -axis and the two ordinates at $x = x_1$ and $x = x_2$ (see Fig. 3.41)

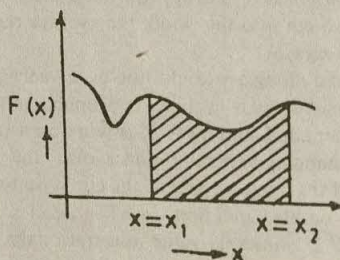


Fig. 3.41

9. *False* The instantaneous acceleration of the body down the plane is $(g \sin \theta - \mu g \cos \theta) = g (\sin \theta - 0.49 \times \cos \theta)$. The body will, therefore, first accelerate (upto $x < \tan \theta / 0.49$), reach a maximum velocity (at $x = \tan \theta / 0.49 = \tan 30^\circ / 0.49 = 1.18$ m), and then start decelerating (for $x > \tan \theta / 0.49$). Thus the maximum velocity is attained after travelling 1.18 m down the plane and not at the bottom of the plane.
10. *False* The insect will crawl up the bowl till the component of its weight down the plane just equals the force of limiting friction. Thus, as shown in Fig. 3.42, we have

$$F = W \sin \alpha \quad \text{and} \quad N = W \cos \alpha \quad \text{so that} \quad \tan \alpha = \frac{F}{N} = \mu = 1/3$$

$$\text{But} \quad \tan (\pi/2 - \alpha) = \cot \alpha = \frac{x}{\sqrt{R^2 - x^2}}$$

$$\text{so that} \quad x = \frac{3}{\sqrt{10}} = 0.949 R$$

$$h = R - x = 0.051 R = 5.1\% \text{ of } R$$

Thus, the insect can climb upto only about 5% of R .

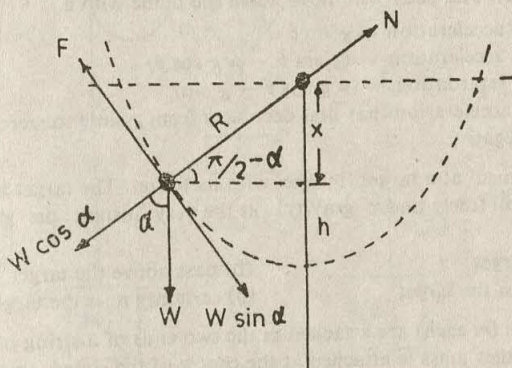


Fig. 3.42

11. *False* The frictional force at first decreases rapidly but later on increases very rapidly. This is because when the two surfaces become so smooth that their surface irregularities are less than the range of intermolecular attraction ($\sim 13^{-8}$ m), there is a 'direct interaction' between the molecules of the two surfaces and we can consider the 'points' of contact of the two surfaces as spread over the whole surface. The force tending to produce relative motion between the surfaces thus encounters large opposition i.e. the frictional force increases.
12. *True* The potential energy ($= mgh$) of the blade is used up in doing work against the resistance of the wood. If the average resistance is F , we have $FS =$ work done $= mgh$. Hence
- $$F = mgh/S$$
13. *True* The work done $W = \text{force} \times \text{displacement} = maS$. Hence, power $P = W/t = maS/t$ so that $a = Pt/mS$. The distance S moved in time t is given by
- $$S = 0 + \frac{1}{2} at^2 = \frac{1}{2} Pt^3/mS$$
- $$S^2 \propto t^3 \quad \text{or} \quad S \propto t^{3/2}$$
14. *True* A mass $M/3$ has its CG raised through a distance $1/2 (L/3) = L/6$. The work done in this process is $M/3g L/6 = Mg L/18$.
15. *True* The effective value of 'g' in this case, being $g + g/8$ or $9/8 g$, the time period will go 'up' by a factor of $\sqrt{8/9}$ or $2\sqrt{2}/3$.

3.1 MULTIPLE-CHOICE QUESTIONS

Select the best alternative in each of the following.

- A bullet hits and gets embedded in a solid block resting on a frictionless surface. In this process
 - momentum is conserved
 - kinetic energy is conserved
 - both momentum and kinetic energy are conserved
 - neither momentum nor kinetic energy is conserved.
- A ball of mass 0.1 kg is thrown against a wall. It strikes the wall normally with a velocity of 30 ms^{-2} and rebounds with a velocity of 20 ms^{-2} in the opposite direction. Then the impulse of the force (in newton-second) by the ball on the wall is
 - 50 N—s
 - 0.5 N—s
 - 1.0 N—s
 - 5 N—s
- A body slides down an inclined plane of inclination θ . The coefficient of friction down the plane varies in direct proportion to the distance moved down the plane ($= kx$). The body will move down the plane with a
 - constant acceleration $= g \sin \theta$
 - constant acceleration $= (g \sin \theta - \mu g \cos \theta)$
 - constant retardation $= (\mu g \cos \theta - g \sin)$
 - variable acceleration that first decreases from $g \sin \theta$ to zero and then becomes negative.
- A gun is 'aimed' at a target 'in line' with its barrel. The target is released (and allowed to fall freely under gravity) at the very instant the gun is fired. The bullet will
 - hit the target
 - pass above the target
 - pass below the target
 - certainly miss the target.
- Equal masses (m each) are attached at the two ends of a string passing over two pulleys. Another mass is attached at the centre of the string. In order that there is no sag in the string, this mass should be
 - m
 - $m/2$
 - $2m$
 - infinite
- A long spring, when stretched by x cm, has a potential energy V . On increasing the stretching to nx cm, the potential energy stored in the spring will be
 - V/n
 - nV
 - $n^2 V$
 - V/n^2
- Two bodies of masses m and $4m$ are moving with equal kinetic energy. The ratio of their linear momenta is
 - 1 : 4
 - 4 : 1
 - 1 : 2
 - 1 : 1
- Two bodies of masses m and $4m$ are moving with equal linear momentum. The ratio of their kinetic energies is
 - 1 : 4
 - 4 : 1
 - 1 : 1
 - 1 : 2
- A jet of water with a cross-sectional area a is striking against a wall at an angle θ to the normal and rebounds elastically. If the velocity of water in the jet is v , the normal force acting on the wall is
 - $2 a v^2 \rho \cos \theta$
 - $a v^2 \rho \cos \theta$
 - $2 a v \rho \cos \theta$
 - $a v \rho \cos \theta$

10. Three different objects of masses m_1 , m_2 and m_3 are allowed to roll from rest from a common point O along three different frictionless paths OP , OQ and OR . The speeds of the three objects on reaching the ground will be in the ratio

(a) $m_1 : m_2 : m_3$

(b) $1 : 1 : 1$

(c) $m_1 OP : m_2 OQ : m_3 OR$

(d) $\frac{OP}{m_1} : \frac{OQ}{m_2} : \frac{OR}{m_3}$

11. A ball is moving to and fro about the lowest point O of a smooth hemispherical bowl. If it is able to rise upto a height of 45 cm on either side of O , its speed at O must be (Take $g = 10 \text{ m s}^{-2}$)

(a) 4.5 m s^{-1}

(b) 9 m s^{-1}

(c) 3 m s^{-1}

(d) 0.045 m s^{-1}

12. In the arrangement shown in Fig. 3.44, masses M_1 and M_2 are both descending with a uniform speed v . The mass M will ascend with a speed

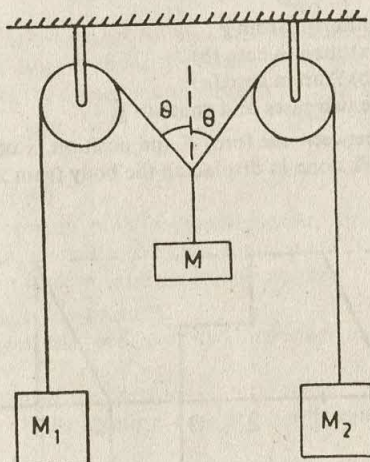


Fig. 3.44

(a) $v \cos \theta$

(b) $2 v \cos \theta$

(c) $2 v / \cos \theta$

(d) $v / \cos \theta$

13. A block of mass M is pulled along a horizontal frictionless surface by a rope of mass m . If a force P is applied at the free end of the rope, the force exerted by the rope on the block will be

(a) $PM / (M + m)$

(b) $Pm / (M + m)$

(c) $Pm / (M - m)$

(d) P

14. A container, filled with water and having a wooden block floating in it, is allowed to fall freely under gravity. During the 'fall', the upthrust on the wooden block will be

(a) equal to the weight of the block in air

(b) equal to the weight of water displaced by the immersed portion of the block

(c) equal to the loss of weight of the block in water

(d) zero

15. Two identical frictionless pulleys are arranged separately as shown in Fig. 3.45 (a) and (b). Assuming that the string has negligible mass, the acceleration of mass m , in the two cases will be

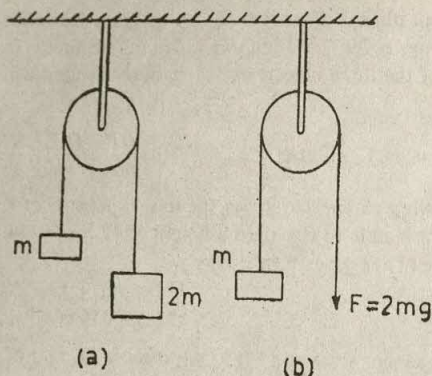


Fig. 3.45 (a) and (b)

- (a) the same but different from g
 (b) more in case (a) than in case (b)
 (c) more in case (b) than in case (a)
 (d) the same in the two cases and equal to g .
16. The relationship between the force F and position x of a body is as shown in Fig. 3.46. The work done in displacing the body from $x = 1$ m to $x = 5$ m will be

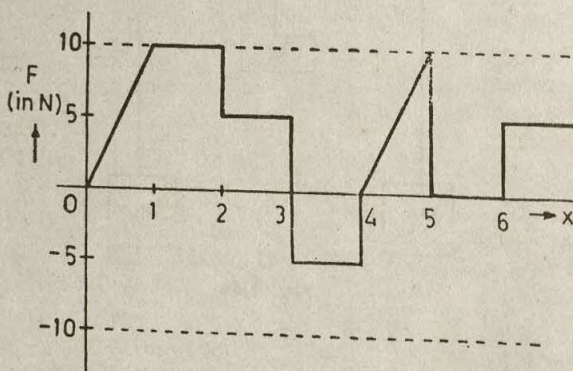


Fig. 3.46

- (a) 20 J
 (b) 15 J
 (c) 25 J
 (d) 30 J
17. A man is pulling on a rope attached to a block on a smooth horizontal table. The tension in the rope will be the same at all points
 (a) if and only if the rope is not accelerated
 (b) if and only if the rope is massless
 (c) if either the rope is not accelerated or is massless
 (d) always
18. A spring balance A and a beam balance B are used to 'weigh' a given object at different points on the earth. Then
 (a) the readings of A and B will be different at different points on the earth
 (b) the readings of A will be different but those of B will be the same at different points on the earth
 (c) the readings of A will be the same but those of B will be different at different points on the earth
 (d) both A and B will have the same readings at different points on the earth.

19. A plumb line is hanging from the ceiling of a car. When the car is moving along a horizontal track with an acceleration a , the plumb line gets inclined at an angle
 (a) $\tan^{-1}(a/g)$ (b) $\tan^{-1}(g/a)$
 (c) $\cos^{-1}(a/g)$ (d) $\sin^{-1}(g/a)$
20. A block can slide on a smooth inclined plane of inclination θ kept on the floor of a lift. When the lift is descending with a retardation a , the acceleration of the block relative to the incline is
 (a) $(g - a) \sin \theta$ (b) $(g + a) \sin \theta$
 (c) $g \sin \theta$ (d) $(g - a)$
21. An automobile moves along a straight horizontal road with a speed (i) v and (ii) $3v$. Assuming identical conditions, the ratio of the shortest distance in which the automobile can be stopped is
 (a) 1 : 3 (b) 1 : 9
 (c) 1 : 27 (d) 1 : 81
22. A given object takes n times as much time to slide down a 45° rough incline as it takes to slide down a perfectly smooth 45° incline. The coefficient of kinetic friction between the object and the incline is given by
 (a) $\mu_k = 1/(1 - n^2)$ (b) $\mu_k = 1 - 1/n^2$
 (c) $\mu_k = \sqrt{\frac{1}{1 - n^2}}$ (d) $\sqrt{1 - 1/n^2}$
23. Several forces varying both in magnitude and direction are used to move a particle along a smooth curved horizontal path. The work done on the particle by the resultant force equals the change in the
 (a) total energy of the particle
 (b) total energy of the particle or its kinetic energy depending on the nature of the path
 (c) kinetic energy of the particle
 (d) total energy or its kinetic energy depending on the magnitude of the resultant force.
24. A body of mass m accelerates uniformly from rest to a speed u in a time T . Then the instantaneous power delivered to the body as a function of time t is given by
 (a) $\frac{1}{2} \frac{m u^2}{T^2} t$ (b) $\frac{1}{2} \frac{m u^2}{T^2} t^2$
 (c) $\frac{m u^2}{T^2} t$ (d) $\frac{m u^2}{T^2} t^2$
25. If g is the acceleration due to gravity on the earth's surface, the gain in potential energy of an object of mass M raised from the surface of the earth to a height equal to the radius R of the earth is
 (a) $MgR/2$ (b) $2MgR$
 (c) MgR (d) $MgR/4$

3.7 NUMERICAL EXERCISES

- The head of a hammer moving at a speed of 10 m s^{-1} strikes a nail and drives it into a wooden block, the duration of the impact is 0.005 s . If the hammer head weighs 1.5 kg , find the average impact force and the distance the nail has penetrated into the wood.
- A disc of mass 20 g is kept floating horizontally by throwing 20 marbles per second against it from below. Assuming that the marbles strike the disc normally and rebound downwards with the same speed, calculate the speed with which the marbles are striking against the disc.

3. A container of volume V cubic metres filled to the brim with water of density ρ kg m^{-3} is counterpoised against suitable weights on a balance. A constant stream of water is poured from a height of h metres into the container at a constant rate of m kg s^{-1} . If the excess water is allowed to overflow horizontally from one side of the balance, what, if any, is the percentage increase in the 'weight' of the container?
4. Two masses m and $2m$ are connected by a massless string which passes over a light frictionless pulley. The masses are initially held with equal lengths of the string on either side and are initially 13.08 m above the ground. Find the velocity of the masses at the instant the lighter mass moves up a distance of 6.54 m. The string is suddenly cut at that instant; calculate the velocities with which the two masses hit the ground. Also calculate the time taken by each mass to reach the ground. (IIT 1977)
5. A ball of mass 100 g is projected vertically upwards from the ground with a velocity of 49 m s^{-1} . At the same time another identical ball is dropped from a height 98 m to fall freely along the same path as that followed by the first ball. After some time, the two balls collide and stick together. Find the time of flight of the balls. (IIT 1985)
6. The human heart forces 4 litres of blood per minute through the arteries under a pressure of 125 mm. Calculate the HP of the heart. Density of blood = 1.03×10^3 kg m^{-3} and 1 HP = 746 W.
7. A block of mass 2 kg slides on an inclined plane which makes an angle of 30° with the horizontal. The coefficient of friction between the block and the surface is $\sqrt{3}/2$
- what force should be applied to the block so that it moves down without any acceleration?
 - what force should be applied to the block so that it moves up without any acceleration?
 - calculate the ratio of the power needed in the above two cases, if the block moves with the same speed in both the cases. (IIT 1976)
8. A body starts rolling down an incline of 1 in 2 from rest. The body is brought to rest just when it reaches the bottom. Show that if the top half of the plane is assumed to be perfectly smooth and the lower half is rough; the force of friction in this rough half equals the weight of the body.
9. In Fig. 3.47 (a) and (b), AC, DG and GF are fixed inclined planes. $BC = EF = x$ and $AB = DE = y$. A small block of mass M is released from rest from the point A. It slides down AC and reaches C with a speed V_c . The same block is released from rest from a point D and slides down DGF and reaches the point F with a speed V_f . The coefficient of kinetic friction between the block and both the surfaces AC and DGF are μ each. Calculate V_c and V_f . (IIT 1980)

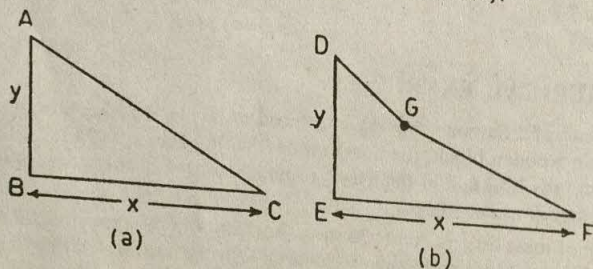


Fig. 3.47 (a) and (b)

10. Two blocks A and B are connected to each other by a string; and a spring; the string passes over a frictionless pulley as shown in Fig. 3.48. Block B slides over the horizontal top surface of a stationary block C and block A slides along the vertical side of C both with the same uniform speed. The coefficient of friction between the surfaces of the blocks is 0.2 and the force constant of the spring is 1960 N m^{-1} . If the mass of block A is 2 kg, calculate the mass of block B and the energy stored in the spring. (IIT 1982)

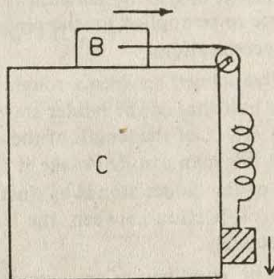


Fig. 3.48

11. Two bodies A and B of masses m and $2m$ respectively are placed on a smooth floor. They are connected by a spring. A third body C of mass m moves with a velocity v_0 along the line joining A and B and collides elastically with A as shown in Fig. 3.49. At a certain instant t_0 after the collision, it is found that the instantaneous velocities of A and B are the same. Further, at this instant, the compression of the spring is found to be x_0 . Determine (a) the constant velocity of A and B at the time t_0 and (b) the spring constant. (IIT 1984)

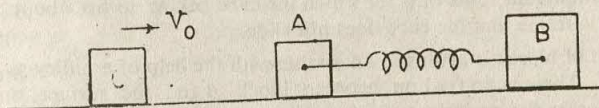


Fig. 3.49

12. Two bodies A and B of equal mass are arranged on a block P as shown in Fig. 3.50. The coefficient of friction between P and either of A or B is μ . Assuming the pulley and the string to be massless and the pulley to be smooth, calculate the minimum horizontal acceleration with which block P should move so that both A and B remain stationary, relative to it.

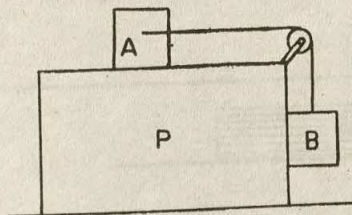


Fig. 3.50

13. A uniform ladder of length l rests on a rough horizontal ground with its upper end projecting slightly over a smooth horizontal rail at a height h . If the ladder is just on the point of slipping, what is the coefficient of friction between the ladder and the rough horizontal ground?

14. A body of mass m rests on a horizontal plane having coefficient of friction μ . At $t = 0$, a horizontal force $\vec{F}$ is applied that varies with time as $\vec{F}(t) = \vec{F}_0 t$, where $\vec{F}_0$ is a constant vector. Find (a) the time instant t_0 at which motion starts and (b) the distance s moved by the body in t seconds ($t > t_0$) after the force begins to act. What is the distance moved in t seconds if $t < t_0$?
15. A block of mass 2 kg slides on an inclined plane making an angle of 30° with the horizontal. The coefficient of friction between the block and the surface is $\sqrt{3}/2$. Calculate the force to be applied to the block so that it moves (a) down and (b) up without any acceleration. (IIT 1978)
16. A ladder is resting equally inclined between a rough vertical wall and ground. If a man whose weight is half that of the ladder starts ascending on it, show that the man can climb up to 71.4% of the length of the ladder before it just starts slipping. Also show that this man can just make it to the top of the ladder if a boy of weight $1/4$ that of the ladder stands against the ground end of the ladder. Take the coefficient of friction between the ladder and (i) ground and (ii) wall as $1/2$ and $1/3$ respectively.
17. A mass of 500 kg is kept in equilibrium on a rough inclined plane of 1 in $\sqrt{3}$ by a horizontal rope. Show that the maximum and minimum tensions in the rope are in the ratio 7 : 3 when the coefficient of friction between the surface is 0.2.
18. A sphere of mass M lying on a rough inclined plane of inclination θ is held in equilibrium by a horizontal string attached to the highest point of the sphere. What is the tension in the string?
19. A uniform cube of side a and mass m rests on a rough horizontal table. A horizontal force $\vec{F}$ is applied normal to one of the faces of the cube at a point that is directly above the centre of the face at a height $3a/4$ above the base. Show that the minimum value of F for which the cube begins to tip about an edge is $2/3 mg$. Assume that the cube does not slide. (IIT 1984)
20. A system of blocks is arranged on a table with the help of a pulley as shown in Fig. 3.51. There is no friction between block B and the surface on which it rests, whereas the coefficient of friction between blocks A and B is 0.2. The system is released from rest. Calculate the distance moved by block B when one-third of block A is still on it. Given length a and mass of block B is four times those of A and block C is identical to block A .
21. A boy pulls a cart of mass m along a horizontal surface at a constant speed. If the coefficient of kinetic friction is μ_k and his 'pull' P is inclined at an angle θ to the horizontal, what work does he do in displacing the cart through a distance x ?

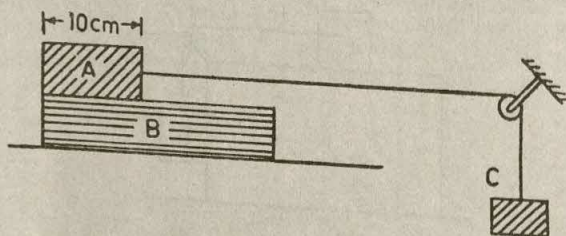


Fig. 3.51

22. The turbine pits at Niagara Falls are 40 m deep. The average horse power developed is 5000; the efficiency being 85%. How much water passes through the turbines per minute? 1 HP = 746 W and $g = 10 \text{ m s}^{-2}$.

23. A box of mass 50 kg is just on the point of sliding down an inclined plane of an incline of 1 in 2. What force must a person apply to make it just move up the plane if he pushes (a) horizontally and (b) up along the plane? The coefficient of friction is 0.5.

ANSWERS

MULTIPLE CHOICE QUESTIONS

- | | | | | |
|---------|---------|---------|---------|---------|
| 1. (c) | 2. (d) | 3. (d) | 4. (a) | 5. (d) |
| 6. (c) | 7. (c) | 8. (b) | 9. (a) | 10. (b) |
| 11. (c) | 12. (b) | 13. (a) | 14. (d) | 15. (c) |
| 16. (b) | 17. (c) | 18. (b) | 19. (a) | 20. (b) |
| 21. (b) | 22. (b) | 23. (c) | 24. (c) | 25. (a) |

NUMERICAL EXERCISES

- 3000 N, 2.5 cm
- 0.49 m s^{-1}
- $\frac{100 \text{ m}}{f V} \sqrt{2 h/g}$
- 20.66 m s^{-1} , 13.07 m s^{-1} , 4.77 s , 2.67 s
- 6.53 s
- $112.76 \times 10^{-6} \text{ HP}$
- 0.6 N, 30.62 N, 0.0196
- $V_e = V_f = \sqrt{2 g (y - \mu x)}$
- 10 kg, 0.098 J
- $v_0/3$, $2 m v_0^2/3 x_0^2$
- $g (1 - \mu)/(1 + \mu)$
- $h \sqrt{l^2 - h^2}/(l^2 + h^2)$
- $\frac{\mu_0 Mg}{F_0}$, $\frac{1}{6} \frac{F_0}{M} (t - t_0)^3$, zero
- 10.99 N, 30.59 N
- $Mg \tan^2 \theta/2$
- 5.25 cm
- $\frac{(\mu_k m g x \cos \theta)}{(\cos \theta \lambda + \mu_k \sin \theta)}$
- 539 metric tons
- 741.9 N, 457.7 N

Circular Motion and Gravitation

4.1 REVIEW OF BASIC CONCEPTS

4.1.1 Angular Quantities A particle in circular motion moves through an angle θ when it moves a distance s measured around the circumference of the circle (Fig. 4.1).

Arc s is called the linear displacement and angle θ the angular displacement. Angle θ is measured in radians. A radian (rad) is the angle whose arc equals the radius of the circle; it is equal to about 57.3° (see Fig. 4.2).

$$\theta = \frac{s}{r} \quad \text{or} \quad s = r\theta \quad (\theta \text{ in radians}) \quad (4.1)$$

The angular velocity is the rate of change of angular displacement:

$$\omega = \lim_{\Delta t \rightarrow 0} \left(\frac{\Delta \theta}{\Delta t} \right) = \frac{d\theta}{dt} \quad (4.2)$$

Linear velocity (v), angular velocity (ω) and the radius of the circle (r) are related as

$$v = r\omega \quad (4.3)$$

where ω is measured in rad s^{-1} .

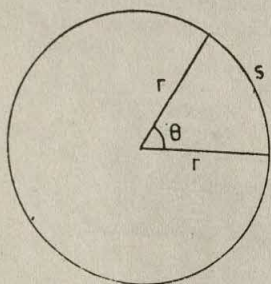


Fig. 4.1

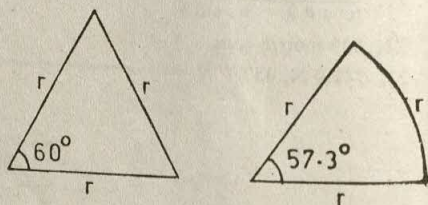


Fig. 4.2 (a) and (b)

Similarly, angular acceleration is the rate of change of angular velocity and is given by

$$\alpha = \lim_{\Delta t \rightarrow 0} \left(\frac{\Delta \omega}{\Delta t} \right) = \frac{d\omega}{dt} \quad (4.4)$$

Linear acceleration a and angular acceleration α are related as

$$a = r \alpha \quad (4.5)$$

where α is measured in rad s^{-2} .

In general, analogous to the equations governing linear quantities, we have the following equations governing angular quantities.

$$\omega = \omega_0 + \alpha t \quad (4.6)$$

$$\theta = \left(\frac{\omega_0 + \omega}{2} \right) t \quad (4.7)$$

$$\omega^2 = \omega_0^2 + 2 \alpha \theta \quad (4.8)$$

$$\theta = \omega_0 t + \frac{1}{2} \alpha t^2 \quad (4.9)$$

Equations (4.6) to (4.9) are valid in any consistent set of angular units. However, Eqs (4.1) and (4.5) require the use of the radian measure, since linear and angular quantities appear in the same equation.

4.1.2 Uniform Circular Motion In this kind of motion, the magnitude of velocity is constant but its direction continually changes along the circular path (see Fig. 4.3). This must be due to the component of the acceleration normal to the velocity. Obviously, the tangential acceleration, in this case, must be equal to zero. The acceleration is directed towards the centre of the circle and is called *centripetal acceleration*.

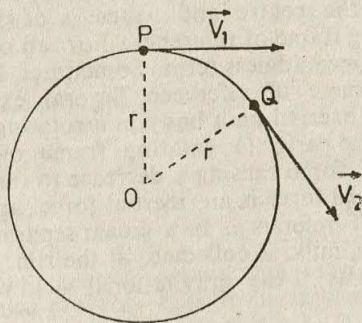


Fig. 4.3

The magnitude of the centripetal acceleration is.

$$a_c = \omega v \quad (4.10)$$

Thus, since $v = r \omega$, we have

$$a_c = \omega v = \omega^2 r \quad (4.11)$$

$$\text{Also } a_c = \omega v = \left(\frac{v}{r} \right) v = \frac{v^2}{r} \quad (4.12)$$

In uniform circular motion, the particle moves through 2π radians (one revolution) in T seconds (periodic time) and hence the angular speed is

$$\omega = \frac{2\pi}{T} \quad (4.13)$$

The distance moved in T seconds is $2\pi r$, therefore, the linear speed in a circular path is given by

$$v = \frac{2\pi r}{T} \quad (4.14)$$

Centripetal acceleration can also be expressed in terms of T :

$$a_c = \omega v = \frac{2\pi}{T} \left(\frac{2\pi r}{T} \right) = \frac{4\pi^2 r}{T^2} \quad (4.15)$$

Since the frequency $\nu = 1/T$, we have

$$a_c = 4\pi^2 r \nu^2 \quad (4.16)$$

4.1.3 Centripetal and Centrifugal Forces As in the case of linear motion, forces are needed to produce acceleration in bodies having mass (Newton's second law of motion). The centripetal force, which causes centripetal acceleration, acts on the body in circular motion and is directed towards the centre of the circle. Any type of force can serve as a centripetal force. Elastic, frictional, gravitational and magnetic forces, etc., can cause uniform circular motion of a body. Numerical examples in this chapter will illustrate this fact.

If there is a force directed toward the centre exerted on a body moving in a circle, by Newton's third law, there must be a force directed away from the centre exerted by the body. This force is equal and opposite to the centripetal force. This is the reaction exerted on the axis of the rotating body or whatever other body supplies the inward force. This force is directed away from the centre and hence is of significance only to an observer in the rotating frame of reference. For an outside observer, the 'centrifugal force' is a meaningless term. Sometimes, however, it is helpful to use the rotating frame of reference. Several examples of centrifugal force (the inertia force exerted on a body in a rotating frame of reference) can be given. We on the earth (a rotating frame of reference) can talk about the centrifugal force causing a decrease in the weight of a body at the equator. The upward force is an inertial force arising from the acceleration of the frame of reference. In a cream separator (a type of centrifuge) the heavier liquid, milk, is collected at the rim of the rotating dish. The effect is the same as if the gravitational field of the earth were increased.

4.1.4 Banking of Curved Tracks When a car (or some other vehicle) moves round a level curve, the centripetal force required to keep the vehicle in motion is provided by the friction between the road and the tyres. If F is the frictional force, we have

$$F = \frac{mv^2}{r} \quad (4.17)$$

where m is the mass of the car, v its speed round the curve and r is the radius of the curved path. The higher the value of F , the faster is the speed at which the curve can be negotiated.

The large amount of friction between the tyres and the road will cause damage to the tyres. To minimize the wearing out of the tyres, the road bed is banked, i.e. the outer part of the road is raised a little so that the road is sloping towards the centre of the curved track.

Suppose a car of mass m is moving round a banked track in a circular path of radius r as shown in Fig. 4.4. Let R_1 and R_2 be the reaction at

each tyre due to the road. Then the total reaction is $R = R_1 + R_2$ acting in the middle of the car. If θ is the angle of banking, the vertical component $R \cos \theta$ supports the weight mg of the car while the horizontal component $R \sin \theta$ provides the necessary centripetal force. Therefore,

$$R \sin \theta = \frac{m v^2}{r}, \quad R \cos \theta = m g$$

so that $\tan \theta = \frac{v^2}{r g}$ (4.18)

Thus, for a given velocity v and radius r , the angle of the banking of the track must be

$$\theta = \tan^{-1} \left(\frac{v^2}{r g} \right) \quad (4.19)$$

so that there is no skidding of the car. For the same reason the outer rail of a curved railway track is raised slightly higher than the inner rail.

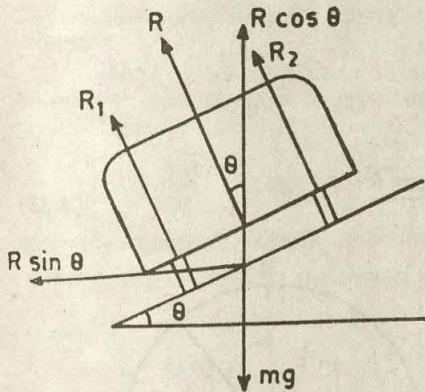


Fig. 4.4

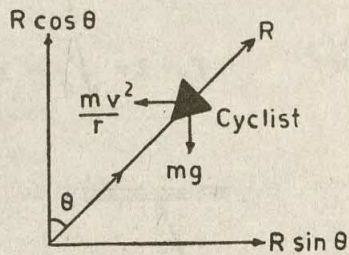


Fig. 4.5

While going round a curve, a cyclist leans inwards to provide the necessary centripetal force which prevents him from falling down. Figure 4.5 shows a cyclist leaning at an angle θ to the vertical. R is the reaction due to the surface inclined at an angle θ and passing through the centre of gravity.

Once again the relations (4.18) and (4.19) hold. If v is larger or r is smaller, angle θ would be larger. In other words, the cyclist must bend more if he wishes to negotiate a curve at a higher speed or if the curve is sharper (i.e. having a smaller radius). Also, note that angle θ is independent of the mass of the vehicle.

The force of friction, F , between the road and the tyres gives the normal reaction, $F = \mu R$, where μ is the coefficient of friction. If the cyclist does not lean ($\theta = 0$); $R = mg$, or

$$F = \mu mg \quad (4.20)$$

The cyclist will, therefore, skid if F is less than $\frac{m v^2}{r}$ or

if $\frac{m v^2}{r} > \mu mg$

$$\text{or } \frac{v^2}{r} > \mu g \quad (4.21)$$

Thus, skidding will occur if the speed is large, the curve is sharp and the coefficient of friction is small.

4.1.5 The Conical Pendulum If a particle of mass m attached at the end of a string of length L suspended from a rigid fixed point is allowed to move in a horizontal circle, the string sweeps over the surface of a cone. If T is the tension in the string, it is clear from Fig. 4.6 that

$$T \sin \theta = \frac{m v^2}{r}, \quad T \cos \theta = m g \quad (4.22)$$

where θ is the angle the string makes with the vertical. Hence

$$\tan \theta = \frac{v^2}{r g} = \frac{r}{l} \quad \text{or} \quad v^2 = \frac{r^2 g}{l}$$

$$\text{But } v = \frac{2 \pi r}{T}$$

where T is the period of revolution.

$$\therefore \frac{4 \pi^2 r^2}{T^2} = \frac{r^2 g}{l}$$

$$\text{or } T = 2 \pi \sqrt{\frac{l}{g}} = 2 \pi \sqrt{\frac{L \cos \theta}{g}} \quad (4.23)$$

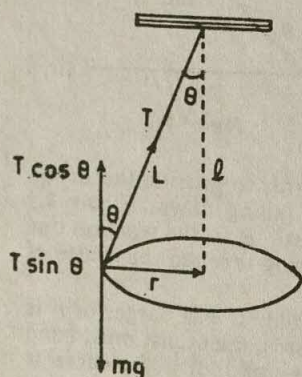


Fig. 4.6

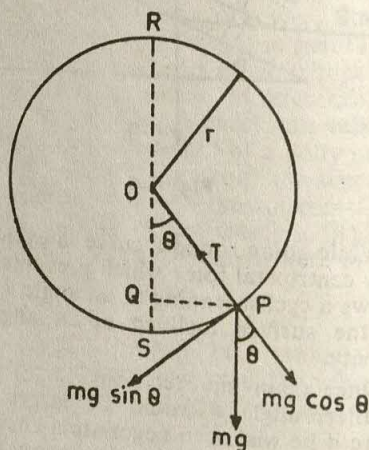


Fig. 4.7

4.1.6 Motion in a Vertical Circle Figure 4.7 shows a particle of mass m attached to the end of a cord OP , moving in a vertical circle about a fixed point O .

This is an example of non-uniform circular motion. Because of gravity the speed increases when the particle is coming down and decreases when it moves up. At point P , the centripetal force is given by

$$\frac{m v^2}{r} = T - m g \cos \theta \quad (4.24)$$

where T is the tension in the cord.

$$T = m \left(\frac{v^2}{r} + g \cos \theta \right) = m a_n \quad (4.25)$$

where a_n is the normal component of acceleration. The tangential force and acceleration respectively are $mg \sin \theta$ and $g \sin \theta$, which is the same as those of a body sliding on a frictionless inclined plane of slope θ .

At the lowest point of the circle $\theta = 0$, then

$$T = m \left(\frac{v^2}{r} + g \right) \quad (4.26)$$

Here the acceleration is purely radial in vertically upward direction.

At the highest point of the circle, $\theta = \pi$ and

$$T = m \left(\frac{v^2}{r} - g \right) \quad (4.27)$$

The acceleration is again purely radial but in the vertically downward direction.

To keep the body of mass m in a circular path the centripetal force must be greater than or equal to mg (the weight of the body) at the highest point. Thus

$$\frac{m v^2}{r} = mg \quad \text{or} \quad v^2 = rg \quad (4.28)$$

gives the minimum velocity in order to complete the circle.

If v_1 is the velocity at the lowest point, then

$$\begin{aligned} v_1^2 &= v^2 + 2 \times 2rg && \text{(using } v^2 - u^2 = 2gh) \\ \therefore v_1^2 &= rg + 4rg = 5rg \end{aligned} \quad (4.29)$$

The minimum velocity the particle should have at the lowest point is

$$v_1 = \sqrt{5rg} \quad (4.30)$$

The tension at this point is given by

$$T_1 = m \left(\frac{v_1^2}{r} + g \right) = 6mg \quad (4.31)$$

In case, $v_1 < \sqrt{5rg}$, the body will either oscillate about the lowest point S or leave the circular path altogether. The tension along the cord is maximum at the bottom and minimum at the top. At no point the tension can be negative, otherwise the body will fall down without completing the circle.

The tension in the cord at the highest point is given by

$$T_2 = \frac{m v_2^2}{r} - mg$$

where v_2 is the velocity at the top. The tension in the cord at the bottom is

$$\begin{aligned} T_1 &= \frac{m v_1^2}{r} + mg \\ T_1 - T_2 &= \frac{m}{r} (v_1^2 - v_2^2) + 2mg \end{aligned}$$

But $v_1^2 - v_2^2 = 2 g h = 4 r g$

$$\therefore T_1 - T_2 = 6 m g \quad (4.32)$$

Thus the difference between the tensions along the cord at the highest and the lowest points is six times the weight of the body.

4.1.7 Gravitation Newton's law of gravitation is expressed by the relation

$$F = G \frac{m_1 m_2}{r^2} \quad (4.33)$$

where F is the force of attraction between two spherical bodies whose masses are m_1 and m_2 , and whose centres are at a distance r apart. The numerical value of the constant G depends on the units used for the measurement of force, mass and distance. In SI units, G has been found to be $6.67 \times 10^{-11} \text{ Nm}^2 \text{ kg}^{-2}$.

The space in which a body experiences a force due to its mass is called *gravitational field*. The force experienced by a unit mass in a gravitational field is known as the *gravitational intensity* or *gravitational field strength*. Gravitational field is said to be uniform if the gravitational intensity has the same value at every point in space.

Considering the earth as an isolated mass, a force is experienced by a body near it. This force is directed towards the centre of the earth and has a magnitude mg , where g is the acceleration due to gravity,

$$F = mg = G \frac{mM}{R^2} \quad (4.34)$$

where M is the mass of the earth and R its radius (nearly constant for a body in the vicinity of the earth)

$$\therefore g = G \frac{M}{R^2} \quad (4.35)$$

This indicates that all bodies near the earth fall with the same acceleration. The acceleration is directed towards the centre of the earth.

The gravitational field of a spherical mass M at a point P , distant r from the centre of the sphere is given by

$$g(P) = G \frac{M}{r^2} \quad (4.36)$$

and is directed towards the centre. The gravitational potential at P is

$$V(P) = - G \frac{M}{r} \quad (4.37)$$

The gravitational potential energy of a small mass m placed at P is

$$P.E. = - G \frac{Mm}{r} \quad (4.38)$$

The total mechanical energy (TME) of a body is the sum of its kinetic energy and potential energy and is given by

$$\text{TME} = \frac{1}{2} mv^2 - G \frac{Mm}{r} \quad (4.39)$$

A body is said to be bound if its TME is negative. The binding energy is the energy supplied to free the body, i.e. to raise its TME to zero. If the only force on the body is the gravitational force, the total mechanical energy remains constant or the TME is conserved. In a gravitational field, a body at a given position has more binding energy when it is at rest than when it is in motion.

4.1.8 Escape Velocity The total work done by a body of mass m in escaping from the earth's surface (i.e. in moving away to an infinite distance from it) is

$$- \left(- G \frac{Mm}{r} \right) = G \frac{Mm}{r}$$

where R is the radius of the earth and M its mass.

If v_e is the escape velocity, then the initial kinetic energy of the body is $\frac{1}{2} m v_e^2$, which must at least be equal to the energy required by the body in freeing itself from the gravitational pull of the earth. Therefore,

$$\frac{1}{2} m v_e^2 = G \frac{Mm}{R}$$

$$\text{or } v_e = \sqrt{\frac{2MG}{R}} \quad (4.40)$$

Equation (4.40) can be obtained by simply putting the TME in Eq. (4.39) equal to zero. This means that the escape velocity is the minimum velocity required to free a body from the binding of the gravitational field. Escape velocity is independent of the mass of the body. However, the force required to accelerate a body until it acquires the escape velocity does depend on the mass of the body.

4.1.9 Satellites A body revolving around a much larger body under the influence of the latter is called a satellite. Thus, moon is a satellite of the earth.

Consider a satellite of mass m moving with a velocity v in a circular orbit of radius r around the earth. Then

$$\text{Kinetic energy (KE)} = \frac{1}{2} m v^2$$

$$\text{Potential energy (PE)} = - G \frac{Mm}{r} \quad (4.41)$$

The centripetal force is given by

$$\frac{mv^2}{r} = G \frac{Mm}{r^2} \quad \text{or} \quad v^2 = G \frac{M}{r} \quad (4.42)$$

$$\text{KE} = G \frac{Mm}{2r} \quad (4.43)$$

Thus the potential energy of a body in a circular orbit is numerically twice its kinetic energy and has the opposite sign. Total mechanical energy in the orbit is

$$\text{TME} = - G \frac{Mm}{r} + \frac{1}{2} G \frac{Mm}{r} = - G \frac{Mm}{2r} \quad (4.44)$$

The orbital velocity of the satellite, from Eq. (4.42) is

$$v = \sqrt{G \frac{M}{r}} = \sqrt{G \frac{M}{(R+h)}} \quad (4.45)$$

where R is the mean radius of the earth and h , the height of the satellite above the earth's surface.

From Eq. (4.35), $GM = gR^2$.

Therefore,

$$v = R \sqrt{\frac{g}{(R+h)}} \quad (4.46)$$

If $h \ll R$ (for a satellite a few hundred km above the earth's surface) we can write

$$v \simeq \sqrt{gR} \quad (4.47)$$

The periodic time T of a satellite is the time it takes to complete one revolution round the earth.

$$T = \frac{2\pi r}{v} = \frac{2\pi r}{R} \sqrt{\frac{(R+h)}{g}} = 2\pi \sqrt{\frac{(R+h)^3}{gR^2}} \quad (4.48)$$

For satellites with $h \ll R$,

$$T \simeq 2\pi \sqrt{\frac{R}{g}} \quad (4.49)$$

Equation (4.46) gives the minimum velocity v_m to maintain the satellite in a circular orbit of radius r . The following conditions hold:

$v < v_m$, body will fall back to earth

$v > v_m$ but $< v_e$, the body will execute an elliptical orbit.

$v \geq v_e$, the body frees itself from the gravitational pull of the earth and moves away along a parabola ($v > v_e$) or a hyperbola ($v = v_e$).

4.1.10 Variation of 'g'

1. *Effect of altitude* For a body at a height h above the surface of the earth, the acceleration due to gravity is given by

$$g_h = G \frac{M}{(R+h)^2} \quad (4.50)$$

Using Eq. (4.35) we get

$$g_h = g \left(\frac{R}{R+h} \right)^2 \quad (4.51)$$

For $h \ll R$, using binomial expansion, we get

$$g_h \simeq g \left(1 - \frac{2h}{R} \right) \quad (4.52)$$

Thus the acceleration due to gravity decreases on moving away from the earth. The relation (4.52) applies only when the distance from the earth is negligible in comparison to its radius.

2. *Effect of depth* Considering the earth to be a homogeneous sphere of uniform density ρ and radius R , the gravitational force on a body of mass m , placed on its surface is

$$mg = G \frac{Mm}{r^2} = G \frac{(4/3)\pi R^3 \rho m}{R^2} = \frac{4}{3}\pi R \rho mG \quad (4.53)$$

If the body lies at a depth d inside the earth, the effective mass of the earth which attracts it is due to a concentric sphere of radius $(R - d)$. If the acceleration due to gravity on the surface of this sphere is g_d , then

$$mg_d = G \frac{[(4/3)\pi (R - d)^3 \rho m]}{(R - d)^2} = \frac{4}{3}\pi (R - d) \rho mG \quad (4.54)$$

From Eqs (4.53) and (4.54) we obtain

$$g_d = g \left(1 - \frac{d}{R}\right) \quad (4.55)$$

Thus g_d is directly proportional to the distance from the centre and hence becomes zero at the centre of the earth ($d = R$). Moreover, the value of g at a distance below the earth's surface is greater than at the same distance above it compare Eqs (4.52) and (4.55). The results for the two cases are illustrated graphically in Fig. 4.8.

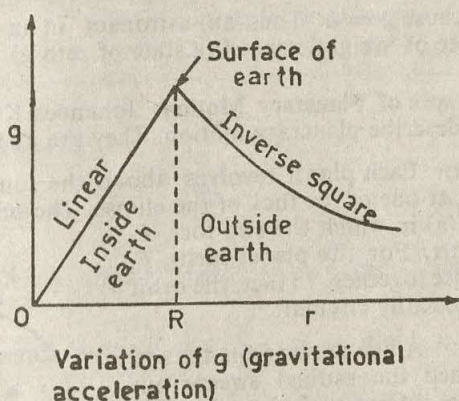


Fig. 4.8

3. *Effect of Rotation* Due to rotation of the earth about its axis from west to east, a body on its surface experiences a centrifugal force on account of which the body can be thrown off the surface. But the gravitational force prevents it. This tendency is the greatest at the equator, where the speed is the greatest and the least at the poles where the speed is the least. The rotation of the earth causes a part of the gravitational force on a body to be used in supplying the centripetal force to it.

Considering the earth to be a homogeneous sphere of radius R the apparent weight of a body of mass m at a point on the equator can be shown to be equal to

$$W = W_0 - \frac{m v^2}{R} \quad (4.56)$$

where v is the speed of a point on the equator. Here W is the earth's gravitational attraction or the true weight. If the earth were not rotating, the free-fall acceleration $g_0 = \frac{W_0}{m}$, then

$$g = g_0 - \frac{v^2}{R} \quad (4.57)$$

Since
$$v = \frac{2\pi R}{T} = \frac{2\pi \times 6.38 \times 10^6}{24 \times 60 \times 60} = 464 \text{ m s}^{-1}$$

we have
$$\frac{v^2}{R} = 0.0337 \text{ m s}^{-2}$$

It can be shown that, at a latitude ϕ , the value of g is given by

$$\begin{aligned} g &= g_0 - \frac{v^2}{R} \cos^2 \phi \\ &= g_0 - 0.0337 \cos^2 \phi \end{aligned} \quad (4.58)$$

Equations (4.56) can be used to discuss 'weightlessness' in satellites. A body moving in an orbit in space is like a freely falling body, with an acceleration a towards the centre of the earth equal to the value of the acceleration due to gravity g at the height of the orbit. The apparent weight W from Eq. (4.56) is given by

$$W = W_0 - m a = m (g - a) \quad (4.59)$$

Hence $W = 0$ because $g = a$. Thus an astronaut in a space vehicle is said to be in a state of 'weightlessness' (a state of zero g).

4.1.11 Kepler's Laws of Planetary Motion Johannes Kepler formulated three laws which describe planetary motion. They are as follows:

1. *Law of orbits:* Each planet revolves about the sun in an elliptical orbit with the sun at one of the foci of the ellipse. The orbit of a planet is shown in Fig. 4.9 (a) in which the two foci, F_1 and F_2 , are far apart. For the planet earth, F_1 and F_2 are very close together. In fact, the orbit of the earth is practically circular.

2. *Law of areas:* A line drawn from the sun to the planet (termed the radius) sweeps out equal areas in equal intervals of time. In Fig. 4.9 (b), P_1, P_2, P_3 and P_4 represent positions of a planet at different times in its orbit and S , the position of the sun.

According to Kepler's second law, if the time interval between P_1 and P_2 equals the time interval between P_3 and P_4 , then area A_1 must be equal to area A_2 . Also the planet has the greater speed in its path from P_1 to P_2 than in its path from P_3 to P_4 .

3. *Law of periods:* The squares of the periods of the planets are proportional to the cubes of their mean distances from the sun. If T_1 represents the period of a planet about the sun, and R_1 its mean distance, then

$$T_1^2 \propto R_1^3$$

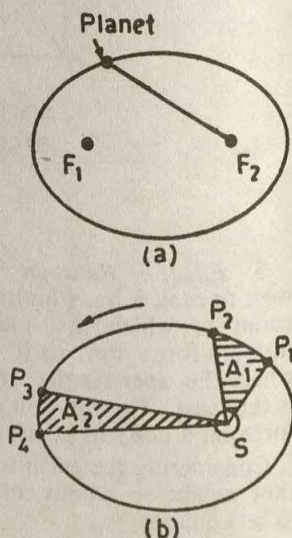


Fig. 4.9 (a) and (b)

If T_2 represents the period of a second planet about the sun, and R_2 its mean distance, then for this planet

$$T_2^2 \propto R_2^3$$

These two relations can be combined since the factor of proportionality is the same for both. Thus

$$\frac{T_1^2}{T_2^2} = \frac{R_1^3}{R_2^3} \quad (4.60)$$

4.1 SI UNITS

Table 4.1

<i>Physical Quantity</i>	<i>SI Units</i>	<i>Symbol</i>
Angular velocity	radian per second	rad s ⁻¹
Angular acceleration	radian per square second	rad s ⁻²
Linear velocity	metre per second	m s ⁻¹
Linear acceleration	metre per square second	m s ⁻²
Time period	second	s
Frequency	hertz	Hz
Gravitational constant (G)	newton metre squared per kilogram squared	N m ² kg ⁻²
Energy	joule	J

4.3 FORMULAE

$$1. \ 1 \text{ radian} = \frac{360^\circ}{2\pi} = \frac{180^\circ}{\pi}$$

$$2. \ (a) \text{ Average angular speed } \bar{\omega} = \frac{\theta}{t} \text{ rad s}^{-1}; \theta \text{ in radian}$$

$$(b) \text{ Angular velocity } \omega = \frac{d\theta}{dt} = \bar{\omega}, \text{ if } \omega \text{ is a constant}$$

$$3. \text{ Linear velocity } v = r \omega$$

$$4. \text{ Linear acceleration } a = r \alpha \quad (\alpha \text{ in rad s}^{-2})$$

$$5. \text{ Angle of rotation } \theta = \frac{\omega_0 + \omega}{2} t; \omega_0, \omega$$

being the initial and final angular speed.

$$6. \text{ Final angular velocity } \omega = \omega_0 + \alpha t \quad (\text{for constant } \alpha)$$

$$7. \text{ Angular acceleration } \alpha = (\omega^2 - \omega_0^2)/2\theta \quad (\text{for constant } \alpha)$$

$$8. \text{ Rotation in time } t, \theta = \omega_0 t + \frac{1}{2} \alpha t^2 \quad (\text{for constant } \alpha)$$

$$9. \text{ Centripetal acceleration } a_c = \omega v = \omega^2 r = v^2/r$$

$$10. \text{ Time period } T = \frac{2\pi r}{v} = \frac{2\pi}{\omega}$$

$$11. \text{ Centripetal force } F_c = \frac{mv^2}{r} = m\omega^2 r$$

12. The proper velocity V on a road banked by an angle θ with the horizontal $V = \sqrt{rg \tan \theta}$

13. Maximum velocity without skidding (on a level road)

$$V = \sqrt{rg \mu}$$

14. Maximum velocity without skidding on a banked road

$$V = \sqrt{\mu rg \tan \theta}$$

15. Time period of a conical pendulum $T = 2\pi \sqrt{\frac{L \cos \theta}{g}}$

16. Minimum speed at the top to complete the circle $v = \sqrt{rg}$

17. Minimum speed needed at the bottom $v = \sqrt{5rg}$

18. Universal law of gravitation $F = G \frac{m_1 m_2}{r^2}$

19. Acceleration due to gravity on the surface of the earth $g = G \frac{M}{R^2}$

20. Gravitational potential $V(P) = -G \frac{M}{r}$

21. Gravitational potential energy $(PE) = -G \frac{Mm}{r}$

22. Intensity of gravitational field $g(P) = G \frac{M}{r^2}$
(P outside the surface).

23. Total mechanical energy of an orbiting body $= -G \frac{Mm}{2r}$

24. Minimum velocity in a circular orbit $v_m = \sqrt{G \frac{M}{(R+h)}}$

25. Time period of a satellite $T = 2\pi \sqrt{\frac{(R+h)^3}{gR^2}}$

26. Value of g at a height h ; $g_h = g \left(\frac{R}{R+h} \right)^2$

27. Value of g at a depth d ; $g_d = g \left(1 - \frac{d}{R} \right)$

28. Value of g at a latitude; $g_\phi = g_0 - 0.0337 \cos^2 \phi$

29. Relation between time periods and orbital radii

$$\frac{T_1^2}{T_2^2} = \frac{R_1^3}{R_2^3}$$

4.4 VALUES OF SOME IMPORTANT CONSTANTS

$$G = 6.67 \times 10^{-11} \text{ Nm}^2 \text{ kg}^{-2}$$

$$g = 9.8 \text{ m s}^{-2} \text{ on earth's surface}$$

Table 4.1 Astronomical Data

	<i>Sun</i>	<i>Earth</i>	<i>Moon</i>
Mass	$1.99 \times 10^{30} \text{ kg}$	$5.98 \times 10^{24} \text{ kg}$	$7.36 \times 10^{22} \text{ kg}$
Radius*	$6.96 \times 10^8 \text{ m}$	$6.38 \times 10^6 \text{ m}$	$1.74 \times 10^6 \text{ m}$
Density*	1460 kg m^{-3}	5500 kg m^{-3}	3340 kg m^{-3}
Radius of orbit*		$1.49 \times 10^{11} \text{ m}$	$3.84 \times 10^8 \text{ m}$
Period of revolution around the sun		$3.16 \times 10^7 \text{ s}$	$2.36 \times 10^6 \text{ s}$

*Average value

4.5 NUMERICAL EXAMPLES

Example 1: A body of mass 2 kg is tied at one end of a string 1 m long. The other end is fixed and the body revolves in a horizontal circle. The maximum tension which the string can withstand is 2000 N. Find the maximum number of revolutions per minute the body will make and its linear velocity when the string just breaks.

Solution: The centripetal force is provided by the tension in the string.

$$\therefore T = m \omega^2 r = (2\pi n)^2 r$$

The tension increases as the speed of revolution increases. If n is the number of revolutions completed when the string just breaks, then

$$2 \times (2\pi n)^2 \times 1 = 2000$$

$$\text{or} \quad n^2 = \frac{2000}{8\pi^2} = 25.33$$

$$\therefore n = 5.03 \text{ rev s}^{-1} \\ = 302 \text{ rpm}$$

Let v be the maximum linear velocity when the string breaks, then

$$v = r \omega \\ = r (2\pi n) \\ = 1 \times 2\pi \times 5.03 \\ = 31.6 \text{ m s}^{-1}$$

After the string breaks, the body will move at a speed of 31.6 m s^{-1} tangential to the circle at that point.

Example 2: A car is travelling at 36 km h^{-1} on a level road and where the coefficient of friction between the tyres and the road is 0.8. Find the minimum turning radius of the car. Take $g = 10 \text{ m s}^{-2}$.

Solution: The maximum centripetal force that the friction can provide is

$$f = \mu mg = \frac{m v^2}{r}$$

$$r = \frac{v^2}{\mu g} = \left(\frac{36 \times 1000}{60 \times 60} \right)^2 \frac{1}{0.8 \times 10} = 12.5 \text{ m}$$

Example 3: Find the proper banking angle for cars moving at 54 km h^{-1} to go round in a curve 300 m in radius. If the curves were not banked, what coefficient of friction would be required between the tyres and the road? Take $g = 10 \text{ m s}^{-1}$.

Solution: $v = \frac{54 \times 1000}{3600} = 15 \text{ m s}^{-1}$

$$\tan \theta = \frac{v^2}{rg} = \frac{15 \times 15}{300 \times 10} = 0.075$$

$$\theta = \tan^{-1}(0.075) = 4.3^\circ$$

The coefficient of friction μ is given by

$$\mu = \frac{v^2}{rg} = 0.075 \quad (\text{see Ex. 2})$$

Example 4: A train has to negotiate a curve of radius 300 m . By how much the outer rail be raised with respect to the inner rail for a speed of 54 km h^{-1} ? The distance between the rails is 1.5 m . Take $g = 10 \text{ m s}^{-2}$.

Solution: Speed of the train

$$= \frac{54 \times 1000}{60 \times 60} = 15 \text{ m s}^{-1}$$

Radius of the curve = 300 m

Let the outer rail be raised to a height h with respect to the inner rail so that the angle of banking is θ (see Fig. 4.10)

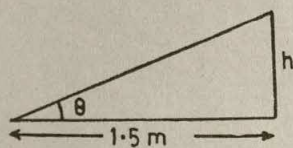


Fig. 4.10

Then $\tan \theta = \frac{h}{1.5} = \frac{v^2}{rg}$

$$h = \frac{1.5 \times 15 \times 15}{300 \times 10}$$

$$= 0.1125 \text{ m} = 11.25 \text{ cm}$$

Example 5: A string of length 1 m is fixed at one end and carries a mass of 100 g at the other end. This string makes $2/\pi$ revolutions per second around a vertical axis passing through its second end. Calculate (a) the angle of inclination of the string with the vertical, (b) the tension in the string and (c) the linear velocity of the mass. (IIT 1976)

Solution: Let T be the tension in the string. Figure 4.11 shows the forces acting on the system.

Tension T can be resolved into two mutually perpendicular components. The horizontal component $T \sin \theta$ provides the centripetal force for the circular motion. The vertical component $T \cos \theta$ is balanced by the weight of the mass.

$$T \cos \theta = mg$$

and $T \sin \theta = m \omega^2 r$, but $r = L \sin \theta$

$$\therefore T \sin \theta = m \omega^2 L \sin \theta$$

or $T = m \omega^2$

$$m = 100 \text{ g} = 0.1 \text{ kg}, L = 1 \text{ m}, n = \frac{2}{\pi} \text{ rev s}^{-1}$$

$$(a) \quad mg = T \cos \theta = m \omega^2 \cos \theta$$

$$\cos \theta = \frac{g}{\omega^2} = \frac{g}{4\pi^2 n^2} = \frac{9.8}{16}$$

$$= 0.6125$$

$$\theta = \cos^{-1}(0.6125)$$

$$= 52.23^\circ = 52^\circ 14'$$

$$(b) \quad T = m \omega^2 = 0.1 \times 4\pi^2 n^2$$

$$= 0.1 \times 4\pi^2 \times \frac{4}{\pi^2}$$

$$= 1.6 \text{ N}$$

(c) Linear velocity

$$V = \omega r$$

$$= 2\pi n L \sin \theta$$

$$= 2\pi \times \frac{2}{\pi} \times 1 \times \sqrt{1 - \cos^2 \theta} = 4 \sqrt{1 - (0.6125)^2}$$

$$= 3.162 \text{ m s}^{-1}$$

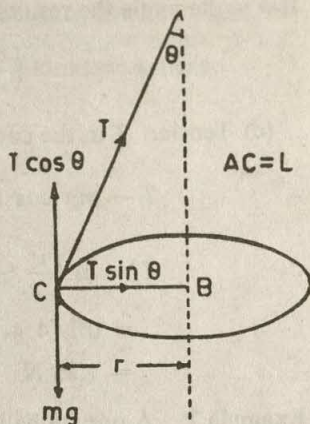


Fig. 4.11

Example 6: A small body of mass $m = 0.1 \text{ kg}$ swings in a vertical circle at the end of a cord of length 1 m . If its speed is 2 m s^{-1} when the cord makes an angle $\theta = 30^\circ$ with the vertical, find

(a) the radial and tangential components of its acceleration at this instant,

(b) the magnitude and direction of the resultant acceleration and

(c) the tension in the cord.

Solution: Force mg can be resolved into two rectangular components as shown in Fig. 4.12.

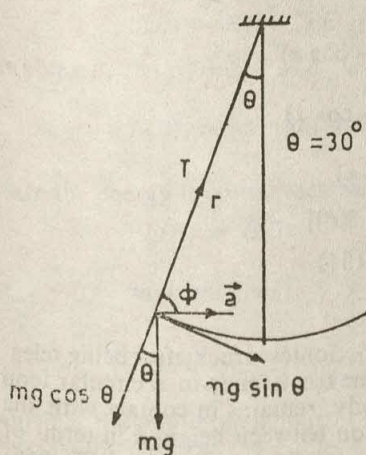


Fig. 4.12

(a) The radial and tangential components of acceleration respectively are

$$a_r = \frac{v^2}{r}$$

$$= \frac{2 \times 2}{1}$$

$$= 4.0 \text{ m s}^{-2}$$

$$a_t = \frac{mg \sin \theta}{m} = g \sin \theta$$

$$= 9.8 \times 0.5 = 4.9 \text{ m s}^{-2}$$

(b) The magnitude of the resultant acceleration is

$$a = \sqrt{a_r^2 + a_t^2}$$

$$= \sqrt{(4.0)^2 + (4.9)^2}$$

$$= 6.33 \text{ m s}^{-2}$$

If ϕ is the angle the resultant makes with the radius at P ,

$$\text{then } \phi = \tan^{-1} \left(\frac{a_t}{a_r} \right) = 50.8^\circ = 50^\circ 48'$$

(c) Tension T in the cord is given by

$$T - mg \cos \theta = \frac{mv^2}{r}$$

$$\begin{aligned} \therefore T &= m \left(\frac{v^2}{r} + g \cos \theta \right) \\ &= 0.1 (4 + 9.8 \times 0.866) \\ &= 1.25 \text{ N} \end{aligned}$$

Example 7: A one kg ball suspended by a thread deflects through an angle of 30° . Find the tension in the thread at the moment the weight passes through the equilibrium position.

Solution: At the moment the weight passes through the equilibrium position, the tension T in thread is given by (see Fig. 4.13)

$$T = mg + \frac{mv^2}{l}$$

where m is the mass of the ball and v its speed.

At position B , the potential energy of the ball is mgh , which is converted into kinetic energy when the ball reaches A .

$$\therefore mgh = \frac{1}{2} mv^2$$

(conservation of energy)

or

$$v = \sqrt{2gh}$$

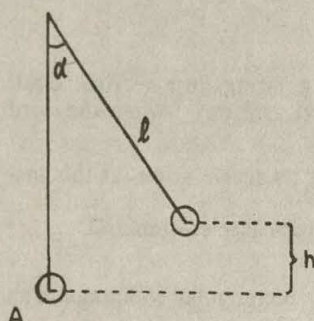


Fig. 4.13

But

$$h = l - l \cos \alpha = l (1 - \cos \alpha)$$

Hence

$$\frac{mv^2}{l} = \frac{m}{l} (2gh) = 2mg (1 - \cos \alpha)$$

$\therefore$

$$\begin{aligned} T &= mg + 2mg (1 - \cos \alpha) \\ &= mg [1 + 2 (1 - \cos 30^\circ)] \\ &= 1 \times 9.8 (1 + 2 \times 0.134) \\ &= 12.4 \text{ N} \end{aligned}$$

Example 8: A small body slides down a frictionless track after being released from rest from a point at a height h . The track ends in a circular loop of diameter D in the vertical plane. The body remains in contact with the loop as it moves along it. Establish a relation between height h in terms of the diameter D of the loop. (IIT 1966)

Solution: The body is released at point P from a height h above the base of the circle. (Fig. 4.14).

At any point on the track, the forces acting on the body are its weight mg and the normal reaction F due to the track. At the highest point on the loop, both mg and F point towards the centre C . Thus we must have

$$F + mg = \frac{mv^2}{r}$$

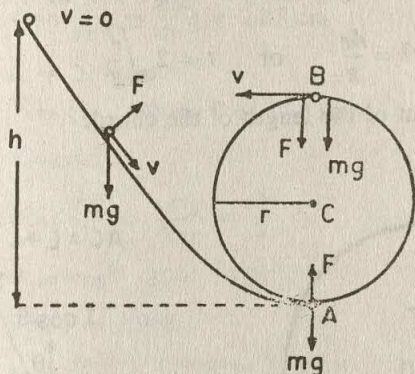


Fig. 4.14

where r is the radius of the track. The minimum velocity of the body at B , if it is to describe a circle, must correspond to $F = 0$ or

$$mg = \frac{mv^2}{r} \quad \text{or} \quad v = \sqrt{gr}$$

If the velocity is less than $\sqrt{gr}$, the downward pull of the weight is greater than the required centripetal force, and the body will separate from the circle before reaching point B .

To obtain the corresponding height h , consider the total energy at point P .

$$E(P) = KE + PE = 0 + mgh \quad (\because v = 0)$$

At point B , $h = 2r$ and $v^2 = gr$

$$E(B) = \frac{1}{2}m(gr) + mg(2r) = \frac{5}{2}mgr$$

Since the energy is conserved, we have

$$E(P) = E(B)$$

$$\text{or} \quad mgh = \frac{5}{2}mgr$$

$$h = \frac{5}{2}r = \frac{5}{4}D$$

which is the minimum height of the starting point of the body if it is to successfully describe the circle.

Example 9: A particle slides from rest from the highest point of a vertical circle of radius r along a smooth chord. Show that the time of descent is the same irrespective of the chord chosen and find it in terms of g and r .

Solution: If AB is the chord along which the particle slides, the component of gravitational acceleration along $AB = g \cos \theta$, as shown in Fig. 4.15.

If r is the radius of the circle, the length of the chord is $2r \cos \theta$. Therefore

$$2r \cos \theta = \left(\frac{1}{2} g \cos \theta\right) t^2 \quad [\because s = ut + \frac{1}{2} at^2]$$

Hence
$$t^2 = \frac{4r}{g} \quad \text{or} \quad t = 2\sqrt{\frac{r}{g}}$$

which is independent of the length of the chord.

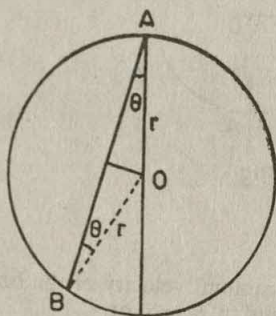


Fig. 4.15

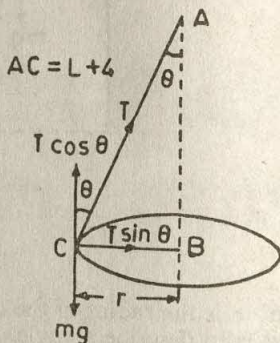


Fig. 4.16

Example 10: A spring which obeys Hooke's law is found to extend by 1 cm when a mass is hung on it. It extends further by 3 cm when the attached mass is moving uniformly in a horizontal circle making two revolutions per second. Find the inclination of the spring to the vertical, the length of the unstretched spring and the radius of the circle.

Solution: According to Hooke's law, the stretching force F is given by

$$F = kx$$

where x is the extension and k is the force constant of the spring. A force mg stretches the spring by 1 cm. When the mass is describing the horizontal circle, the total extension in the spring is 1 cm + 3 cm = 4 cm. Hence the force when the mass is describing the circle must be $4 mg$.

Let T be the tension in the spring. It can be resolved into horizontal and vertical components. The vertical component $T \cos \theta$ must balance the weight mg because there is no acceleration in this direction (see Fig. 4.16). The horizontal component $T \sin \theta$ provides the necessary centripetal force for the circular motion.

$$\therefore T \sin \theta = \frac{mv^2}{r} = m\omega^2 r$$

where r is the radius of the circle. Now

$$\omega = 2\pi n = 4\pi \text{ rad s}^{-1}$$

Let L be the length of the unstretched spring. Then

$$r = (L + 4) \sin \theta$$

$$T \sin \theta = m \times 16\pi^2 \times (L + 4) \sin \theta$$

or
$$T = 16\pi^2 m (L + 4) = 4 mg$$

$\therefore L + 4 = \frac{g}{4\pi^2} = \frac{980}{4\pi^2} = 24.8 \text{ cm}$

$$L = 20.8 \text{ cm}$$

$$r = (L + 4) \sin \theta = 24.8 \sin \theta$$

$$T \cos \theta = mg$$

$$\cos \theta = \frac{mg}{T} = \frac{mg}{4mg} = \frac{1}{4}$$

$$\theta = \cos^{-1} \left(\frac{1}{4} \right) = 75.52^\circ$$

$$\sin (75.52^\circ) = 0.968$$

$\therefore r = 24.8 \times 0.968 = 24.0 \text{ cm}$

Example 11: A mass of 2.9 kg is suspended from a string of length 50 cm and is at rest. Another body of mass 100 g moving horizontally with a velocity of 150 m s^{-1} , strikes and sticks to it. (a) What is the tension in the string when it makes an angle of 60° with the vertical? (b) Will it complete a circle?

Solution: The total mass after the two masses join together is

$$2.9 + 0.100 = 3.0 \text{ kg}$$

Initial momentum $= 0.100 \times 150 = 15 \text{ N s}$

If v is the velocity of the composite mass, its momentum $= 3.0 v \text{ N s}$

Since momentum is conserved

$$3v = 15$$

$\therefore v = 5 \text{ m s}^{-1}$

(a) When the body hits the stationary mass, the composite mass starts executing circular motion in a vertical plane. The situation when the string makes an angle 60° with the vertical is shown in Fig. 4.17.

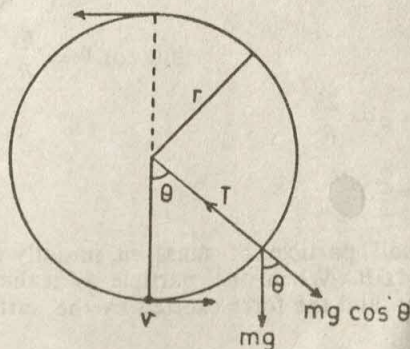


Fig. 4.17

The force mg on the mass can be resolved into two components. If T is the tension in the string, we have

$$\begin{aligned} T &= \frac{mv^2}{r} - mg \cos \theta \\ &= m \left(\frac{v^2}{r} - g \cos \theta \right) \\ &= 3 \left(\frac{25}{0.5} - 9.8 \times \frac{1}{2} \right) \\ &= 3 (50 - 4.9) = 135.3 \text{ N} \end{aligned}$$

(b) At the lowest point, the velocity is 5 m s^{-1} . In order to complete the circle, the minimum velocity at the bottom of the circle is given by

$$v_1 = \sqrt{5rg} = \sqrt{5 \times 0.5 \times 9.8} = 4.95 \text{ m s}^{-1}$$

which is less than 5 m s^{-1} . Hence, the body will complete the circle.

Example 12: A boy is seated on the top of a hemispherical mound of ice. He is given a very small push and starts sliding down the ice. Show that he leaves the ice at a point whose height is $2R/3$ if the ice is frictionless.

Solution: As the boy is given a small push, his initial speed can be taken to be zero. Let v be the speed with which he leaves the ice. The energy conservation gives,

$$\frac{1}{2} mv^2 = mg(R - h)$$

The forces acting on the boy are the weight mg and the normal reaction F , as shown in Fig. 4.18.

Resolving mg into its rectangular components, we find that at any point P ,

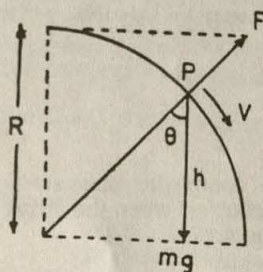


Fig. 4.18

$$mg \cos \theta - F = \frac{mv^2}{R}$$

Combining the above two equations, we get

$$mg \cos \theta - F = 2mg \left(\frac{R - h}{R} \right)$$

As the boy leaves the mound, the normal force vanishes. Thus putting $F = 0$, we have

$$\cos \theta = 2 \left(\frac{R - h}{R} \right),$$

$$\text{but } \cos \theta = \frac{h}{R}$$

$$\therefore \frac{h}{R} = 2 - \frac{2h}{R}$$

$$\text{or } h = \frac{2}{3} R$$

Example 13: A small particle of mass m , initially at A , slides down a frictionless surface ADB . When the particle is at the point C , show that the angular velocity and the force exerted by the surface are: (see Fig. 4.19)

$$\omega = \sqrt{2g \sin \alpha / r}, \quad F = 3mg \sin \alpha$$

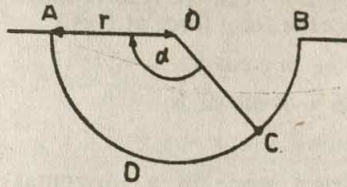


Fig. 4.19

Solution: At the point A, the potential energy of the particle is mgr where r is the radius of the semi-circular surface ADB. Draw a line OD perpendicular to AOB. Then angle $AOD = \pi/2$ and we call $\angle DOC = \beta$. (See Fig. 4.20).

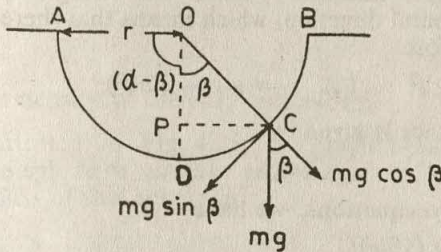


Fig. 4.20

As the particle slides down, some of the potential energy is converted into kinetic energy

At the point C, the total energy is

$$KE + PE = mg \times PD + \frac{1}{2} mv^2$$

where v is the linear velocity at this point.

The energy conservation gives us

$$\begin{aligned} mgr &= mg \cdot PD + \frac{1}{2} m r^2 \omega^2 \\ &= mg (OD - OP) + \frac{1}{2} m r^2 \omega^2 \\ &= mg (r - r \cos \beta) + \frac{1}{2} m r^2 \omega^2 \end{aligned}$$

$$\therefore gr = gr (1 - \cos \beta) + \frac{1}{2} r^2 \omega^2$$

$$\begin{aligned} \text{or } \omega^2 r^2 &= 2gr - 2gr (1 - \cos \beta) \\ &= 2gr \cos \beta \end{aligned}$$

$$\begin{aligned} \omega^2 r^2 &= 2gr \sin (\beta + \pi/2) \\ &= 2gr \sin (\beta + \alpha - \beta) \\ &= 2gr \sin \alpha \end{aligned}$$

$$\therefore \omega = \sqrt{\frac{2g \sin \alpha}{r}}$$

$$\text{Also } m \omega^2 r = 2mg \sin \alpha$$

Now, at point C , the weight mg can be resolved into two components $mg \cos \beta$ and $mg \sin \beta$. Hence the total force at C is

$$\begin{aligned} F &= m \omega^2 r + mg \cos \beta \\ &= 2mg \sin \alpha + mg \sin \alpha \\ F &= 3mg \sin \alpha \end{aligned}$$

Example 14: A boy whirls a stone in a horizontal circle 2 m above the ground by means of a string 1.5 m long. The string breaks and the stone flies off horizontally, striking the ground 10 m away. What is the centripetal acceleration during circular motion?

Solution: Centripetal acceleration

$$a_c = \frac{v^2}{r}$$

To obtain a_c we need to calculate v . When the string breaks, the stone is projected in a horizontal direction, which means that there is no initial vertical velocity. Therefore

$$2 = \frac{1}{2} g t^2 \quad (\because s = ut + \frac{1}{2} a t^2)$$

The horizontal distance is given by

$$10 = v t$$

Combining these two equations, we have

$$2 = \frac{1}{2} g (10/v)^2$$

$$\text{or} \quad v^2 = \frac{1}{4} \times 9.8 \times 100 = 245$$

$$\therefore a_c = \frac{v^2}{r} = \frac{245}{1.5} = 163.3 \text{ m s}^{-2}$$

Example 15: A rocket is launched vertically from the surface of the earth with an initial speed of 10 km s^{-1} . How far above the surface of the earth would it go? Radius of earth = 6400 km. Ignore the atmospheric resistance.

Solution: On the surface of the earth, the total energy is

$$KE + PE = \frac{1}{2} m v^2 - G \frac{mM}{R}$$

At the highest point

$$v = 0; PE = -G \frac{mM}{(R + h)}$$

where h is the maximum height attained.

From the law of conservation of energy we have

$$\frac{1}{2} m v^2 - G \frac{mM}{R} = -G \frac{mM}{(R + h)}$$

$$\frac{1}{2} v^2 = G \frac{M}{R} \left(1 - \frac{R}{R + h} \right)$$

$$= \left(G \frac{M}{R^2} \right) \frac{hR}{R + h}$$

$$= g \frac{Rh}{R + h}$$

where g is the acceleration due to gravity on the surface of the earth.

Therefore $\frac{R+h}{h} = \frac{2gR}{v^2}$

or
$$h = R \left(\frac{2gR}{v^2} - 1 \right)^{-1}$$

$$= R \left(\frac{2 \times 9.8 \times 6.4 \times 10^6}{10^8} - 1 \right)^{-1}$$

$$h = 3.93 R$$

$$= 3.93 \times 6.4 \times 10^3 \text{ km}$$

$$h = 2.5 \times 10^4 \text{ km}$$

Example 16: Suppose a hole were drilled completely through the earth along a diameter. Show that the force on a mass m at a distance r from the centre of the earth is

$$F = - \frac{mgr}{R}$$

assuming that the density of the earth is uniform.

Solution: As illustrated in Fig. 4.21, the gravitational force on mass m arises only from a sphere of radius r enclosing the mass. The mass of this sphere is

$$M' = \frac{4}{3} \pi r^3 \rho$$

where ρ is the density of the earth. Hence the force on the mass m is

$$F = - G \frac{M' m}{r^2}$$

The negative sign indicates that the force is attractive. This force is directed towards the centre of the earth. Rewriting the above equation, we have

$$F = - \left(\frac{G M}{R^2} \right) \frac{M' m R^2}{r^2}$$

where M is the total mass of the earth.

But
$$G \frac{M}{R^2} = g, \quad \frac{M'}{M} = \frac{(4/3) \pi r^3 \rho}{(4/3) \pi R^3 \rho} = \frac{r^3}{R^3}$$

$\therefore$
$$F = - g \frac{r^3}{R^3} \frac{m R^2}{r^2} = - m g \frac{r}{R}$$

Example 17: Compute the mass of a planet that has a satellite whose time period is T and orbital radius is r .

Solution: Suppose that a satellite of mass m describes a circular orbit around a planet of mass M . The force of attraction between the planet and its satellite is

$$F = G \frac{mM}{r^2}$$

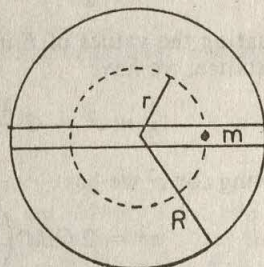


Fig. 4.21

This force must be mass times the centripetal acceleration, i.e.

$$\frac{v^2}{r} = \omega^2 r$$

Thus, $m \omega^2 r = \frac{4\pi^2 mr}{T^2} = G \frac{mM}{r^2}$

or $M = \frac{4\pi^2 r^3}{GT^2}$

Example 18: A body is released at a distance r from the centre of the earth. Compute the velocity of the body when it strikes the surface of the earth.

Solution: Since the initial velocity of the body is zero, its total energy is

$$E = -G \frac{Mm}{r}$$

where m is the mass of the body and M is the mass of the earth.

When the body reaches the earth's surface, its velocity is v and its distance from the centre of the earth is the earth's radius R . Therefore,

$$E = \frac{1}{2} m v^2 - G \frac{Mm}{R}$$

Equating the values of E in the above two equations and neglecting the air resistance, we have

$$\frac{1}{2} m v^2 - G \frac{Mm}{R} = -G \frac{mM}{r}$$

Solving for v^2 we have

$$v^2 = 2GM \left(\frac{1}{R} - \frac{1}{r} \right)$$

Also $g = GM/R^2$ (see Eq. 4.35)

$$\therefore v^2 = 2R^2 g \left(\frac{1}{R} - \frac{1}{r} \right) \quad \text{or} \quad v = R \left[2g \left(\frac{1}{R} - \frac{1}{r} \right) \right]^{1/2}$$

This expression can also be used for finding the distance r reached by a body thrown vertically with a velocity v from the surface of the earth.

If the body is released at a great height so that $1/r$ is negligible compared with $1/R$, we get

$$v_{\infty} = \sqrt{2Rg}$$

which is the expression for the escape velocity.

Example 19: A satellite is to be put into an orbit 600 km above the surface of the earth. If its vertical velocity after launching is 2400 ms^{-1} at this height, calculate the magnitude and direction of the impulse required to put the satellite directly into orbit. The mass of the satellite is 60 kg and the radius of the earth is 6400 km. Assume that $g = 10 \text{ ms}^{-2}$.

Solution: If v is the initial velocity required for an orbit of radius r , then

$$\frac{mv^2}{r} = G \frac{mM}{r^2} = \left(\frac{GM}{R^2} \right) \frac{R^2}{r^2} m = g \frac{R^2}{r^2} m$$

$$\therefore v^2 = gR^2/r$$

Here $r = 600 + 6400 = 7000 \text{ km}$

$$v^2 = \frac{9.8 \times (6400 \times 10^3)^2}{7000 \times 10^3}$$

$$\therefore v = 7572.6 \text{ m s}^{-1}$$

Referring to Fig. 4.22, the vertical component of momentum at this height is given by

$$\begin{aligned} U_y &= mv = 60 \times 2400 \\ &= 1.44 \times 10^5 \text{ kg ms}^{-1} \end{aligned}$$

The horizontal component of momentum is

$$\begin{aligned} U_x &= 60 \times 7572.6 \\ &= 4.5436 \times 10^5 \text{ kg ms}^{-1} \end{aligned}$$

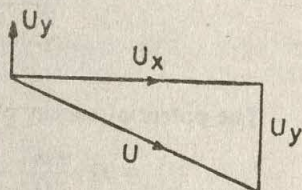


Fig. 4.22

The resultant of the two is the impulse required. Thus

$$\begin{aligned} U &= \sqrt{U_x^2 + U_y^2} \\ &= [(1.44)^2 + (4.5436)^2]^{1/2} \times 10^5 \\ &= 4.77 \times 10^5 \text{ kg ms}^{-1} \end{aligned}$$

The angle θ made by the total impulse with the horizontal is given by

$$\tan \theta = \frac{U_y}{U_x} = 0.317$$

$$\theta = 17.6^\circ$$

Example 20: Two equal masses of 6.40 kg are separated by a distance of 0.16 m. A small body is released from a point P equidistant from the two masses and at a distance 0.06 m from the line joining them (see Fig. 4.23). (a) Calculate the velocity of this body when it passes through Q . (b) Compute the acceleration of this body at P and Q if its mass is 0.1 kg.

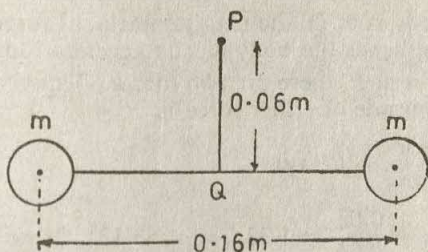


Fig. 4.23

Solution: Let the mass of the body be m . Then the total energy of this mass at P is equal to its potential energy. Therefore

$$E = -G \frac{m m'}{r} - G \frac{m m'}{r} = -2G \frac{m m'}{r}$$

where r is the distance of the point P from either mass.

From Fig. 4.24, the distance r is given by

$$r = \sqrt{(0.06)^2 + (0.08)^2} = 0.1 \text{ m}$$

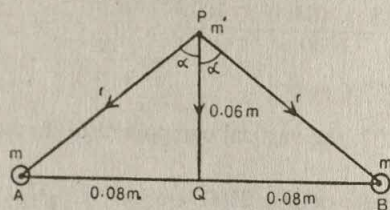


Fig. 4.24

The potential energy of the mass m' at point Q is

$$-G \frac{mm'}{AQ} - G \frac{mm'}{BQ} = -2G \frac{mm'}{0.08}$$

If v is the velocity of this body at Q , its kinetic energy is

$$\frac{1}{2} m' v^2$$

(a) Total energy at Q must be equal to the energy at P

$$\begin{aligned} \therefore \quad \frac{1}{2} m' v^2 - 2G \frac{mm'}{0.08} &= -2G \frac{mm'}{r} \\ &= -2G \frac{mm'}{0.1} \end{aligned}$$

$$\therefore \quad \frac{1}{2} m' v^2 = 2G mm' \left(\frac{1}{0.08} - \frac{1}{0.1} \right)$$

$$\begin{aligned} \text{or} \quad v^2 &= 4G m (12.5 - 10) = 10 Gm \\ &= 10 \times 6.67 \times 10^{-11} \times 5.4 \end{aligned}$$

$$v = \sqrt{6.67 \times 6.4 \times 10^{-10}} = 6.53 \times 10^{-5} \text{ ms}^{-1}$$

(b) When the body is at Q , the net gravitational force due to the masses at A and B is zero; hence the body has no acceleration at this point.

When the body is at P , there are two forces of equal magnitude along PA and PB . The magnitude of either force is

$$G \frac{mm'}{r^2} = f \text{ (say)}$$

$$\tan \alpha = \frac{0.08}{0.06} = 1.333, \alpha = 53.13^\circ, 2\alpha = 106.26^\circ$$

The resultant force F on m is along PQ and its magnitude is given by

$$F = \sqrt{f^2 + f^2 - 2f^2 \cos 2\alpha}$$

$$= \sqrt{2f^2 (1 - \cos 106.26^\circ)}$$

$$= f \sqrt{2(1 - \cos 73.74^\circ)} = 1.2 f$$

$$\therefore \quad F = \frac{1.2 \times 6.67 \times 10^{-11} \times 0.1 \times 6.4}{0.01} = 5.12 \times 10^{-9} \text{ N}$$

The acceleration of the body at P

$$\begin{aligned} &= F/m' \\ &= \frac{5.12 \times 10^{-9}}{0.1} \\ &= 5.12 \times 10^{-8} \text{ m s}^{-2} \end{aligned}$$

Example 21: What would be the speed of rotation of the earth in order that a body on the equator has no weight? Determine the apparent weights of the bodies situated at a latitude of 60° and at the poles. The radius of the earth = 6400 km and $g = 9.8 \text{ m s}^{-2}$.

Solution: The body will become weightless if the gravitational force mg on it is entirely used up in providing the centripetal acceleration for the rotation of the earth. Then

$$mg = \frac{mv^2}{R} = m\omega^2 R$$

$$\begin{aligned} \therefore \omega^2 &= \frac{g}{R} = \frac{9.8}{6400 \times 10^3} \\ \omega &= 1.237 \times 10^{-3} \text{ rad s}^{-1} \end{aligned}$$

If the earth rotates at this speed, the bodies on the equator will have no weight.

At a latitude ϕ , the apparent weight W_A is given by

$$W_A = mg \left(1 - \frac{\omega^2 R}{g} \cos^2 \phi \right)$$

Here, however, $g = \omega^2 R$

Therefore, $W_A = mg (1 - \cos^2 \phi)$

When $\phi = 60^\circ$, $\cos \phi = \frac{1}{2}$

$$\therefore W_A = mg \left(1 - \frac{1}{4} \right) = \frac{3}{4} \times \text{true weight}$$

At poles, $\phi = \pi/2$

$$\therefore W_A = mg = \text{true weight}$$

Thus, a body situated on the poles remains unaffected, whatever the speed of rotation of the earth.

Example 22: Calculate the minimum energy required to launch a 250 kg satellite from earth's surface in a circular orbit at an altitude of $2R$, where R is the radius of the earth and is equal to 6400 km.

Solution: The total energy of a satellite of mass m in a circular orbit of radius r is

$$E = \frac{1}{2} mv^2 - G \frac{mM}{r}$$

where r is measured from the centre of the earth.

From Eq. (4.44) we have

$$E = G \frac{mM}{2r} - G \frac{mM}{r} = -G \frac{mM}{2r}$$

Here, $r = 2R + R = 3R$

$$\therefore E = -G \frac{mM}{6R}$$

The potential energy on the surface of the earth $= -G \frac{Mm}{R}$

$$\text{Minimum energy required} = -\frac{1}{6} G \frac{mM}{R} - \left(-G \frac{Mm}{R} \right)$$

$$= \frac{5}{6} G \frac{mM}{R}$$

$$= \frac{5}{6} mgR$$

$$= \frac{5}{6} \times 250 \times 9.8 \times 6.4 \times 10^6$$

$$= 1.3 \times 10^{10} \text{ J}$$

Example 23: An artificial satellite is describing an equatorial orbit at 1600 km above the surface of the earth. Calculate its orbital speed and the period of revolution. If the satellite is travelling in the same direction as the rotation of the earth (i.e. from west to east), calculate the interval between two successive times at which it will appear vertically overhead to an observer at a fixed point on the equator. Radius of earth = 6400 km.

Solution: Radius of the orbit $= 6400 + 1600 = 8000 \text{ km}$

Period of the satellite is

$$T = 2\pi \left(\frac{r^3}{gR^2} \right)^{1/2} \quad (\text{see Eq. 4.48})$$

$$T = \left[\frac{4\pi^2 \times (8000 \times 10^3)^3}{9.8 \times (6400 \times 10^3)^2} \right]^{1/2} = 7096 \text{ s}$$

Orbital speed $v = \sqrt{G/Mr}$ (see Eq. 4.45)

$$v = \left(\frac{gR^2}{r} \right)^{1/2} = 6400 \times 10^3 \sqrt{\frac{9.8}{8000 \times 10^3}} = 7083.5 \text{ ms}^{-1}$$

Let t be the time interval between two successive moments at which the satellite is overhead to an observer at a fixed position on the equator. During time t , let

α_1 = angle through which the earth rotates,

α_2 = angle through which the satellite revolves

Then
$$\alpha_2 = 2\pi + \alpha_1 \quad \frac{\alpha_2}{t} = \frac{2\pi}{t} + \frac{\alpha_1}{t}$$

$$\frac{\alpha_2}{t} = \omega_2, \text{ angular speed of the satellite}$$

and
$$\frac{\alpha_1}{t} = \omega_1, \text{ the angular speed of the earth.}$$

$$\therefore \omega_2 - \omega_1 = \frac{2\pi}{t}$$

or
$$\frac{1}{t} = \frac{\omega_2}{2\pi} - \frac{\omega_1}{2\pi} = \frac{1}{T_2} - \frac{1}{T_1}$$

where T_2 is the period of the satellite.

$$T_2 = 7096 \text{ s}$$

$$T_1 = \text{period of earth's rotation}$$

$$= 24 \times 60 \times 60 \text{ s} = 8.64 \times 10^4 \text{ s}$$

$$t = \frac{T_1 T_2}{T_1 - T_2} = \frac{7096 \times 8.64 \times 10^4}{7.93 \times 10^4}$$

$$= 7731 \text{ s} \approx 129 \text{ min.}$$

4.6 TRUE-FALSE STATEMENTS WITH REASONS

Given below are a few statements. We have to decide whether they are true or false, giving the reason for our answer in each case.

1. Drops of water or mud fly off a rapidly rotating wheel in a radial direction.
2. The outside 'horses' on a merry-go-round get more acceleration than the inside ones.
3. A simple pendulum with a bob of mass m swings with an angular amplitude of 40° . When its angular displacement is 20° , the tension in the string is greater than $mg \cos 20^\circ$. (IIT 1984)
4. It is possible to put an artificial satellite into orbit in such a way that it will always remain directly over New Delhi. (IIT 1984)
5. The gravitational attraction of the moon is much less than that of the earth.
6. When water in a bucket is whirled fast overhead, the water does not fall out at the top of the circular path because the centripetal force on water is less than the weight of water.
7. If the earth stops rotating about its axis, the value of g at equator will increase by $\omega^2 R$ where R is earth's radius and ω its velocity of rotation.
8. An earth satellite continues to move in an orbit due to the centripetal force provided by the burning of fuel in its engine.
9. Two identical trains are moving on rails along the equator on the earth in opposite directions with the same speed. They will exert the same pressure on the rails.

ANSWERS

1. *False*. Drops of water or mud fly off in a direction tangential to any point on the wheel because that is the direction of the velocity at every point in circular motion.
2. *True*. This is so because the linear acceleration is directly proportional to the distance from the centre (see Eq. 4.5).
3. *True*. The tension T in the string is given by

$$T = mg \cos \theta + \frac{mv^2}{r}$$

where r is the length of the string making an angle θ with the vertical and v is the velocity with which the bob moves. For $\theta = 40^\circ$, $v = 0$. Tension at 20° is given by

$$T(20^\circ) = mg \cos 20^\circ + \frac{mv^2}{r}$$

$$T(20^\circ) - mg \cos 20^\circ = \frac{mv^2}{r} > 0$$

$$\therefore T(20^\circ) > mg \cos 20^\circ$$

4. *False* To put an artificial satellite into an orbit such that it remains directly overhead at a fixed location, its time period should be related to that of the earth in the equatorial plane; a geostationary satellite cannot be used for New Delhi.
5. *True* The gravitational force on a given mass depends upon M/r^2 which is smaller for moon (see the Table 4.1).
6. *True* The water does not fall from the bucket when it is at the top of its circular path because the centripetal force on the water is less than its weight at this position.
7. *True* $g = g_0 - \omega^2 R$ at the equator where g_0 is the value at the poles. If $\omega = 0$, the value at the equator will increase by $\omega^2 R$.
8. *False* The centripetal force is provided by the gravitational force. Fuel is needed only at the time of launching of the satellite to provide the energy to overcome the gravitational pull of the earth.
9. *False* Due to the rotation of the earth, the effective weight of a body of mass m is equal to $m(g - \omega^2 r)$. Let v be the effective velocity of the body located at a distance r from the centre of the earth. The velocity of the earth will be added to the velocity of one of the trains moving in the same direction and it will be subtracted from the velocity of the second train. The effective value of velocity v therefore, will be different. This results in a different value for the weight $m(g - \omega^2 r)$ and hence the effective value of pressure on the rails will be different.

4.7 MULTIPLE CHOICE QUESTIONS

Choose the correct answer from the given alternatives.

1. When a fan turns through an integral number of revolutions, the angle in radians is an even multiple of

(a) $\pi/2$	(b) π
(c) 2π	(d) $3\pi/2$
2. In the equation for the linear acceleration $a = \alpha r$, where α is uniform angular acceleration, the radian measure for α

(a) must be used	(b) may be used
(c) cannot be used	(d) none of these
3. In applying the equation for motion with uniform angular acceleration, $\omega = \omega_0 + \alpha t$, the radian measure

(a) must be used for both ω and α	(b) may be used for both ω and α
(c) may be used for ω but not for α	(d) cannot be used for both α and ω
4. Centrifugal force is an inertial force when considered by

(a) an observer at the centre of the circular motion	(b) an outside observer
(c) an observer who is moving with the particle that experiences the force	(d) none of the above
5. On a railway curve the outside rail is laid higher than the inside one so the resultant force exerted on the wheels of the railcar by the tops of the rails will

(a) have a horizontal inward component	(b) be vertical
(c) equilibrate the centripetal force	(d) be decreased

6. Mars has about $1/10$ th as much mass as the earth and half as great a diameter. The acceleration of a falling body on Mars is about
- (a) 9.8 m s^{-2} (b) 1.96 m s^{-2}
(c) 3.92 m s^{-2} (d) 4.9 m s^{-2}
7. If M is the mass of the earth and R its radius, the ratio of the gravitational acceleration and the gravitational constant is
- (a) R^2/M (b) M/R^2
(c) MR^2 (d) M/r
8. An object weighs W newton on earth. It is suspended from the lower end of a spring balance whose upper end is fixed to the ceiling of a space capsule in a stable orbit around the earth. The reading of the spring balance will be
- (a) W (b) less than W
(c) more than W (d) zero
9. The escape velocity from the earth is V_e . What is the escape velocity from a planet whose mass and radius are twice those of the earth?
- (a) V_e (c) $2 V_e$
(b) $4 V_e$ (d) $16 V_e$
10. In order to cause a moving body to pursue a circular path, it is necessary to apply
- (a) inertial force (b) gravitational force
(c) centripetal force (d) centrifugal force
11. A body travelling in a circle at a constant speed
- (a) has a constant velocity
(b) is not accelerated
(c) has an inward radial acceleration
(d) has an outward radial acceleration
12. A 2 kg stone at the end of a string 1 m long is whirled in a vertical circle at a constant speed. The speed of the stone is 4 ms^{-1} . The tension in the string will be 52 N when the stone is
- (a) at the top of the circle
(b) halfway down
(c) at the bottom of the circle
(d) none of the above.
13. A hole is drilled through the earth along a diameter and a stone is dropped into it. When the stone is at the centre of the earth, it has a finite
- (a) mass (b) weight
(c) acceleration (d) potential energy
14. If g is the acceleration due to gravity on the earth's surface, the gain in the potential energy of an object of mass m raised from the earth's surface to a height equal to the radius R of the earth, is
- (a) $1/2 mgR$ (b) $2 mgR$
(c) mgR (d) $1/4 mgR$
15. The escape velocity for a body projected vertically upward from the surface of the earth is 11.2 km s^{-1} . If the body is projected in a direction making an angle 45° with the vertical, the escape velocity will be
- (a) $11.2/\sqrt{2} \text{ km s}^{-1}$ (b) $11.2 \times \sqrt{2} \text{ km s}^{-1}$
(c) $11.2 \times 2 \text{ km s}^{-1}$ (d) 11.2 km s^{-1}

4.8 NUMERICAL EXERCISES

Unless otherwise stated, assume that $g = 9.8 \text{ m s}^{-2}$. If required, the values of other constants may be taken from Table 4.1?

1. A sphere of mass 200 g is attached to an inextensible string of length 130 cm whose upper end is fixed to the ceiling. The sphere is made to describe a horizontal circle of radius 50 cm .
(a) Calculate the time to complete one revolution
(b) What is the tension in the string? (IIT 1974)
2. A spaceman in training is rotated in a seat at the end of a horizontal rotating arm of length 5 m . If he can withstand acceleration up to $9g$, what is the maximum number of revolutions per second permissible? The acceleration of free fall (g) may be taken as 10 m s^{-2} .
3. A string 0.5 m long is used to whirl a 1 kg stone in a vertical circle at a uniform velocity of 5 m s^{-1} . What is the tension in the string
(a) when the stone is at the top of the circle?
(b) when the stone is at the bottom of the circle?
4. A particle of mass 100 g is suspended from the end of a weightless string of length 1 m and is allowed to swing in a vertical plane. The speed of the mass is 2 m s^{-1} when string makes an angle of $\theta = 60^\circ$ with the vertical.
Determine
(a) the tension in the string at $\theta = 60^\circ$
(b) the speed of the particle when it is in the lowest position. (IIT 1971)
5. A plane comes out of a power dive, turning upward in a curve whose centre of curvature is 1200 m above the plane. The speed of the plane is 300 m s^{-1} . Calculate the upward force on the 80 kg pilot of the plane. What will be the upward internal force on 0.08 kg sample of blood in the pilot's brain?
6. A cylindrical drum of radius 9.8 cm rotates about its vertical axis. A small body is in contact with the drum. The frequency of revolution is more than 200 rpm but falls at lower frequencies. Determine the coefficient of friction between the body and the surface of the drum.
7. A stone tied to a string 50 cm long rotates uniformly in a vertical plane. At what number of revolutions per second will the string break if it is known to do so under a load equal to ten times the weight of the stone?
8. A particle is placed inside a hemispherical bowl which rotates about its vertical axis with constant angular velocity ω . It is just prevented from sliding down when OP is inclined at 45° with the axis. (O is the centre of the sphere and P is the position of the particle). The radius of the bowl is $10\sqrt{2} \text{ m}$, coefficient of friction between the particle and the bowl is 0.5 . Find the magnitude of ω ($g = 10 \text{ m s}^{-2}$).

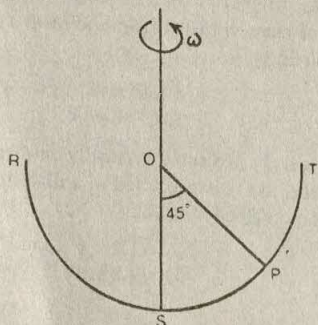


Fig. 4.25

9. A nail is located at a certain distance vertically below the point of suspension of a simple pendulum. The pendulum bob is released from a position where the string makes an angle 60° with the vertical. Calculate the distance of the nail from the point of suspension such that the bob will just perform revolutions with the nail as centre. Assume the length of the pendulum to be 1 m. (IIT 1975)
10. A body of mass 5 kg is attached to the end of a string and is at rest on a smooth horizontal table. The other end of the string passes through a small hole in the table and carries another weight of 0.5 kg. A length 1 m of the string is on the table. Find the number of revolutions per minute with which the body on the table has to revolve so as to balance the other weight.
11. Determine the internal gravitational potential energy of 8 particles, each of mass m , located at the vertices of a cube of side r .
12. An artificial satellite moving in a circular orbit around earth has period of revolution equal to that of its rotation about its own axis. Find its distance from earth's surface. Radius of earth = 6400 km.
13. Find the velocity acquired by a body in falling to the surface of earth from
 - (a) far away distance
 - (b) a point distant twice the earth's radius from its surface. $R = 6400$ km.
14. A stone of mass 0.5 kg is attached to a string 0.5 m long which will break if the tension in it exceeds 20 N. The stone is whirled in a vertical circle, the axis of rotation being 1 m above the ground. The angular speed is slowly increased until the string breaks. In what position is this break most likely to occur and at what angular speed? Where will the stone hit the ground?
15. A research satellite of mass 200 kg circles the earth in an orbit of average radius $3R/2$ where R is the radius of the earth. Assuming the gravitational pull on a mass 1 kg at the earth's surface to be 10 N, calculate the pull on the satellite.
16. Taking the earth to be a uniform sphere of radius 6400 km, and the value of g at the surface to be 10 m s^{-2} , calculate the total energy needed to raise a satellite of mass 2000 kg to a height of 800 km above the ground and to set it into a circular orbit at that altitude.
17. A highway has a curve with a bank of 10° , the radius of the curve being 100 m. What speed is the curve intended for?
18. The combined mass of a motorcycle and its rider is 100 kg. What is the necessary force of friction if the rider is to go around a curve of 80 m radius at 20 m s^{-1} ? If the coefficient of friction is 0.6, will the rider negotiate the curve successfully? At what angle should he lean over to avoid a fall?
19. In an ultracentrifuge a G -field of 300,000 times gravity is realized (this means that the acceleration is $300,000 g$). If the rotor is 10 cm in radius, what must be the rotational speed?
20. At what distance r from the centre of the moon is the point at which the strength of the resultant of the earth's and moon's gravitational field is equal to zero, if the earth's mass is assumed to be $\eta = 81$ times that of the moon and the distance between their centres, $n = 60$ times greater than the radius R of the earth?
21. A satellite revolving in a circular equatorial orbit of radius $r = 2 \times 10^4$ km from west to east appears over a certain point at the equator every $\tau = 11.6$ hours. Calculate M , the mass of the earth, the value of gravitational constant $G = 6.67 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}$.

MULTIPLE CHOICE QUESTIONS

- | | | | | |
|---------|---------|---------|---------|---------|
| 1. (c) | 2. (a) | 3. (b) | 4. (c) | 5. (a) |
| 6. (c) | 7. (b) | 8. (d) | 9. (a) | 10. (c) |
| 11. (c) | 12. (c) | 13. (a) | 14. (a) | 15. (d) |

NUMERICAL EXERCISES

1. (a) 2.2 s, (b) 2.12 N
2. 0.675 rev s⁻¹
3. (a) 40.2 N, (b) 59.8 N
4. (a) 0.89 N, (b) 3.71 m s⁻¹
5. 6784 N, 6.8 N
6. 0.228
7. 2.1 rev s⁻¹
8. $\omega = \frac{1}{\sqrt{3}} \text{ rad s}^{-1}$
9. 0.8 m
10. $n = 9.45 \text{ rpm}$
11. $-22.8 \text{ Gm}^2/r$
12. $3.594 \times 10^4 \text{ km}$
13. (a) $1.12 \times 10^4 \text{ m s}^{-1}$, (b) $9.145 \times 10^3 \text{ m s}^{-1}$
14. Vertically below suspension, 7.7 rad s⁻¹, 1.22 m away
15. 889 N
16. $7.1 \times 10^{10} \text{ J}$
17. 47 km h⁻¹
18. 588 N, the rider will be able to negotiate the turn, $\theta = 27^\circ$
19. $5.18 \times 10^4 \text{ rpm}$
20. $r = nR/(1 + \sqrt{\eta}) = 3.8 \times 10^4 \text{ km}$
21. $M = \left(\frac{4\pi^2 R^3}{GT^2} \right) \left(1 + \frac{T}{\tau} \right)^2 = 6 \times 10^{24} \text{ kg}$,
where T is the period of earth's rotation about its own axis

Collision and Rigid Body Rotation

5.1 REVIEW OF BASIC CONCEPTS

5.1.1 System of Particles A number of particles taken collectively constitute a system of particles. The choice of a system is not unique. Everything other than the system chosen is known as the surroundings. The choice as to what will constitute the system and what constitutes the surroundings varies from problem to problem and also depends on the approach to the problem.

5.1.2 Centre of Mass For a system of particles, the centre of mass is defined as that point where the entire mass of the system is imagined to be concentrated, for considerations of its translational motion.

In a uniform gravitational field, the centre of mass and the centre of gravity of a system are coincident.

The concept of centre of mass is useful because a problem involving many particles (or a continuous distribution of mass) can be tackled in the same manner as that a point mass.

5.1.3 Collisions A collision is said to take place between two particles when the two come into contact (not necessarily a physical contact) with one another for a short duration of time and then move away. A collision involves an abrupt change in the motion of the particles. It is possible to clearly separate out the parameters of motion 'before' and 'after' a collision.

5.1.4 Elastic and Inelastic Collisions If there are no external forces acting, the total linear momentum is conserved in a collision. Collisions may be *elastic* or *inelastic*. In an elastic collision the total kinetic energy of the system is conserved, whereas, in an inelastic collision, the total kinetic energy of the system is not conserved.

5.1.5 One-dimensional Elastic Collision In a one-dimensional elastic collision, the speeds before and after collision are related to one another as

$$v_1' = \left(\frac{M_1 - M_2}{M_1 + M_2} \right) v_1 + \left(\frac{2 M_2}{M_1 + M_2} \right) v_2$$

and
$$v_2' = \left(\frac{2 M_1}{M_1 + M_2} \right) v_1 + \left(\frac{M_2 - M_1}{M_1 + M_2} \right) v_2$$

where M_1 and M_2 are the masses of the two particles; v_1 and v_2 are the speeds of the two particles before collision and v_1' and v_2' their speed after collision.

5.1.6 Rigid Body Bodies of a finite size can be regarded as made up of a collection of a large number of point masses. If the relative orientation of the point masses constituting a body does not change under external forces, the body is said to be *rigid*. No object is perfectly rigid. Generally solids can be regarded as rigid bodies.

5.1.7 Axis of Rotation A rigid body is said to be *rotating* if every point mass that makes it up, describes a circular path of a different radius but the same angular speed. The circular paths of all the point masses have a common centre. A line passing through this common centre is the *axis of rotation*.

5.1.8 Moment of Inertia The *rotational inertia* of a rigid body is referred to as its *moment of inertia*. The moment of inertia of a rigid body, about a given axis, is a measure of the inertness of a body to changes in its state of rotational rest or uniform rotation. It is numerically defined as

$$I = \frac{\tau}{\alpha}$$

where τ is the couple producing an angular acceleration α .

Numerically, moment of inertia is equal to the couple required to produce a unit angular acceleration. The moment of inertia of a body depends on (i) the axis of rotation and (ii) the distribution of mass in the body.

Moment of inertia, I has dimensions $M L^2$ and its SI unit is kg-m^2 .

5.1.9 Radius of Gyration For a body having moment of inertia I , about a given axis, it is possible to define a distance k , such that if a point mass equal to M , is imagined to be placed at a distance k from the axis of rotation, the moment of inertia of the point mass is the same as that of the actual body. This distance k is known as the *radius of gyration*. Obviously,

$$I = Mk^2$$

5.1.10 Theorem of Parallel Axes Let I be the moment of inertia of a body about an axis XY passing through O as shown in Fig. 5.1. If G is the position of the centre of mass of the body and $X'Y'$ is the axis parallel to XY and passing through G , then

$$I = I_G + Mh^2$$

where I is the moment of inertia of the body about XY and M is the mass of the body. The above statement is known as the *theorem of parallel axes*.

5.1.11 Analogy between Rotational and Translational Motion The rotational angular momentum L of a body having moment of inertia I and angular speed ω is

$$L = I\omega$$

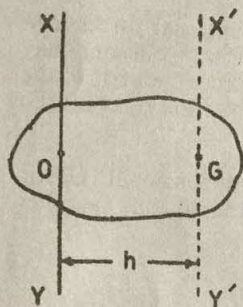


Fig. 5.1

In an exactly similar manner, the rotational kinetic energy is $\frac{1}{2} I \omega^2$. In fact, for every parameter in linear motion, there is a corresponding parameter in rotational motion. The definitions of translational and rotational parameters are similar to one another.

5.1.12 Law of Conservation of Angular Momentum The law of conservation of angular momentum states as follows.

If there is no external couple acting, the total angular momentum of a rigid body or a system of particles is conserved.

5.2 SI UNITS

Table 5.1

Physical Quantity	SI Unit	Symbol
Angular displacement	radian	rad
Angular speed	radian per second	rad s ⁻¹
Angular acceleration	radian per second squared	rad s ⁻²
Torque or couple	newton metre	N m
Moment of inertia	kilogram metre squared	kg m ²
Radius of gyration	metre	m
Angular momentum	kilogram (metre) ² radian per second	kg-m ² rad s ⁻¹
Rotational kinetic energy	joule	J

5.3 FORMULAE

- The position vector of centre of mass ($\vec{R}$) is determined from the formula

$$\vec{R} = \frac{\sum_{j=1}^N m_j \vec{r}_j}{\sum_{j=1}^N m_j}$$

or

$$\vec{R} = \frac{\int \vec{r} \, dm}{\int dm}$$

where $\vec{r}_j$ = position vector of the j -th particle

m_j = mass of the j -th particle

dm = mass of an element around the point having position vector $\vec{r}$.

- Moment of inertia I is defined by

$$I = \sum_{j=1}^N m_j r_j^2$$

where m_j = mass of the j -th particle

r_j = distance of the j -th particle from the axis of rotation

- The radius of gyration k is defined by

$$k = \sqrt{I/M}$$

$$= \sqrt{\frac{\sum mr^2}{\sum m}}$$

where m is the point mass and r is the distance of the point mass from the axis of rotation

4. Relation between angular momentum and moment of inertia is

$$L = I \omega$$

5. Rotational kinetic energy is given by

$$K_{\text{rot}} = \frac{1}{2} I \omega^2$$

6. The expressions for moment of inertia of a few bodies of regular shape about some common axis of rotation are tabulated below.

Table 5.2

Shape of body	Axis of Rotation	Expression for moment of Inertia
1. Circular ring of mass M and radius R	(i) through centre, perpendicular to plane of ring	$M R^2$
	(ii) any diameter	$\frac{1}{2} M R^2$
	(iii) any tangent in the plane of ring	$\frac{3}{2} M R^2$
	(iv) any tangent perpendicular to plane of ring	$2 M R^2$
2. Circular disc of mass M and radius R	(i) through centre, perpendicular to plane of disc	$\frac{1}{2} M R^2$
	(ii) any diameter	$\frac{1}{4} M R^2$
	(iii) tangent in the plane of the disc	$\frac{5}{4} M R^2$
	(iv) tangent perpendicular to plane of disc	$\frac{3}{2} M R^2$
3. Sphere of mass M and radius R	(i) any diameter	$\frac{2}{5} M R^2$
	(ii) any tangent plane	$\frac{7}{5} M R^2$
4. Cylinder of mass M radius R and length L	(i) own axis	$\frac{1}{2} M R^2$
	(ii) through centre perpendicular to length	$M (R^2/4 + L^2/12)$
	(iii) through end faces and $\perp$ to length	$M (R^2/4 + L^2/3)$
5. One dimensional rod of mass M and length L	(i) centre of rod and $\perp$ to length	$M L^2/12$
	(ii) one end and $\perp$ to length	$M L^2/3$
6. Rectangular lamina of mass M , length L and breadth B	(i) length of lamina and in its plane	$M B^2/3$
	(ii) breadth of lamina and in its plane	$M L^2/3$
	(iii) centre of lamina and parallel to length or breadth in its plane	$\frac{M B^2}{12}$ or $\frac{M L^2}{12}$
	(iv) centre of lamina and $\perp$ to its plane	$M \left(\frac{L^2 + B^2}{12} \right)$
	(v) centre of length and $\perp$ to its plane	$M \left(\frac{L^2}{12} + \frac{B^2}{3} \right)$
	(vi) centre of breadth and $\perp$ to its plane	$M \left(\frac{L^2}{3} + \frac{B^2}{12} \right)$

(Contd.)

7. Rectangular block of mass M , length L , breadth B and height H
- (i) through centre of block and parallel to length or breadth or height of the block
- $$M \left(\frac{B^2 + H^2}{12} \right) \text{ or } M \left(\frac{H^2 + L^2}{12} \right) \text{ or } M \left(\frac{L^2 + B^2}{12} \right)$$
- (ii) through end face and parallel to length or breadth or height of the block
- $$M \left(\frac{H^2}{3} + \frac{B^2}{12} \right) \text{ or } M \left(\frac{L^2}{3} + \frac{H^2}{12} \right) \text{ or } M \left(\frac{B^2}{3} + \frac{L^2}{12} \right)$$

5.4 NUMERICAL EXAMPLES

Example 1: Particles of masses 1 kg, 1 kg, 2 kg and 2 kg are placed at the corners A, B, C, D of a square of side L as shown in Fig. 5.2. Find the centre of mass of the system.

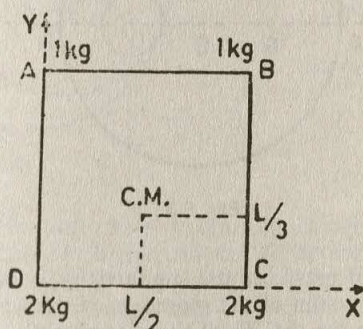


Fig. 5.2

Solution: Referring to Fig. 5.2, let us choose the sides DC and DA to define a Cartesian co-ordinate system. The position vectors of the various masses are

$$\vec{r}_A = O\hat{i} + L\hat{k}$$

$$\vec{r}_B = L\hat{i} + L\hat{k}$$

$$\vec{r}_C = L\hat{i} + L\hat{k}$$

$$\vec{r}_D = O\hat{i} + o\hat{k}$$

Let $\vec{R} = X\hat{i} + Y\hat{j}$ be the position co-ordinates of the centre of mass. By definition,

$$\vec{R} = \frac{\sum m\vec{r}}{\sum m}$$

$$\begin{aligned} \text{or } X\hat{i} + Y\hat{j} &= \frac{1(L\hat{k}) + 1(L\hat{i} + L\hat{k}) + 2(L\hat{i}) + 2 \times 0}{1 + 1 + 2 + 2} \\ &= \frac{3L\hat{i} + 2L\hat{k}}{6} \end{aligned}$$

Comparing coefficients of $\hat{i}$ and $\hat{k}$ we obtain

$$X = \frac{L}{2}; \quad Y = \frac{L}{3}$$

The position of the centre of mass is shown in Fig. 5.2.

Example 2: A circular hole of radius 1 m is cut off from a disc of radius 6 m. The centre of the hole is 3 m from the centre of the disc. Find the centre of mass of the remaining disc.

Solution: Let O be the centre of the disc and O' that of the hole (Fig. 5.3).

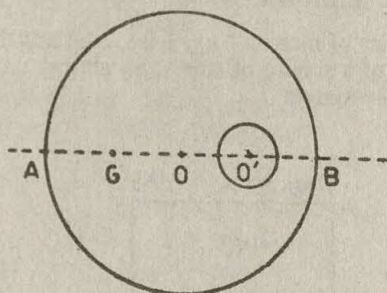


Fig. 5.3

To find the centre of mass, we use the fact that a body balances at this point, i.e. the algebraic sum of the moments of the weights about the centre of gravity is zero. The weight W_1 of the disc acts at point O . The hole can be regarded as a negative weight W_2 acting at O' . If X is the distance of the centre of gravity of the combination, then

$$X = \frac{W_1 \times O + (-W_2) \times 3}{W_1 + (-W_2)}$$

Also

$$W_1 = \rho\pi \times 6^2 = 36\rho\pi$$

$$W_2 = \rho\pi \times 1^2 = \rho\pi$$

where ρ is the mass per unit area of the disc.

Substituting the values of W_1 and W_2 , we get

$$X = \frac{\rho\pi \times 3}{36\rho\pi - \rho\pi} \text{ m} = -3/35 \text{ m}$$

The negative sign indicates that the centre of gravity is to the left of point O .

Example 3 A circular plate of uniform thickness has a diameter of 56 cm. A circular portion of diameter 42 cm is removed from one edge of the plate as shown in Fig. 5.4. Find the centre of mass of the remaining portion.

(IIT 1980)

Solution Referring to Fig. 5.4, let $AB = 56$ cm be the diameter of the circular plate. $BC = 42$ cm is the diameter of the circular hole.

The centres of the two circles are at O and O' . Obviously,

$$\begin{aligned} OO' &= OB - O'B \\ &= (28 - 21 \text{ cm}) = 7 \text{ cm} \end{aligned}$$

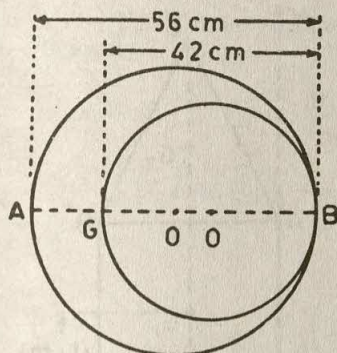


Fig. 5.4

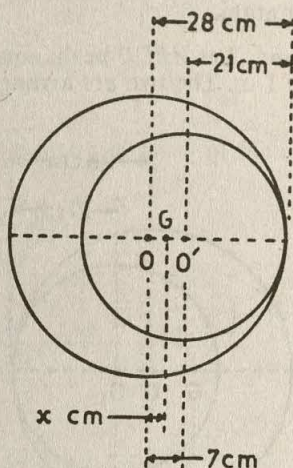


Fig. 5.5

If the plate and the hole were taken separately, the centre of mass of each would be at their respective centres. To determine the centre of mass of the “left over” portion, we imagine a mass M_1 and a mass $-M_2$ placed at O and O_1 and take their moments about G . Since the body balances, the total moment about G must be zero. Now,

$$\begin{aligned} M_1 &= \text{mass of disc of radius 28 cm} \\ &= \pi (28)^2 \times \rho \end{aligned}$$

$$\begin{aligned} \text{and } M_2 &= \text{mass of circular hole of radius 21 cm} \\ &= \pi (21)^2 \times \rho \end{aligned}$$

where ρ is the mass per unit area of the disc.

Let the distance of G from O be x . Then

$$GO_1 = 7 - x$$

Taking the moments of M_1 and M_2 placed at O and O_1 respectively, about G , we have

$$M_1 x = -M_2 (7 - x)$$

$$\begin{aligned} \text{or } \frac{x}{7 - x} &= \frac{-M_2}{M_1} = -\left(\frac{21}{28}\right)^2 \\ &= -9/16 \end{aligned}$$

$$\text{or } 16x = -63 + 9x$$

$$\text{or } 7x = -63$$

$$\text{or } x = -9 \text{ cm}$$

This means that actually, the centre of mass of the remaining portion is to the left of point O and at a distance of 9 cm from it as shown in Fig. 5.6.

Example 4: An equilateral triangle stands on a square. The sides of the triangle and the square are equal to 1 m. Find the centre of gravity of the combination.

Solution: Let $ABCD$ be the square and ABE be the equilateral triangle each of side 1 m. The two are arranged as shown in Fig. 5.7.

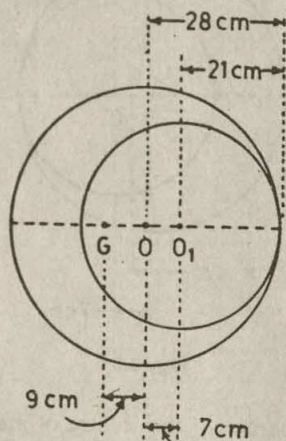


Fig. 5.6

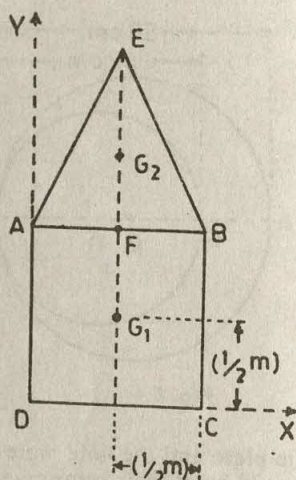


Fig. 5.7

Let M_1 and M_2 be the masses of the square and the triangle respectively. Obviously,

$$M_1 = \rho \times 1^2$$

$$M_2 = \rho \times \left(\frac{1}{2} \times 1 \times 1 \times \cos 60^\circ \right) = \rho \frac{\sqrt{3}}{4}$$

where ρ is the mass per unit area

Let us choose a rectangular co-ordinate system with point D as the origin and AD and DC as the y and x axes respectively. The centre of mass of the square is given by

$$\vec{r}_1 = \frac{1}{2} \hat{i} + \frac{1}{2} \hat{j}$$

The C.G. of the triangle is at a point G_2 such that

$$EG_2 = \frac{2}{3} h$$

where $h = EF = \frac{\sqrt{3}}{2} \text{ m}$

Let $\vec{r}_2$ be the position vector of G_2 . Then

$$\vec{r}_2 = \frac{1}{2} \hat{i} + (1 + G_2 F) \hat{j}$$

Also

$$\begin{aligned}
 G_2 F &= EF - EG_2 \\
 &= \frac{\sqrt{3}}{2} - \frac{2}{3} \frac{\sqrt{3}}{2} = \frac{1}{3} \frac{\sqrt{3}}{2} \\
 &= \frac{1}{2} \frac{1}{\sqrt{3}}
 \end{aligned}$$

$$\therefore \vec{r}_2 = \frac{1}{2} \hat{i} + \left(1 + \frac{1}{2\sqrt{3}}\right) \hat{j}$$

Let $\vec{R} = X\hat{i} + Y\hat{j}$ be the co-ordinates of the centre of mass of the combination. Obviously,

$$\vec{R} = X\hat{i} + Y\hat{j} = \frac{M_1 \vec{r}_1 + M_2 \vec{r}_2}{M_1 + M_2}$$

$$\text{or } X\hat{i} + Y\hat{j} = \frac{\rho \left(\frac{1}{2}\hat{i} + \frac{1}{2}\hat{j}\right) + \rho \frac{\sqrt{3}}{4} \left(\frac{1}{2}\hat{i} + \left[1 + \frac{1}{2\sqrt{3}}\right]\hat{j}\right)}{\rho + \rho \frac{\sqrt{3}}{4}}$$

$$= \left(\frac{1}{2}\right)\hat{i} + \frac{\frac{1}{2} + \frac{\sqrt{3}}{4}\left(1 + \frac{1}{2\sqrt{3}}\right)}{1 + \frac{\sqrt{3}}{4}}\hat{j}$$

$$= \frac{1}{2}\hat{i} + \frac{5 + 2\sqrt{3}}{2(4 + \sqrt{3})}\hat{j}$$

Comparing the coefficients of $\hat{i}$ and $\hat{j}$, we obtain

$$X = 1/2$$

$$\begin{aligned}
 Y &= \frac{5 + 2\sqrt{3}}{2(4 + \sqrt{3})} = \frac{(5 + 2\sqrt{3})}{2(4 + \sqrt{3})} \times \frac{(4 - \sqrt{3})}{(4 - \sqrt{3})} \\
 &= \frac{14 + 3\sqrt{3}}{26}
 \end{aligned}$$

Example 5: A solid cone and a hemisphere have a common base. Calculate the ratio of the height of the cone and the radius of the hemisphere so that the centre of mass of the common structure coincides with the centre of the common base.

Solution: Let h and R denote the height of the cone and the radius of the hemisphere respectively. If ρ is the density of the material, we have

$$\begin{aligned}
 M_1 &= \text{mass of the cone} \\
 &= \rho \times \text{volume of the cone} \\
 &= \rho \times \frac{\pi R^2 h}{3}
 \end{aligned}$$

and

$$\begin{aligned}
 M_2 &= \text{mass of the hemisphere} \\
 &= \rho \times \left(\frac{2\pi}{3} R^3\right)
 \end{aligned}$$

Let us choose a cartesian co-ordinate system as shown in Fig. 5.8. Let G_1 and G_2 be the centre of mass of the cone and hemisphere respectively. We know that

$$A G_1 = 3/4 h$$

and

$$D G_2 = 3/8 r$$

The position co-ordinates $\vec{r}_1$ and $\vec{r}_2$ of G_1 and G_2 are

$$\vec{r}_1 = \frac{R}{2} \hat{i} + \left(\frac{1}{4} h\right) \hat{j}$$

$$\vec{r}_2 = \frac{R}{2} \hat{i} - \left(\frac{3}{8} R\right) \hat{j}$$

We want the centre of mass of combination to be at the point D . Obviously, if $\vec{R}$ is the position vector of the centre of mass

$$\vec{R} = \frac{R}{2} \hat{i} + 0 \hat{j}$$

Also by definition

$$\begin{aligned} \vec{R} &= (M_1 \vec{r}_1 + M_2 \vec{r}_2) / (M_1 + M_2) \\ &= \frac{(R/2) \hat{i} (M_1 + M_2) + (1/4) h M_1 - (3/8) R M_2}{M_1 + M_2} \hat{j} \\ &= \frac{R}{2} \hat{i} + \left((1/4) h M_1 - (3/8) R M_2 \right) \hat{j} / (M_1 + M_2) \end{aligned}$$

Comparing coefficients of $\hat{i}$ and $\hat{j}$

$$R/2 = R/2$$

and

$$\frac{1/4 h M_1 - 3/8 R M_2}{M_1 + M_2} = 0$$

$$\therefore (1/4 h) / (3/8 R) = M_2 / M_1 = \frac{\left(\frac{2\pi}{3} R^3 \rho \right)}{\left(\frac{1}{3} \pi R^2 h \right) \rho}$$

$$\therefore \frac{2}{3} \frac{h}{R} = \frac{2 R}{h}$$

$$\text{or } h^2 = 3 R^2$$

$$\text{or } h/R = \sqrt{3}$$

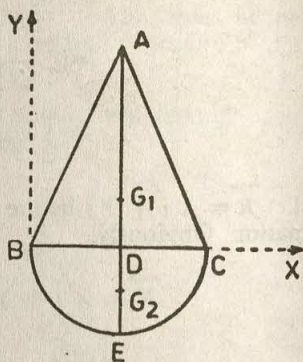


Fig. 5.8

Example 6: Calculate the centre of mass of a non-uniform rod, whose mass per unit length ρ varies as

$$\rho = \rho_0 \frac{x^2}{L}$$

where ρ_0 is a constant, L is the length of rod and x is distance of any point on rod measured from one end.

Solution: Let the rod OA be placed along the x -axis and its end O be chosen as the origin of the x -axis (see Fig. 5.9). Consider a small element

PQ of the rod lying between x and $x + dx$. Let dm be the mass of this element. Obviously,

$$\begin{aligned} dm &= \rho \, dx \\ &= \frac{\rho_0}{L} x^2 \, dx \end{aligned}$$

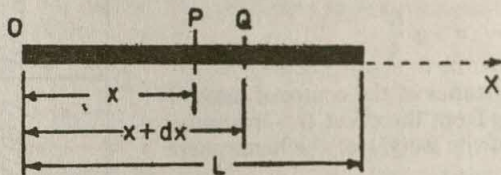


Fig. 5.9

If X is the distance of the centre of mass from end O ; then, by definition,

$$\begin{aligned} X &= \frac{\left(\int_0^L x \, dm \right)}{\left(\int_0^L dm \right)} \\ &= \frac{\frac{\rho_0}{L} \int_0^L x^3 \, dx}{\frac{\rho_0}{L} \int_0^L x^2 \, dx} \\ &= \frac{\frac{L^4}{4}}{\frac{L^3}{3}} \\ &= \frac{3}{4} L \end{aligned}$$

Example 7: Obtain the centre of mass of a uniform hemispherical ball of radius R .

Solution: Let ABC be the section of the hemispherical ball of radius R in the plane of the page (Fig. 5.10). Let the centre O of the hemisphere be chosen as the origin of a co-ordinate system as shown in Fig. 5.10

It is obvious from symmetry that the centre of mass must lie on the x -axis. To locate its position, consider a section of the ball by two planes DE and EF at distances x and $x + dx$ from O . It is clear that this section is a circular disc of radius y and thickness dx . The CG of this disc lies along the x -axis at the centre of the disc, i.e. at point H .

$$\begin{aligned} \text{Mass of circular disc} &= \rho (\pi y^2) \times dx \\ &= \pi \rho (R^2 - x^2) dx \end{aligned}$$

The moment of the weight of the circular disc about O

$$= \pi \rho g (R^2 - x^2) x \, dx$$

The algebraic sum of the moment of weights of all circular discs that make the hemisphere

$$\begin{aligned}
 &= \int_{x=0}^{x=R} \pi \rho g (R^2 - x^2) x \, dx \\
 &= \pi \rho g \left[\frac{R^2 x^2}{2} - \frac{x^4}{4} \right]_0^R \\
 &= \pi \rho g \frac{R^4}{4}
 \end{aligned}$$

Let x be the distance of the centre of mass of the hemisphere from the point O . Obviously, the moment of the weight of the hemisphere about O

$$= \frac{2\pi}{3} R^3 \rho g \times x$$

Hence, $\pi \rho g \frac{R^4}{4} = \frac{2\pi}{3} R^3 \rho g x$

or $x = \frac{3}{8} R$

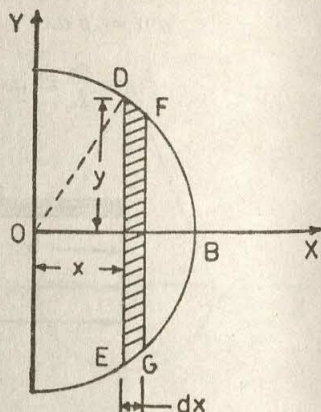


Fig. 5.10

Example 8: A ball A of mass 2.5 kg undergoes an elastic collision with another ball B that is at rest. After collision, ball A continues moving in its original direction with a speed $1/5$ of its original speed. Determine the mass of ball B .

Solution: The situation before and after collision is shown in Fig. 5.11 (a) and (b) respectively.

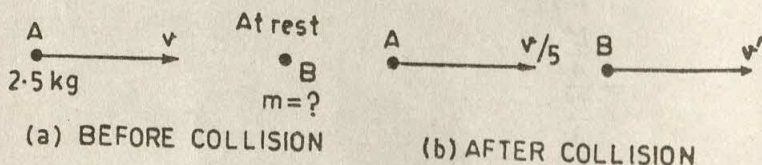


Fig. 5.11

Since the collision is elastic, both energy and momentum must be conserved. The law of conservation of momentum gives

$$2.5 V = 2.5 \frac{V}{5} + m v'$$

or $2V = mv'$ (1)

The law of conservation of energy yields

$$\frac{1}{2} \times 2.5 V^2 = \frac{1}{2} \times 2.5 \left(\frac{V}{5} \right)^2 + \frac{1}{2} m v'^2$$

or $2.4 V^2 = m v'^2$ (2)

Combining Eqs (1) and (2), we get

$$2V = m \left(\frac{2.4}{m} \right)^{1/2} V$$

$$\text{or} \quad 2 = m^{1/2} (2.4)^{1/2}$$

$$\text{or} \quad m = 4/2.4 \text{ kg} = 1.667 \text{ kg.}$$

Example 9: A particle of mass M moving with a speed V undergoes an elastic (head-on) collision with a stationary particle of mass m . After the collision, both particles move along the initial direction of motion. Calculate their speeds after collision. Show that

(a) If $M = m$, the ball of mass M comes to rest whereas the ball of mass m starts moving with a speed V .

(b) If $m \gg M$, the ball of mass M bounces back with speed V whereas the ball of mass m remains stationary.

(c) If the ball of mass M is a neutron ($M = 1.8 \times 10^{-27} \text{ kg}$) and the ball of mass m is a deuteron ($m = 3.6 \times 10^{-27} \text{ kg}$), and $V = 10^6 \text{ m s}^{-1}$, calculate the speeds of the neutron and the deuteron after the collision.

Solution: Let A and B be the particles of masses M and m respectively, Figure 5.12 (a) shows the situation before the collision. After the collision let the speeds of A and B be V' and v' respectively, as shown in Fig. 5.12 (b)

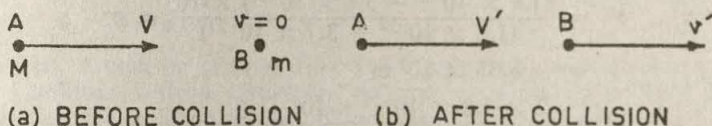


Fig. 5.12

Since the collision is perfectly elastic, both energy and momentum must be conserved.

The conservation of linear momentum gives

$$MV + m \times 0 = MV' + mv'$$

$$\text{or} \quad MV = MV' + mv' \quad (1)$$

From the law of conservation of energy, we have

$$\frac{1}{2} MV^2 = \frac{1}{2} M V'^2 + \frac{1}{2} m v'^2$$

$$\text{or} \quad M V^2 = M V'^2 + m v'^2 \quad (2)$$

Solving Eqs (1) and (2), we get

$$V' = \left(\frac{M - m}{M + m} \right) V \quad (3)$$

$$\text{and} \quad v' = \frac{2M}{M + m} V \quad (4)$$

(a) When $M = m$, we have from Eq. (3)

$$V' = 0$$

and from Eq. (4),

$$v' = V$$

In other words, after the collision, the particle of mass M becomes stationary and that of mass m starts moving with a speed V .

(b) If $m \gg M$

$$M - m \simeq -m$$

$$M + m \simeq m$$

Therefore, Eqs (3) and (4) reduce to

$$V' \simeq -V$$

$$\text{and } v' \simeq \frac{M}{m} V \simeq 0 \quad \left(\because \frac{M}{m} \ll 1 \right)$$

In other words, after collision, the particle of mass M bounces back with the same speed and the particle of mass m remains stationary.

$$(c) \quad M = 1.8 \times 10^{-27} \text{ kg}$$

$$m = 3.6 \times 10^{-27} \text{ kg}$$

$$V = 10^6 \text{ m s}^{-1}$$

Substituting these value in Eqs (3) and (4) we obtain

$$\begin{aligned} V' &= \frac{(1.8 \times 10^{-27} - 3.6 \times 10^{-27}) \times 10^6}{(1.8 \times 10^{-27} + 3.7 \times 10^{-27})} \text{ m s}^{-1} \\ &= -3.33 \times 10^5 \text{ m s}^{-1} \end{aligned}$$

$$\begin{aligned} \text{and } v' &= \frac{2 \times 1.8 \times 10^{-27} \times 10^6}{(1.8 + 3.6) \times 10^{-27}} \text{ m s}^{-1} \\ &= 6.66 \times 10^5 \text{ m s}^{-1} \end{aligned}$$

The negative sign of V' indicates that the direction of motion of the neutron is reversed after collision.

Example 10: Blocks A and B of masses 1 kg and 4 kg respectively rest on a smooth horizontal table as shown in Fig. 5.13. Initially blocks A and B are moving with speeds of 20 ms^{-1} and 5 m s^{-1} respectively. A massless spring of spring constant 10^3 N m^{-1} is attached to block B . The blocks collide and stick together. Assuming that the spring obeys Hooke's law, calculate the maximum compression of the spring.

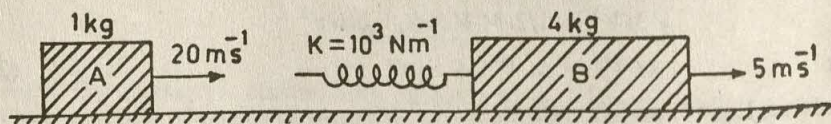


Fig. 5.13

Solution: Let v be the speed of the combination of the blocks A and B stuck together after the collision. Applying the law of conservation of linear momentum, we have

$$1 \times 20 + 4 \times 5 = 5 \times V$$

$$\therefore V = 8 \text{ m s}^{-1}$$

The total initial kinetic energy of the system

$$\begin{aligned} &= \frac{1}{2} \times 1 \times (20)^2 + \frac{1}{2} \times 4 \times (5)^2 \\ &= 250 \text{ J} \end{aligned}$$

The total kinetic energy after collision

$$= \frac{1}{2} \times 5 \times 8^2$$

$$= 160 \text{ J}$$

The energy difference of 90 J must be stored in the spring as potential energy. Let a be the maximum compression of the spring. The potential energy stored in the spring

$$= \frac{1}{2} k a^2$$

$$= \frac{1}{2} \times 10^3 \times a^2 \text{ J}$$

$$\therefore \frac{1}{2} \times 10^3 \times a^2 = 90$$

$$\text{or } a^2 = 180/1000 \text{ m}^2$$

$$= 0.18 \text{ m}^2$$

$$\therefore a = \sqrt{0.18} \text{ m}^2$$

$$\text{or } a = 0.416 \text{ m}$$

Example 11: A neutron of Mass $1.8 \times 10^{-27} \text{ kg}$ moving with a velocity of 10^6 m s^{-1} collides with a stationary deuteron of mass $3.6 \times 10^{-27} \text{ kg}$. Assuming that the collision is headon and elastic, determine the speed of each particle after the collision.

Solution: Let v_n and v_d respectively be the velocities of neutron and deuteron after the collision. Since the collision is headon and elastic, we have, from the laws of conservation of linear momentum and conservation of kinetic energy,

$$m_n \times 10^6 + m_d \times 0 = m_n v_n + m_d v_d$$

$$\text{and } \frac{1}{2} \times m_n \times (10^6)^2 + 0 = \frac{1}{2} m_n v_n^2 + \frac{1}{2} m_d v_d^2$$

where m_n = mass of the neutron and m_d = mass of the deuteron. Substituting the values of the masses, we have

$$1.8 \times 10^{-27} \times 10^6 = 1.8 \times 10^{-27} v_n + 3.6 \times 10^{-27} v_d$$

$$\text{or } 1.8 v_n + 3.6 v_d = 1.8 \times 10^6$$

$$\text{or } v_n + 2 v_d = 10^6$$

$$\therefore v_n = (10^6 - 2 v_d)$$

$$\text{and } 1.8 \times 10^{-27} \times 10^{12} = 1.8 \times 10^{-27} v_n^2 + 3.6 \times 10^{-27} v_d^2$$

$$\text{or } v_n^2 + 2 v_d^2 = 10^{12}$$

$$\therefore (10^6 - 2 v_d)^2 + 2 v_d^2 = 10^{12}$$

$$\text{or } 6 v_d^2 - 4 v_d \times 10^6 = 0$$

$$\text{or } 2 v_d (3 v_d - 2 \times 10^6) = 0$$

$$\therefore v_d = 2/3 \times 10^6 \text{ m s}^{-1} = 0.67 \times 10^6 \text{ m s}^{-1}$$

$$\text{and } v_n = \left(1 - \frac{4}{3}\right) \times 10^6 \text{ m s}^{-1} = -0.33 \times 10^6 \text{ m s}^{-1}$$

Thus, after the collision, the deuteron moves forward with a speed of 2/3rds of the original speed of the neutron and the neutron rebounds with 1/3rd of its original speed.

Example 12: Three balls A , B and C are arranged over a smooth horizontal surface as shown in Fig. 5.14. The ball A moves with a speed v_a towards balls B and C that are at rest and undergoes an elastic collision. Show that

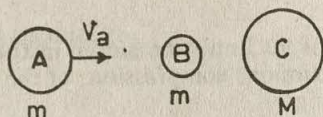


Fig. 5.14

(a) if $M \leq m$, there are two collisions and

(b) if $M > m$, there are three collisions. In each case, find out all the final speeds.

Solution: The situation before collision is shown in Fig. 5.15 (a). The first collision will be between balls A and B . Since both have the same mass, after collision, A will come to rest and B will move with a speed of v_a . This is shown in Fig. 5.15 (b). Now ball B moving with initial speed v_a will collide with the stationary ball C . After this second collision, let the speeds of balls B and C be v'_b and v'_c respectively. This is shown in Fig. 5.15 (c).

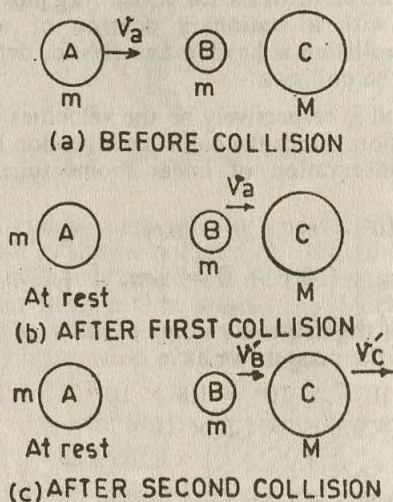


Fig. 5.15

Since the collisions are elastic, we have

$$v'_b = \left(\frac{m - M}{m + M} \right) v_a \quad (1)$$

and
$$v'_c = \frac{2m}{m + M} v_a \quad (2)$$

(a) If $M \leq m$, $(m - M) \geq 0$. In other words, after the second collision, ball B will be either at rest or move in the direction of v_c . Also from Eq. (2), it is clear that $v'_c > v'_b$ and both have the same direction as that of v_a . Since $v'_b < v'_c$, ball B cannot collide with ball C again. Therefore, there are only two collisions.

(b) If $M > m$, we have from Eq. (1)

$$v_b' = - \frac{(M - m)}{(M + m)} v_a$$

After the second collision, ball B moves in a direction opposite to that before the collision occurred. This ball B can, therefore, make another collision with ball A which is lying at rest after the first collision. Hence there are three collisions in all.

Example 13: A ball A of mass 10 kg and another ball B of unknown mass are placed on a horizontal frictionless table which rests against a rigid wall as shown in Fig. 5.16. The ball A moves towards the stationary ball B with a velocity v . What should be the mass of B such that both A and B move with the same speed after A has undergone a collision with ball B and the wall? All collisions are assumed to be elastic.

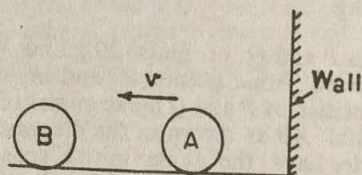


Fig. 5.16

Solution: Figure 5.17 (a) shows the positions of balls A and B before the collision takes place. After the collision has taken place, let V' and v' be the respective speeds of balls A and B (see Fig. 5.17 (b)). The ball A starts moving towards the wall and undergoes an elastic collision with the rigid wall. After collision with the wall, the velocity of A will be reversed. We are given that

$$V' = v'$$

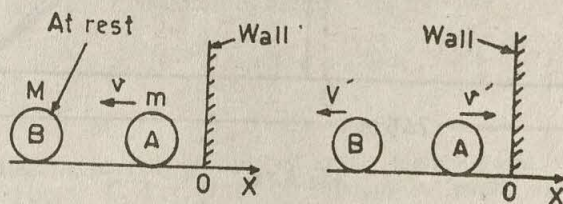


Fig. 5.17

Applying the law of conservation of linear momentum, we have

$$M \times 0 - m \times v = -M V' + m v'$$

or

$$m v = (M - m) v' \quad (\because V' = v') \quad (1)$$

The law of conservation of energy gives

$$\frac{1}{2} m v^2 = \frac{1}{2} M V'^2 + \frac{1}{2} m v'^2$$

or

$$m v^2 = (M + m) v'^2 \quad (2)$$

Combining Eqs (1) and (2) we obtain

$$m v^2 = (M + m) \frac{m^2 v^2}{(M - m)^2}$$

or $(M - m)^2 = m (M + m)$

We are given that $m = 10$ kg. Therefore,

$$(M - 10)^2 = 10 (M + 10)$$

or $M^2 - 30M = 0$

or $M(M - 30) = 0$

Since $M \neq 0$, $M - 30 = 0$

$\therefore M = 30$ kg

Example 14: Particles P and Q of mass 20 g and 40 g respectively are simultaneously projected from points A and B on the ground (Fig. 5.18). The initial velocities of P and Q make angles of 45° and 135° respectively with the horizontal AB as shown in the diagram. The separation AB is 245 m. Both particles have the same initial speed of 49 m s^{-1} . Both particles travel in the same vertical plane and undergo a collision. After the collision, P retraces its path. Determine the position of Q when it hits the ground. How much time after the collision does the particle Q take to reach the ground? Take $g = 9.8 \text{ m s}^{-2}$. (IIT 1982)

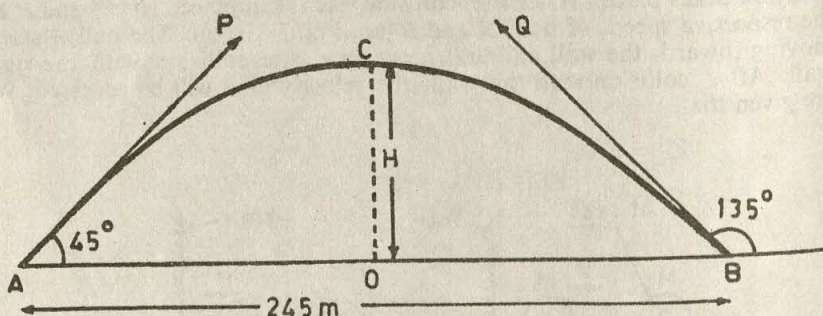


Fig. 5.18

Solution: It is clear from Fig. 5.18 that both P and Q have the same projection velocity and the same inclination of 45° to the horizontal. The range R of both is

$$R = \frac{u^2 \sin 2\theta}{g}$$

Since $u = 49 \text{ m s}^{-1}$ and $\theta = 45^\circ$, we have

$$\begin{aligned} R &= \frac{(49)^2 \sin 90^\circ}{9.8} = 5 \times 49 \\ &= 245 \text{ m} \end{aligned}$$

It is clear that both P and Q will meet exactly midway between A and B . When they meet, the horizontal distance travelled by each is $245/2$ m. If the particles meet t_1 seconds after they were projected,

$$\frac{245}{2} = (u \cos \theta) \times t_1$$

$$= 49 \cos 45^\circ \times t_1$$

$$\therefore t_1 = \frac{245}{2} \times \frac{1}{49} \times \sqrt{2} = \frac{5}{\sqrt{2}} \text{ s}$$

At the moment P and Q meet, they are both at the highest point in their respective parabolic paths. Their vertical component of velocity is zero and the horizontal component of velocity is $u \cos 45^\circ = \frac{49}{\sqrt{2}} \text{ m s}^{-1}$. However, the directions of the horizontal components of velocities of P and Q are opposite to each other. At this moment P and Q collide.

$\therefore$ Total momentum of P and Q before collision

$$= 20 \times 10^{-3} \times \frac{49}{\sqrt{2}} - 40 \times 10^{-3} \times \frac{49}{\sqrt{2}}$$

$$= -20 \times 10^{-3} \times \frac{49}{\sqrt{2}} \text{ kg m s}^{-1}$$

After collision, particle P retraces its path, i.e. the velocity of P reverses sign. Therefore, the velocity of P after collision will be $-\frac{49}{\sqrt{2}} \text{ m s}^{-1}$. Let V be the velocity of Q after collision. Then

Total momentum of P and Q after collision

$$= -20 \times 10^{-3} \times \frac{49}{\sqrt{2}} + 40 \times 10^{-3} V$$

From the law of conservation of momentum, we have

$$-20 \times 10^{-3} \left(\frac{49}{\sqrt{2}} \right) = -20 \times 10^{-3} \left(\frac{49}{\sqrt{2}} \right) + 40 \times 10^{-3} V$$

$$\therefore V = 0$$

After collision, particle Q is at rest at the highest point of its parabolic path, i.e. at point C . Let H be the maximum height attained by each particle during their parabolic paths. Clearly,

$$H = (u \sin \theta) t_1 - \frac{1}{2} g t_1^2$$

$$\begin{aligned} H &= \frac{49}{\sqrt{2}} \times \frac{5}{\sqrt{2}} - \frac{1}{2} \times 9.8 \times \left(\frac{5}{\sqrt{2}} \right)^2 \\ &= 61.25 \text{ m} \end{aligned}$$

If Q reaches ground t_2 seconds after collision, then t_2 is the time taken by it to fall freely through a vertical distance of 61.25 m, starting with zero initial speed. Hence

$$61.25 = 0 \times t_2 + \frac{1}{2} \times 9.8 \times t_2^2$$

$$\therefore t_2^2 = \frac{61.25 \times 2}{9.8} \text{ s}^2$$

$$\therefore t_2 = \sqrt{122.5/9.8} \text{ s}$$

$$= 3.53 \text{ s}$$

Example 15: A ball moving with a speed of 9 m s^{-1} strikes an identical stationary ball such that after the collision the direction of each ball makes an angle of 30° with the original line of motion. Find the speeds of the two balls after the collision. Is kinetic energy conserved in this collision process? (IIT 1975)

Solution: Let M be the mass of each ball. The situation before collision is shown in Fig. 5.19 (a). The total initial momentum in horizontal direction

$$= m \times 9 + m \times 0$$

$$= 9 \text{ m kg ms}^{-1}$$

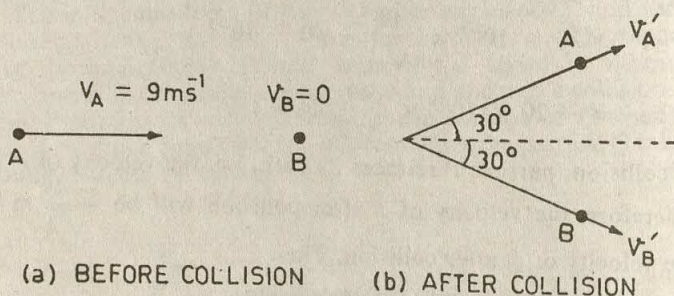


Fig. 5.19

The situation after the collision is shown in Fig. 5.19 (b). Resolve the velocities of A and B after collision into horizontal and vertical components. The total final momentum, in horizontal direction, after collision

$$= m v'_A \cos 30^\circ + m v'_B \cos 30^\circ$$

$$= m \frac{\sqrt{3}}{2} (v'_A + v'_B)$$

The law of conservation of linear momentum in the horizontal direction yields

$$9m = m \frac{\sqrt{3}}{2} (v'_A + v'_B)$$

or $v'_A + v'_B = 6\sqrt{3}$ (1)

After collision, the momentum of the particles in the vertical direction

$$= m v'_A \sin 30^\circ - m v'_B \sin 30^\circ$$

$$= \frac{m}{2} (v'_A - v'_B)$$

Since initially the momentum in the vertical direction is zero, the law of conservation of momentum requires that

$$\frac{m}{2} (v'_A - v'_B) = 0$$

or $v'_A - v'_B = 0$ (2)

Solving Eqs (1) and (2) we get

$$v'_A = v'_B = 3\sqrt{3} \text{ m s}^{-1}$$

The total kinetic energy of the system, before collision,

$$\begin{aligned} &= \frac{1}{2} m (9)^2 + \frac{1}{2} m (0)^2 \\ &= \frac{81}{2} m \end{aligned}$$

The total kinetic energy of the system, after collision

$$\begin{aligned} &= \frac{1}{2} m (3\sqrt{3})^2 + \frac{1}{2} m \left(\frac{3\sqrt{3}}{2} \right)^2 \\ &= 27 m \end{aligned}$$

Since the kinetic energy before and after the collision is not the same, the collision is an *inelastic collision*.

Example 16: In a billiards game a stationary red ball is aimed at a white ball that had been given a speed of 10 m s^{-1} . After the collision, the red ball moves at an angle of 30° with the original direction of motion of the white ball. Assuming the two balls to have the same mass, calculate the speed of the two balls and the direction of motion of the white ball. The collision is assumed to be elastic.

Solution: Let m be the mass of each ball and v_1 and v_2 the speeds of the red and the white balls respectively after the collision has taken place (Fig. 5.20).

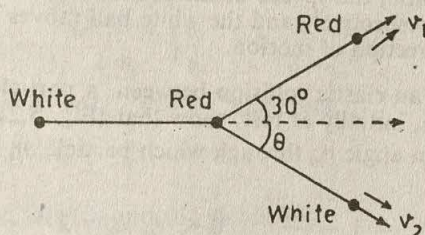


Fig. 5.20

If θ is the angle at which the white ball moves after impact, we have (equating the parallel and perpendicular components of momentum)

$$m \times 10 = m v_1 \cos 30^\circ + m v_2 \cos \theta$$

$$0 = m v_1 \sin 30^\circ - m v_2 \sin \theta$$

or $v_1 \frac{\sqrt{3}}{2} + v_2 \cos \theta = 10$

and $v_1 \cos \theta - v_2 \sin \theta = 0$

$\therefore \sin \theta = v_1/2v_2$ and $\cos \theta = \frac{20 - \sqrt{3}v_1}{2v_2}$

Using $\sin^2 \theta + \cos^2 \theta = 1$,

we get $\frac{v_1^2}{4v_2^2} + \frac{400 + 3v_1^2 - 40\sqrt{3}v_1}{4v_2^2} = 1$

This gives $v_1^2 - 10\sqrt{3}v_1 + 100 = v_2^2$ (1)

Again from the law of conservation of kinetic energy, we have

$$1/2 m \times 10^2 = 1/2 m v_1^2 + 1/2 m v_2^2$$

$\therefore v_1^2 + v_2^2 = 100$

or $v_2^2 = 100 - v_1^2$ (2)

From Eqs (1) and (2) we get

$$v_1^2 - 10\sqrt{3}v_1 + 100 = 100 - v_1^2$$

or $2v_1(v_1 - 5\sqrt{3}) = 0$

either $v_1 = 0$ or $v_1 = 5\sqrt{3} \text{ m s}^{-1}$

Since $v_R \neq 0$, we have $v_1 = 5\sqrt{3} = 8.66 \text{ m s}^{-1}$

Also $v_2^2 = 100 - v_1^2 = 100 - (5\sqrt{3})^2$
 $= 100 - 75 = 25$

or $v = \sqrt{25} = 5 \text{ m s}^{-1}$

And $\sin \theta = \frac{v_1}{2v_2} = \frac{5\sqrt{3}}{2 \times 5} = \sqrt{3}/2$

$\therefore \theta = 60^\circ$

Thus, after collision, the speeds of the red and the white balls are 8.66 m s^{-1} and 5 m s^{-1} respectively and the white ball moves at an angle of 60° with its original direction of motion.

Example 17: For an elastic collision between a particle of mass m_1 and another of mass m_2 , initially at rest, show that (Fig. 5.21)

(a) the maximum angle θ_m through which particle m_1 can be deflected is given by

$$\cos^2 \theta_m = 1 - \frac{m_2^2}{m_1^2} \quad 0 \leq \theta_m \leq \pi/2$$

if $m_1 > m_2$

(b) θ_1 can take all possible values lying between 0 and π if $m_1 < m_2$.

(c) $\theta_1 + \theta_2 = \pi/2$, if $m_1 = m_2$

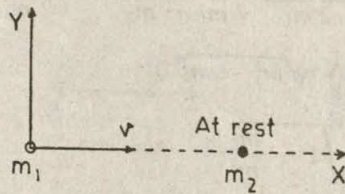
Solution: Since the collision is elastic, the linear momentum and the kinetic energy are both conserved. Applying the law of conservation of linear momentum to x and y components of momentum, we get

$$m_1 v_1 = m_1 v_1' \cos \theta_1 + m_2 v_2' \cos \theta_2 \quad (1)$$

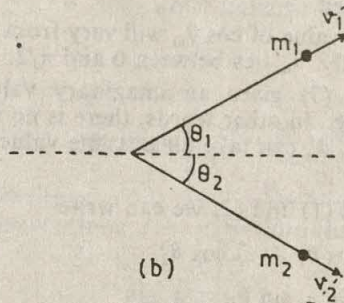
and $m_1 v_1' \sin \theta_1 = m_2 v_2' \sin \theta_2 \quad (2)$

The law of conservation of kinetic energy gives

$$\frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2 \quad (3)$$



(a)



(b)

Fig. 6.21

Equations (1) and (2) may be rewritten as

$$m_2 v_2' \cos \theta_2 = m_1 v_1 - m_1 v_1' \cos \theta_1$$

and

$$m_2 v_2' \sin \theta_2 = m_1 v_1' \sin \theta_1$$

Squaring and adding these equations, we obtain, on simplification,

$$m_2 v_2'^2 = m_1^2 v_1^2 + m_1 v_1'^2 - 2 m_1^2 v_1 v_1' \cos \theta_1 \quad (4)$$

Eliminating v_2' from Eqs (3) and (4), we have

$$\cos \theta_1 = \left(\frac{m_1 - m_2}{2 m_1} \right) \frac{v_1}{v_1'} + \frac{v_1 (m_1 + m_2)}{2 m_1 v_1} \quad (5)$$

For a fixed value of v_1 , $\cos \theta_1$ will be maximum if

$$\frac{d(\cos \theta_1)}{d v_1'} = 0 \quad (6)$$

The combination of Eqs (5) and (6) yields

$$-\left(\frac{m_1 - m_2}{2 m_1} \right) v_1 v_1'^{-2} + \left(\frac{m_1 + m_2}{2 m_1 v_1} \right) = 0$$

$$\text{or} \quad \frac{(m_1 - m_2) v_1}{2 m_1 v_1'^2} = \frac{m_1 + m_2}{2 m_1 v_1}$$

$$\text{or} \quad v_1'^2 = \left(\frac{m_1 - m_2}{m_1 + m_2} \right) v_1^2$$

$$\text{or} \quad v_1' = \sqrt{\frac{m_1 - m_2}{m_1 + m_2}} v_1 \quad (7)$$

The substitution of v_1' as given by Eq. (7) in Eq. (5) gives $\cos \theta_m$ which reads

$$\begin{aligned}\cos \theta_m &= \frac{m_1 + m_2}{2m_1} \sqrt{\frac{m_1 + m_2}{m_1 - m_2}} + \frac{m_1 - m_2}{2m_1} \sqrt{\frac{m_1 - m_2}{m_1 + m_2}} \\ &= \frac{1}{m_1} \sqrt{m_1^2 - m_2^2} \\ &= \sqrt{1 - \frac{m_2^2}{m_1^2}}\end{aligned}\quad (8)$$

which gives the maximum value of the angle through which the particle of mass m_1 can be deflected.

(a) If $m_1 > m_2$, the value of $\cos \theta_m$ will vary from 1 to a finite value less than zero. Consequently, θ_m lies between 0 and $\pi/2$.

(b) If $m_1 < m_2$, Eq. (7) gives an imaginary value of $\cos \theta_m$. This is physically unacceptable. In other words, there is no maximum limit on the value of θ_m . Therefore, θ_m can take all possible values lying between 0 and π .

(c) Rearranging Eqs (1) and (2) we can write

$$v_1 = v_1' \cos \theta_1 + v_2' \cos \theta_2$$

$$v_1' \sin \theta_1 = v_2' \sin \theta_2$$

$$\therefore \tan \theta_2 = \left(\frac{\sin \theta_1}{\frac{v_1}{v_1'} - \cos \theta_1} \right) \quad (9)$$

Since $m_1 = m_2$, Eq. (5) yields

$$\cos \theta_1 = \frac{v_1}{v_1'} \quad (10)$$

Combining Eqs (9) and (10) we get

$$\begin{aligned}\tan \theta_2 &= \frac{\sin \theta_1}{(1/\cos \theta_1 - \cos \theta_1)} = \frac{\sin \theta_1 \cos \theta_1}{(1 - \cos^2 \theta_1)} \\ &= \cot \theta_1 \\ &= \tan (\pi/2 - \theta_1)\end{aligned}$$

$$\therefore \theta_2 = \pi/2 - \theta_1$$

$$\text{or} \quad \theta_1 + \theta_2 = \pi/2$$

Example 18: A deuteron of mass 4×10^{-27} kg, accelerated to a speed of 10^7 m s⁻¹ collides with another deuteron at rest. (a) The two deuterons stick together forming a nucleus of helium atom. What is the speed of the helium nucleus formed? (b) The helium nucleus formed above breaks up into a neutron of mass 2×10^{-27} kg and an isotope of helium of mass 6×10^{-27} kg. The neutron produced flies in a direction at right angles to the original one at a speed of 5×10^6 m s⁻¹. Find the velocity of the helium isotope.

Solution: (a) Figure 5.22 (a) shows the two deuterons before collision. After the collision, the two combine together to form a helium nucleus as shown in Fig. 5.22 (b).

Let V be the speed of the helium nucleus formed. The law of conservation of linear momentum gives

$$4 \times 10^{-27} \times 10^7 = 8 \times 10^{-27} \times V$$

$$\therefore V = 5 \times 10^6 \text{ m s}^{-1}$$

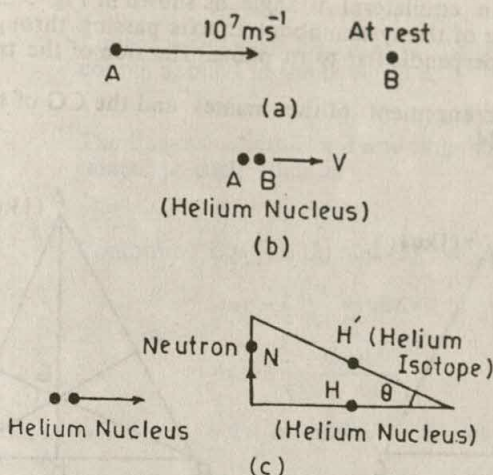


Fig. 5.22

(b) When the helium nucleus disintegrates, let the neutron produced and the isotope of helium move as shown in Fig. 5.22 (c). Let p , p_n and p' be the momentum of helium nucleus, the neutron and the isotope of helium respectively. Obviously,

$$p'^2 = p_n^2 + p^2$$

Let V' be the speed of helium isotope of mass m . Then

$$V' = p'/m' = \pm \frac{\sqrt{p_n^2 + p^2}}{m'} = \sqrt{\frac{m_n^2 v_n^2 + m^2 V^2}{m'^2}}$$

We are given that

$$m_n = 2 \times 10^{-27} \text{ kg}, \quad m' = 6 \times 10^{-27} \text{ kg}$$

$$m = 8 \times 10^{-27} \text{ kg}, \quad V = 5 \times 10^6 \text{ m s}^{-1}$$

and $v_n = 5 \times 10^6 \text{ m s}^{-1}$

Substituting these values, we have

$$\begin{aligned} V' &= \sqrt{\left(\frac{1}{3} \times 5 \times 10^6\right)^2 + \left(\frac{4}{3} \times 5 \times 10^6\right)^2} \\ &= 6.9 \times 10^6 \text{ m s}^{-1} \end{aligned}$$

Now, referring to Fig. 5.22 (c), we have

$$\tan \theta = p_n/p$$

$$= \frac{m_n v_n}{m V}$$

$$= \frac{2 \times 10^{-27} \times 5 \times 10^6}{8 \times 10^{-27} \times 5 \times 10^6}$$

$$= 1/4$$

Therefore, $\theta = \tan^{-1} 0.25$

$$\approx 14^\circ$$

Example 19: Particles of masses 1, 2 and 3 kg respectively are arranged at the corners of an equilateral triangle as shown in Fig. 5.23. Obtain the moment of inertia of the system about an axis passing through the CG of the triangle and perpendicular to its plane. The side of the triangle is 1 m.

Solution: The arrangement of the masses and the CG of the triangle is shown in Fig. 5.24.

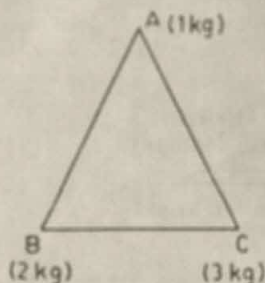


Fig. 5.23

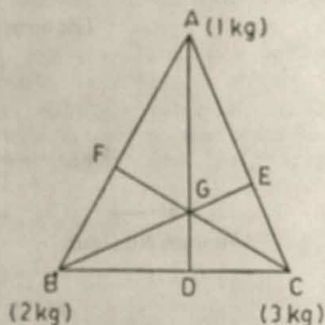


Fig. 5.24

The axis of rotation passes through G , the centre of gravity and is perpendicular to the plane of the triangle. Obviously, the perpendicular distances of A , B and C from the axis of rotation are AG , BG and CG . Also, for an equilateral triangle,

$$AG = BG = CG$$

Hence, the moment of inertia of the system is

$$\begin{aligned} I &= 1 \times AG^2 + 2 \times BG^2 + 3 \times CG^2 \\ &= 6 (AG)^2 \end{aligned}$$

It is well known that

$$AG = \frac{2}{3} AD$$

and

$$\begin{aligned} AD &= \sqrt{1^2 - 0.5^2} \\ &= \sqrt{0.75} \end{aligned}$$

$$\text{Therefore, } I = 6 \times \frac{4}{9} \times 0.75$$

$$= 2 \text{ kg-m}^2$$

Example 20: The moment of inertia of a solid flywheel about its axis is 0.1 kg-m^2 . A tangential force of 2 kgwt is applied round the circumference of the flywheel with the help of a string and mass arrangement as shown

in Fig. 5.25. If the radius of the wheel is 0.1 m, find the acceleration of the mass.

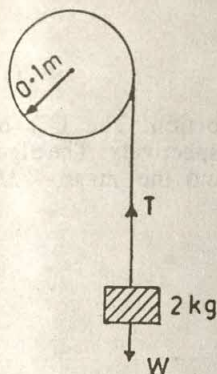


Fig. 5.25

Solution: Let a be the linear acceleration of the mass and T the tension in the string. It is clear that

$$mg - T = ma \quad (1)$$

Let the angular acceleration of the flywheel be α . The couple applied to the flywheel is

$$I\alpha = TR \quad (2)$$

The linear acceleration a and angular acceleration are related to each other as

$$a = R\alpha \quad (3)$$

Combining Eqs (1), (2) and (3), we get

$$mg - \frac{I\alpha}{R} = mR\alpha$$

$$\therefore \alpha = \frac{mgR}{(I + mR^2)} \quad (4)$$

It is given that $m = 2 \text{ kg}$, $R = 0.1 \text{ m}$ and $I = 0.1 \text{ kg-m}^2$. Substituting these values, we get

$$\begin{aligned} \alpha &= \frac{2 \times 9.8 \times 0.1}{(0.1 + 2 \times 0.1^2)} \text{ rad s}^{-2} \\ &= 16.7 \text{ rad s}^{-2} \end{aligned}$$

Example 21: A uniform disc of radius R has a cut as shown in Fig. 5.26. The mass of the remaining portion of the disc is m . Find the moment of inertia of such a disc about an axis passing through the centre of mass of the remaining portion and perpendicular to the plane of the disc.

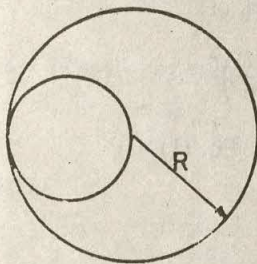


Fig. 5.26

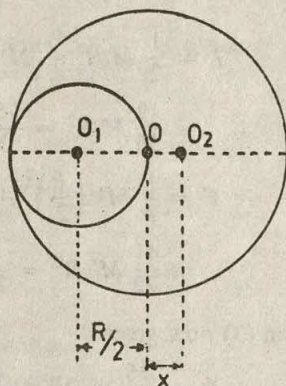


Fig. 5.27

Solution: Let M be the mass of the circular disc of radius R . The radius of the hole made is $R/2$. Let the mass of the hole that has been cut out be M' . Then

$$M = \pi R^2 \rho$$

and

$$M' = \pi \left(\frac{R}{2}\right)^2 \rho = \frac{\pi R^2 \rho}{4}$$

Hence, the mass of the remaining portion (m) = $M - M'$
 $= 3/4 \pi R^2 \rho$

Obviously,

$$M = \frac{4}{3} m; \text{ and } M' = \frac{m}{3} \quad (1)$$

Let O_2 be the position of the CG of the remaining portion. The CG of the disc of radius R and of the hole are at O and O_1 respectively. The algebraic sum of the moments of the mass M (at O) and the mass $-M'$ (at O_1) about O_2 must be zero.

Therefore,

$$M x - M' \left(\frac{R}{2} + x \right) = 0$$

$$\text{or } \pi R^2 \rho x - \frac{\pi R^2 \rho}{4} \left(\frac{R}{2} + x \right) = 0$$

$$\text{or } x - \frac{1}{4} \left(\frac{R}{2} + x \right) = 0$$

$$\therefore x = R/6 \quad (2)$$

Let I and I' respectively be the moments of inertia of the disc of radius R and of the hole about O_2 . Then the moment of inertia of the remaining portion about the desired axis is

$$I_1 = I - I' \quad (3)$$

To write I and I' we recall that the moment of inertia of the disc about an axis passing through its centre and perpendicular to its plane is $1/2 \text{ mass} \times (\text{radius})^2$. Combining this result with the theorem of parallel axes, we have

$$\begin{aligned} I &= \frac{1}{2} M R^2 + M \left(\frac{R}{6} \right)^2 \\ &= \frac{19}{36} M R^2 = \frac{76}{108} m R^2 \text{ cf Eq. (1)} \end{aligned}$$

$$\begin{aligned} \text{and } I' &= \frac{1}{2} M' \left(\frac{R}{2} \right)^2 + M' \left(\frac{R}{2} + \frac{R}{6} \right)^2 \\ &= \frac{41}{72} M' R^2 = \frac{41}{216} m R^2 \text{ cf Eq. (1)} \end{aligned}$$

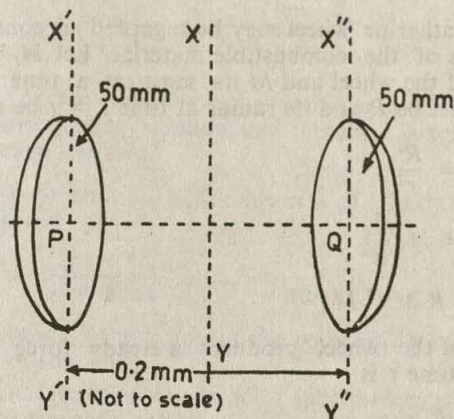
Equation (3) now gives

$$\begin{aligned} I_1 &= \frac{76}{108} m R^2 - \frac{41}{216} m R^2 \\ &= \frac{37}{72} m R^2 \end{aligned}$$

Example 22: Two discs of mass 0.1 kg and radius 50 mm is placed at the two ends of a rod 0.2 m long and 10 mm in diameter. Calculate the moment of inertia of the system about an axis passing through the centre of the rod and perpendicular to its length. The density of the material of the rod is $7.8 \times 10^3 \text{ kg m}^{-3}$.

Solution: Let the moment of inertia of each disc about the axis of rotation be I_1 and that of the rod be I_2 . Then I , the moment of inertia of the system is

$$I = 2 I_1 + I_2$$



AXIS OF ROTATION

Fig. 5.28

Referring to Fig. 5.28, let $X'Y'$ be the line passing through the centre of the disc and parallel to XY . Obviously $X'Y'$ is along a diameter of the disc. Using the theorem of parallel axes, we have

$$\begin{aligned} I &= \frac{1}{4} \times 0.1 \times (5 \times 10^{-2})^2 + 0.1 \times (0.1)^2 \\ &= 1.0625 \times 10^{-3} \text{ kg-m}^2 \end{aligned}$$

The moment of inertia I_2 of the rod about axis XY (i.e. the axis passing through its mid-point and perpendicular to its length) is

$$I_2 = M \left(\frac{R^2}{4} + \frac{L^2}{12} \right)$$

The rod is regarded as a cylinder of radius R and length L . Also,

$$\begin{aligned} M &= \pi R^2 L \rho \\ &= \pi \times (5 \times 10^{-3})^2 \times 0.2 \times 7.8 \times 10^3 \text{ kg} \end{aligned}$$

Therefore,

$$\begin{aligned} I_2 &= \pi \times (5 \times 10^{-3})^2 \times 0.2 \times 7.8 \times 10^3 \\ &\quad \times \left[\frac{(5 \times 10^{-3})^2}{4} + \frac{(0.2)^2}{12} \right] \\ &= 0.409484 \times 10^{-3} \text{ kg-m}^2 \end{aligned}$$

Hence

$$I = 2 I_1 + I_2 \approx 2.53 \times 10^{-3} \text{ kg-m}^2$$

Example 23: A uniform Catherine wheel is mounted in a vertical plane and is free to rotate about an axis passing through its centre and perpendicular to its plane. The wheel is made up of combustible material and burns at a constant rate of $\alpha \text{ kg s}^{-1}$. The thrust produced due to combustion is

steady and equals F . If the wheel starts from rest, show that the angular speed acquired when half of its mass has burnt is

$$\frac{4F}{\alpha R} (\sqrt{2} - 1)$$

Solution: The Catherine wheel may be regarded as consisting of a large number of turns of the combustible material. Let M_i be the initial mass (at time $t = 0$) of the wheel and M its mass at a time t . Let the initial radius of the wheel be R and its radius at time $t = t$ be r . Obviously,

$$\frac{M_i}{M} = \frac{R^2}{r^2}$$

or
$$M = M_i \frac{r^2}{R^2}$$

$$\therefore r = R M^{1/2} M_i^{-1/2} \quad (1)$$

The burning of the wheel produces a steady force F . Therefore, the couple acting at time t is

$$\begin{aligned} \tau &= F \times r \\ &= I \frac{d\omega}{dt} \end{aligned} \quad (2)$$

where I is the moment of inertia and ω the angular speed at $t = t$. Regarding the wheel as a circular disc, its moment of inertia is

$$I = \frac{1}{2} M r^2 \quad (3)$$

From Eqs. (2) and (3) we have

$$F \times r = \frac{1}{2} M r^2 \frac{d\omega}{dt}$$

or
$$\begin{aligned} \frac{d\omega}{dt} &= \frac{2F}{Mr} \\ &= \frac{2F M_i^{1/2}}{R M^{3/2}} \end{aligned}$$

$$\therefore \frac{d\omega}{dM} \frac{dM}{dt} = \frac{2F M_i^{1/2}}{R M^{3/2}}$$

We are given that

$$\frac{dM}{dt} = -\alpha$$

$$\therefore \frac{d\omega}{dM} = -\frac{2F M_i^{1/2}}{R \alpha} \frac{1}{M^{3/2}}$$

$$\therefore d\omega = -\frac{2F M_i^{1/2}}{R \alpha} \frac{dM}{M^{3/2}}$$

Integrating we get

$$\int_{\omega=0}^{\omega} d\omega = -\frac{2F M_i^{1/2}}{R \alpha} \int_{M=M_i}^{M=M_i/2} \frac{dM}{M^{3/2}}$$

$$\begin{aligned}
 \text{or } \omega_1 &= \frac{2 F M_i^{1/2}}{R \alpha} \left[2 M^{-1/2} \frac{M_i}{M_i} \right] \\
 &= \frac{4 F M_i^{1/2}}{R \alpha} \left\{ \left(\frac{M_i}{2} \right)^{-1/2} - (M_i)^{-1/2} \right\} \\
 &= \frac{4 F}{R \alpha} (\sqrt{2} - 1) \simeq \frac{1.656 F}{R \alpha}
 \end{aligned}$$

Example 24: You are given two spheres of the same size, mass and appearance. One of the spheres is hollow and the other is solid. How will you distinguish between the two?

Solution: We know that the acceleration a of a body rolling down an inclined plane is given by

$$a = \frac{g \sin \theta}{(1 + K^2/R^2)} \quad (1)$$

where K is the radius of gyration of the body and R is the radius of the circular section of the body.

Let a_s be the acceleration of the solid sphere when it rolls down the inclined plane. For the solid sphere, the moment of inertia is $\frac{2}{5} MR^2$. If K_F is the radius of gyration of the solid sphere, then

$$\frac{2}{5} M R^2 = M K_F^2$$

$$\text{or } K_F^2 = \frac{2}{5} R^2 \quad (2)$$

$$\therefore a_s = \frac{g \sin \theta}{(1 + 2/5)} = \frac{5}{7} g \sin \theta \quad (3)$$

The moment of inertia of a hollow sphere of internal and external radii r and R is $\frac{2}{5} M \frac{R^5 - r^5}{R^3 - r^3}$. If K_H is the radius of gyration of the hollow sphere, then

$$\frac{2}{5} M \frac{R^5 - r^5}{R^3 - r^3} = M K_H^2$$

$$\text{or } K_H^2 = \frac{2}{5} R^2 \frac{\left[1 - \left(\frac{r}{R} \right)^5 \right]}{\left[1 - \left(\frac{r}{R} \right)^3 \right]}$$

Since $r < R$,

$$\frac{1 - \left(\frac{r}{R} \right)^5}{1 - \left(\frac{r}{R} \right)^3} > 1$$

$$\therefore K_H^2 > K_S^2 \quad (4)$$

The acceleration a_H of the hollow sphere when rolled down the same inclined plane, is

$$a_H = \frac{g \sin \theta}{[1 + K_H^2/R^2]} \quad (5)$$

Using condition (4), it follows from Eqs. (3) and (5) that

$$a_S > a_H$$

Thus, if both solid and hollow spheres are rolled down the same inclined plane, the solid sphere (i.e. the one having a larger acceleration) will reach the bottom first.

Example 25: A cylinder rolls down an inclined plane of inclination α . Show that the cylinder will slip if μ , the coefficient of friction is less than $1/3 \tan \alpha$.

Solution: Figure 5.29 shows the various forces acting on the cylinder rolling down the inclined plane.

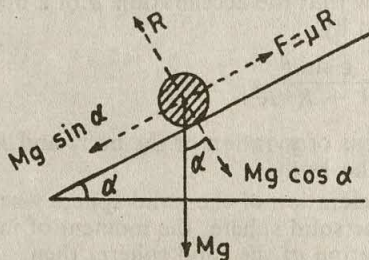


Fig. 5.29

Let a be the linear acceleration of the cylinder down the inclined plane. Obviously,

$$Ma = Mg \sin \alpha - \mu R$$

$$\text{or} \quad a = g \sin \alpha - g \cos \alpha \quad (1)$$

Let α be the angular acceleration of the cylinder. If the cylinder rolls down without slipping, its linear acceleration a' is given by

$$a' = R \alpha$$

The couple acting on the cylinder is, τ , where

$$\begin{aligned} \tau &= f \times R \\ &= (\mu Mg \cos \alpha) \times R \end{aligned} \quad (3)$$

The moment of inertia (I) of the cylinder about its own axis is $1/2 MR^2$. The angular acceleration (α) is

$$\begin{aligned} \alpha &= \tau/I \\ &= \frac{\mu Mg \cos \alpha R}{1/2 MR^2} \\ &= \frac{2 \mu g \cos \alpha}{R} \end{aligned} \quad (4)$$

Using Eqs (2) and (4) we have

$$a' = 2 \mu g \cos \alpha \quad (5)$$

The cylinder will start slipping if

$$a' < a$$

$$\text{or } 2\mu g \cos \alpha < (g \sin \alpha - \mu g \cos \alpha)$$

$$\text{or } \mu < \frac{1}{3} \tan \alpha$$

Example 26: A small sphere A of radius r , rolls without slipping, inside a large hemispherical bowl of radius R as shown in Fig. 5.30. The sphere A starts from rest at the top point of the hemisphere.

- (a) What fraction of the total energy is translational and rotational when the small sphere is at the bottom of the hemisphere?
 (b) Find the normal force exerted by the small sphere on the hemisphere when it is at the bottom of the hemisphere.

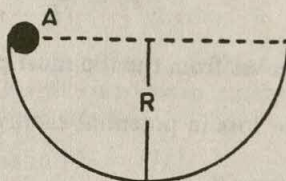


Fig. 5.30

Solution: (a) Let V be the linear speed of the small sphere at the bottom of the hemisphere. The translational kinetic energy is

$$K_1 = \frac{1}{2} m V^2 \quad (1)$$

where m is the mass of the small sphere. If ω is the angular speed and I the moment of inertia of the small sphere, the rotational energy is given by

$$\begin{aligned} K_2 &= \frac{1}{2} I \omega^2 \\ &= \frac{1}{2} \left(\frac{2}{5} m r^2 \right) \omega^2 \end{aligned} \quad (2)$$

Since the small sphere rolls down without slipping, we have

$$V = r \omega$$

$$\text{Therefore } K_2 = \frac{1}{5} m r^2 \frac{V^2}{r^2} = \frac{1}{5} m V^2 \quad (3)$$

The total energy (kinetic) of the small sphere at the bottom of the hemisphere is

$$\begin{aligned} K &= K_1 + K_2 \\ &= \frac{1}{2} m V^2 + \frac{1}{5} m V^2 = \frac{7}{10} m V^2 \end{aligned} \quad (4)$$

Hence, the fractional translational kinetic energy is

$$\frac{K_1}{K} = \frac{5}{7}$$

and the fractional rotational kinetic energy is

$$\frac{K_2}{K} = \frac{2}{7}$$

(b) Figure 5.31 shows the small sphere at the bottom of the hemisphere. The forces acting are

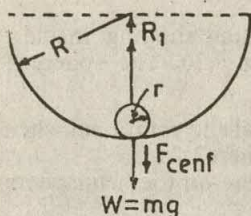


Fig. 5.31

- (i) Weight $W = mg$ of the small sphere acting vertically downwards.
- (ii) The centrifugal force, F_c
- (iii) R_1 , the reaction force acting vertically upwards.

To calculate F_c , we note that the small sphere moves along a circle of radius $(R - r)$. Therefore,

$$F_c = \frac{m V^2}{(R - r)} \quad (6)$$

As the small sphere moves from the top most point of the hemisphere to the bottom,

Gain in kinetic energy = loss in potential energy

$$\text{or} \quad \frac{7}{10} m V^2 = mg (R - r)$$

$$\text{or} \quad \frac{m V^2}{(R - r)} = \frac{10}{7} mg \quad (7)$$

Combining Eqs (5), (6) and (7) we have

$$\begin{aligned} R_1 &= mg + \frac{10}{7} mg \\ &= \frac{17}{7} mg \end{aligned}$$

Example 27: Figure 5.32 shows a Yo-Yo of mass M having a shaft of radius r around which a string is wound. The Yo-Yo starts from rest and unwinds itself. Show that the tension in the string during descent is given by

$$\frac{M g I}{(M r^2 + I)}.$$

where I is the moment of inertia of the Yo-Yo. What will be the tension in the string during ascent?

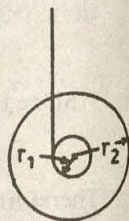


Fig. 5.32

Solution: Figure 5.33 shows the two positions A and B of the Yo-Yo. At position A , the string is completely wound around the shaft of radius r . In position B , the Yo-Yo has descended through a vertical height H .

Let T be the tension in the string. The couple τ acting on the Yo-Yo is due to this tension. Since the string is wound round the shaft of radius r ,

$$\tau = T \times r \quad (1)$$

If I is the moment of inertia of the Yo-Yo and α its angular acceleration, we have

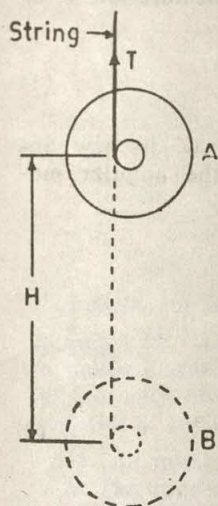
$$\alpha = \tau / I \quad (2)$$

Since the Yo-Yo moves without slipping,

$$a = r \alpha \quad (3)$$

Combining Eqs (1), (2) and (3) we have

$$T = \frac{I a}{r^2} \quad (4)$$



To calculate a we note that when the Yo-Yo descends from position A to B , its acceleration is a . Since the Yo-Yo starts from rest and descends a vertical height H , its velocity at position B is given by

$$V^2 = 2 a H$$

$$\therefore a = \frac{V^2}{2 H} \quad (5)$$

To obtain V we note that as the Yo-Yo moves from A to B , there is a loss in potential energy and a gain in kinetic energy.

$$\text{Loss in PE} = M g H$$

$$\text{Gain in KE} = \text{translational KE} + \text{rotational KE}$$

$$= \frac{1}{2} M V^2 + \frac{1}{2} I \omega^2$$

Fig. 5.33

Since there is no slipping during the movement of the Yo-Yo,

$$V = r \omega$$

$$\begin{aligned} \text{Gain in KE} &= \frac{1}{2} M V^2 + \frac{1}{2} I \frac{V^2}{r^2} \\ &= \frac{1}{2} \left(M + \frac{I}{r^2} \right) V^2 \end{aligned}$$

The law of conservation of energy demands

$$M g H = \frac{1}{2} \left(M + \frac{I}{r^2} \right) V^2$$

$$\text{or} \quad \frac{V^2}{2} = \frac{M g r^2 H}{(M r^2 + I)} \quad (6)$$

From Eqs. (5) and (6) we have

$$a = \frac{M g r^2}{(M r^2 + I)} \quad (7)$$

Using the value of a from Eq. (7) in Eq. (4) we get

$$T = \frac{M g a}{(M r^2 + I)}$$

Since T is independent of H , the tension during ascent will be the same as that during descent.

Example 28: If the earth were to suddenly contract to one-third of its present size without any change in its mass, calculate

- the duration of the new day
- the change in the weight of a body at the equator
- the ratio of the kinetic energies of the earth before and after contraction.

Solution: Let M and R be the initial mass and the initial radius of the earth. If ω is the angular speed of rotation of the earth, the duration T of the day is

$$T = \frac{2\pi}{\omega} \quad (1)$$

Let R' be the radius of the earth after contraction and ω' its new angular speed. Since the contraction takes place suddenly, the angular momentum is conserved. Therefore,

$$I \omega = I' \omega'$$

$$\begin{aligned} \text{or} \quad \frac{2}{5} M R^2 \omega &= \frac{2}{5} M R'^2 \omega' \\ &= \frac{2}{5} M \frac{R^2}{9} \omega' \end{aligned}$$

$$\text{or} \quad \omega' = 9 \omega \quad (2)$$

(a) The duration T' of the new day is

$$\begin{aligned} T &= \frac{2\pi}{\omega'} = \frac{2\pi}{9\omega} \\ &= \frac{1}{9} T \\ &= \frac{24}{9} \text{ h} = 2.667 \text{ h} \end{aligned}$$

(b) The weight W of a body, at the equator, initially is

$$W = m(g - \omega^2 R) \quad (3)$$

where g is the acceleration due to gravity and is given by

$$g = \frac{GM}{R^2}$$

When the earth contracts, the new weight W' will be

$$W = m(g' - \omega'^2 R') \quad (4)$$

$$\text{Also} \quad g' = \frac{GM}{R'^2} = 9g$$

$$\begin{aligned} \therefore W' &= m \left[9g - (9\omega)^2 \frac{R}{3} \right] \\ &= m[9g - 27\omega^2 R] \end{aligned}$$

Change in weight = $W' - W$

$$= m[8g - 26\omega^2 R]$$

(c) Initially, the rotational kinetic energy of the earth is

$$\begin{aligned} K &= \frac{1}{2} I \omega^2 \\ &= \frac{1}{2} \left(\frac{2}{5} MR^2 \right) \omega^2 \end{aligned}$$

After contraction, the rotational kinetic energy will be

$$\begin{aligned}
 K' &= \frac{1}{2} \left(\frac{2}{5} M R'^2 \right) \omega'^2 \\
 &= \frac{1}{2} \left(\frac{2}{5} M \left(\frac{R}{3} \right)^2 \right) (9 \omega)^2 \\
 &= \frac{9}{2} \left(\frac{2}{5} M R^2 \right) \omega^2 \\
 &= 9 \frac{1}{2} \left(\frac{2}{5} M R^2 \right) \omega^2
 \end{aligned}$$

Therefore, $K'/K = 9$

Example 29: Two bugs each of mass 10^{-3} kg move along a circular track of radius 2 m with identical speeds of 10 m s^{-1} . Initially, the two start from the two ends of a diameter. A thin massless rod is now placed along the diameter of the circular track. Obtain

- The angular speed and the angular momentum of the bugs.
- The bugs start moving along the rod and reduce their separation to 3 m. What is the angular speed of the bugs?
- Find the ratio of the rotational kinetic energy at the start and after the bugs have moved closer.

Solution: (a) Figure 5.34 (a) shows the movement of the two bugs along the circular track. Let ω be their angular speed; then

$$\omega = \frac{v}{r}$$

$$\text{or } \omega = \frac{10}{2} = 5 \text{ rad s}^{-1} \quad (1)$$

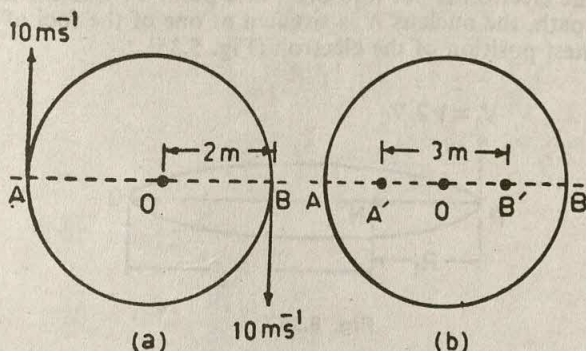


Fig. 5.34

Let I be the moment of inertia of the two bugs about an axis passing through O . Then

$$\begin{aligned}
 I &= (10^{-3}) \times 2^2 + (10^{-3}) \times 2^2 \\
 &= 8 \times 10^{-3} \text{ kg-m}^2
 \end{aligned}$$

The angular momentum L of the system is

$$L = I \omega = 8 \times 10^{-3} \times 5 = 4 \times 10^{-2} \text{ SI units}$$

(b) When the bugs reduce the separation between them to 3 m, the moment of inertia I of the system is

$$\begin{aligned} I' &= 2 \times 10^{-3} \times (1.5)^2 \\ &= 4.50 \times 10^{-3} \text{ kg-m}^2 \end{aligned}$$

As the bugs move along the rod, the torque applied to the system is zero and hence the angular momentum of the system must be conserved, i.e.

$$L = L'$$

or $I\omega = I'\omega'$

$$\begin{aligned} \omega' &= \frac{I\omega}{I'} \\ &= \frac{8 \times 10^{-3} \times 5}{4.5 \times 10^{-3}} \text{ rad s}^{-1} \\ &= 8.89 \text{ rad s}^{-1} \end{aligned}$$

(c) The ratio of the rotational kinetic energies is

$$\begin{aligned} \frac{K'}{K} &= \frac{I\omega^2}{I'\omega'^2} \\ &= \frac{(8 \times 10^{-3} \times 5^2)}{(4.5 \times 10^{-3} \times (8.89)^2)} \\ &= 0.56 \end{aligned}$$

Example 30: An electron is moving in an elliptical orbit around a hydrogen nucleus. It is put into orbit at a point P with a speed 1.2 times the speed required for a circular orbit at that position (Fig. 5.35). Find the greatest distance of the electron from the hydrogen nucleus.

Solution: The electron is put into orbit at a point P . Since it moves along an elliptical path, the nucleus N is situated at one of the foci of the ellipse. Q is the farthest position of the electron (Fig. 5.35).

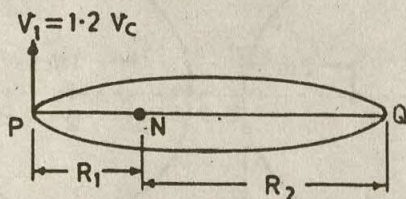


Fig. 5.35

Let m and M be the respective masses of the electron and the nucleus. R_1 and R_2 represent, respectively, the closest and the farthest distances of the electron from the nucleus.

The velocity v_c for a circular orbit at position P is given by

$$\frac{m v_c^2}{R_1} = K \frac{Ze e}{R_1^2}$$

or

$$v_c^2 = K \frac{Z e^2}{m R_1} \quad (1)$$

Let E_1 and L_1 stand for the total energy and angular momentum of the electron at P , then

$$\begin{aligned} E &= KE + PE \\ &= \frac{1}{2} m v_1^2 - K \frac{Z e^2}{R_1} \\ &= \frac{1}{2} m (1.2)^2 K \frac{Z e^2}{R_1} - \frac{K Z e^2}{R_1} = -0.28 K \frac{Z e^2}{R_1} \end{aligned} \quad (2)$$

and $L = m v_1 R_1$ (3)

At point Q , let the electron move with a speed v_2 and let E_2 and L_2 stand for the total energy and angular momentum there. Then

$$E_2 = \frac{1}{2} m v_2^2 - K \frac{Z e^2}{R_2} \quad (4)$$

and $L_2 = m v_2 R_2$

Since the motion of the electron obeys the inverse square law, (radial force), the angular momentum and the total energy are both conserved. Therefore,

$$L_1 = L_2$$

or $m v_1 R_1 = m v_2 R_2$

or $v_2 = v_1 \frac{R_1}{R_2}$

Also
$$\begin{aligned} -0.28 \frac{K Z e^2}{R_1} &= \frac{1}{2} m v_2^2 - \frac{K Z e^2}{R_2} \\ &= \frac{1}{2} m v_1^2 \left(\frac{R_1}{R_2} \right)^2 - \frac{K Z e^2}{R_2} \\ &= \frac{1}{2} m \left(\frac{R_1}{R_2} \right)^2 (1.2 v_c)^2 - \frac{K Z e^2}{R_2} \\ &= \frac{1}{2} \left(\frac{R_1}{R_2} \right)^2 (1.2)^2 \frac{K Z e^2}{R_1} - \frac{K Z e^2}{R_2} \end{aligned}$$

or
$$-\frac{0.28}{R_1} = 0.72 \frac{R_1}{R_2^2} - \frac{1}{R_2}$$

or
$$-0.28 = 0.72 \left(\frac{R_1}{R_2} \right)^2 - \frac{R_1}{R_2}$$

Put $\frac{R_1}{R_2} = X$. We have now

$$-0.28 = 0.72 x^2 - x$$

or
$$0.72 x^2 - x + 0.28 = 0$$

$$X = \frac{1 \pm \sqrt{1^2 - 4 \times 0.72 \times 0.28}}{2 \times 0.72}$$

$$= \frac{1 \pm 0.44}{1.44}$$

Since R_2 is greater than R_1 , X should be less than unity. We, therefore, select the negative sign in the above equation. Thus,

$$X = \frac{0.56}{1.44}$$

or
$$R_2 = \frac{1.44}{0.56} R_1$$

$$= 2.57 R_1$$

5.5 TRUE-FALSE STATEMENTS WITH REASONS

We are given some statements. We have to decide whether they are true or false giving reasons, in brief, in support of our answer.

1. The general form of Newton's law of motion is $\vec{F}_{\text{ext}} = m \vec{a}$.
2. The location of the centre of mass of a system of particles is independent of the reference frame used to locate it.
3. A sailboat can be propelled by air blown at the sails from a fan attached to the boat.
4. A body can have energy without having momentum and vice versa.
5. A quick collision between two bodies is more violent than a slow collision even when the initial and the final velocities are identical.
6. A jet of high pressure water can even break through the wall of a building.
7. n identical marbles each of mass m are lined up straight on a perfectly frictionless horizontal surface. If p similar marbles ($p < n$) each moving with a velocity u collide with this row of n marbles from the left, then after the collision, p marbles from the right will roll out with a speed u each and the remaining $(n - p)$ marbles will continue to be at rest.
8. The relative velocity of two particles in a 'head-on' collision is unchanged both in magnitude and direction.
9. When a particle moves under a central force (i.e. a force whose line of action passes through the reference point), the angular momentum about the centre is constant and vice versa.
10. A shell at the origin explodes into three pieces of mass 1 kg, 2 kg and m kg. The 1 kg and the 2 kg pieces fly off with speeds of 12 m s^{-1} along the x -axis and 8 m s^{-1} along the y -axis respectively. If the m kg piece flies off with a speed of 40 m s^{-1} , the total mass of the shell must be 3.5 kg. Also the velocities of the pieces as observed from the centre of mass frame will be unchanged.

ANSWERS

1. *False* More generally, we must write

$$\vec{F}_{\text{ext}} = \frac{d}{dt} \vec{P} = \frac{d}{dt} (m \vec{V}) = m \frac{d\vec{V}}{dt} + \vec{V} \frac{dm}{dt}$$

The given form is thus valid only if $\frac{dm}{dt} = 0$, i.e. the mass of the system does not change.

2. *True* The centre of mass of a system of particles is given by

$$\vec{r} = \frac{\sum m_i \vec{r}_i}{\sum m_i}$$

and depends only on the masses of the particles and their positions relative to one another.

3. *False* The force applied being internal to the system, it cannot change the state of motion of the system.
4. *False* Whereas a body at rest may have potential energy and yet no momentum, a body having momentum must necessarily have kinetic energy.
5. *True* The rate of change of momentum which determines the force is greater in the first case.
6. *True* The push of a water stream arises from the momentum transferred and this can be really large when a jet of high pressure water is stopped by the wall of a building.
7. *True* This will be consistent with the laws of conservation of linear momentum and kinetic energy.
8. *False* Applying the laws of conservation of linear momentum and kinetic energy, we find that

$$(v_2 - v_1) = -(u_2 - u_1)$$

Thus, the relative velocity is unchanged in magnitude but gets reversed in direction.

9. *True* We know that the rate of change of angular momentum equals the torque (or the momentum of the force) about the reference point:

$$\frac{d\vec{J}}{dt} = \vec{\tau} = \vec{r} \times \vec{F}$$

For a central force, $\vec{r}$ and $\vec{F}$ are in the same direction so that $\vec{r} \times \vec{F} = 0$. Hence, for a central force,

$$\frac{d\vec{J}}{dt} = 0$$

or $\vec{J} = \text{constant}$

10. *True* The application of the law of conservation of linear momentum shows that the x and y components of the momentum of the third piece have magnitudes of 12 kg m s^{-1} and 16 kg m s^{-1} respectively. The total momentum of this piece thus has a magnitude

$$\sqrt{(12)^2 + (16)^2} = 20 \text{ kg m s}^{-1}$$

The mass of this piece is, therefore, $20/40 \text{ kg}$ or 0.5 kg .

There being no external force acting on the system, the centre of mass is at rest and the velocities observed from the centre of mass remain unchanged.

5.6 MULTIPLE CHOICE QUESTIONS

Choose the best alternative in each of the following;

1. A shell fired from a gun at an angle to the horizontal explodes in midair. Then the centre of mass of the shell fragments will move
 - (a) vertically down
 - (b) horizontally
 - (c) along the same parabolic path along which the 'intact' shell was moving
 - (d) along the tangent to the parabolic path of the 'intact' shell, at the point of explosion.

2. A shell of mass $2m$ fired with a speed v at an angle θ to the horizontal explodes at the highest point of its motion into two fragments of mass m each. If one fragment, whose initial speed is zero, falls vertically, the distance at which the other fragment falls from the gun is given by

(a) $\frac{3}{2} \frac{u^2 \sin 2\theta}{g}$ (b) $\frac{2}{g} \frac{u^2 \sin 2\theta}{g}$
 (c) $\frac{u^2 \sin 2\theta}{g}$ (d) $\frac{3}{g} \frac{u^2 \sin 2\theta}{g}$

3. The centre of mass of a system of two particles is

- (a) on the line joining them and midway between them
 (b) on the line joining them at a point whose distance from each particle is proportional to the square of the mass of that particle
 (c) on the line joining them at a point whose distance from each particle is proportional inversely to the mass of that particle
 (d) on the line joining them at a point whose distance from each particle is proportional to the mass of that particle.

4. Of the two given spheres of the same size, mass and appearance, one is hollow and the other is solid. The two spheres are simultaneously rolled down an inclined plane. Then the

- (a) hollow sphere will reach the bottom first
 (b) solid sphere will reach the bottom first
 (c) both will reach the bottom together
 (d) hollow sphere will reach the bottom just a little before the solid sphere.

5. A 'Yo-Yo' is a toy in which a string is wound round a central shaft as shown in Fig. 5.36. The string unwinds and rewinds itself alternately making the Yo-Yo rise and fall. The ratio of the tensions in the string during descent and ascent is

- (a) $1 : 1$ (b) $r_2 : r_1$
 (c) $r_1 : r_2$ (d) $r_1 : l_1$

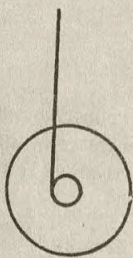


Fig. 5.36

6. If the earth were to suddenly contract to $1/n$ th of its present size without any change in its mass, the duration of the new day will be nearly
 (a) $24/n$ hours (b) $24n$ hours
 (c) $24/n^2$ hours (d) $24n^2$ hours
7. When a U-238 nucleus, originally at rest, decays by emitting an alpha particle, say with a speed of $v \text{ m s}^{-1}$, the recoil speed of the residual nucleus is (in m s^{-1})
 (a) $v/4$ (b) $-4v/238$
 (c) $-4v/234$ (d) $+4v/238$
8. If sand is allowed to drop from a stationary hopper onto a conveyor belt at a rate given by dM/dt , the force required to keep the belt moving at a constant speed v is
 (a) $v^2 dM/dt$ (b) $1/v dM/dt$
 (c) $1/v^2 dM/dt$ (d) $v dM/dt$

9. A bird is in a wire cage hanging from a spring balance. The reading of the balance is taken when the bird is flying about in the cage and when the bird is at rest in the cage. The first reading will be
- (a) much greater than the second
 - (b) greater than the second
 - (c) less than the second
 - (d) the same as the second
10. A loaded spring gun of mass M fires a 'shot' of mass m with a velocity v at an angle of elevation θ . The gun is initially at rest on a horizontal frictionless surface. After firing, the centre of mass of the gun-shot system
- (a) moves with a velocity vm/M
 - (b) moves with a velocity $\frac{vm}{M} \cos \theta$ in the horizontal direction
 - (c) remains at rest
 - (d) moves with a velocity $\frac{v(M-m)}{(M+m)}$ in the horizontal direction.
11. If a man of mass M jumps to the ground from a height h and his centre of mass moves a distance x in the time taken by him to 'hit' the ground, the average force acting on him (assuming his retardation to be constant during his impact with the ground) is
- (a) $Mg h/x$
 - (b) $Mg x/h$
 - (c) $Mg (h/x)^2$
 - (d) $Mg (x/h)^2$
12. A truck moving on a smooth horizontal surface with a uniform speed u is carrying stone-dust. If a mass Δm of the stone-dust 'leaks' from the truck in a time Δt , the force needed to keep the truck moving at its uniform speed is
- (a) $u \Delta m / \Delta t$
 - (b) $\Delta m du / dt$
 - (c) $u \frac{\Delta m}{\Delta t} + (\Delta m) \frac{du}{dt}$
 - (d) zero
13. Three particles each of mass m are located at the vertices of an equilateral triangle ABC . They start moving with equal speeds v each along the medians of the triangle and collide at its centroid G . If, after collision, A comes to rest and B retraces its path along GB , then C
- (a) also comes to rest
 - (b) moves with a speed v along CG
 - (c) moves with a speed v along BG
 - (d) moves with a speed v along AG .
14. A body of mass M moving with a speed u has a 'head-on' collision with a body of mass m originally at rest. If $M \gg m$, the speed of the body of mass m after collision, will be nearly
- (a) $u m/M$
 - (b) $u M/m$
 - (c) $u/2$
 - (d) $2u$
15. A particle A suffers an oblique elastic collision with a particle B that is at rest initially. If their masses are the same, then, after the collision,
- (a) they will move in opposite directions
 - (b) A continues to move in the original direction while B remains at rest
 - (c) they will move in mutually perpendicular directions
 - (d) A comes to rest and B starts moving in the direction of the original motion of A .

16. The motion of the centre of mass of a system of two particles is unaffected by their internal forces
- irrespective of the actual directions of the internal forces
 - only if they are along the line joining the particles
 - only if they are at right angles to the line joining particles
 - only if they are obliquely inclined to the line joining the particles.
17. The ratio of the radii of gyration of a circular disc and a circular ring of the same radii about a tangential axis is
- 1 : 2
 - 5 : 6
 - 2 : 3
 - 2 : 1
18. A massive circular hoop of radius r oscillates in its own plane about a horizontal axis at a distance x above the centre of the hoop. The period of oscillation is a minimum when x equals
- r
 - $r/2$
 - $r/3$
 - zero

5.7 NUMERICAL EXERCISES

- Six particles of masses 1, 2, 3, 4, 5 and 6 kg respectively are placed at the corners of a hexagon of side L . Locate the centre of mass of the system.
- From a square sheet of uniform density, a portion is removed shown shaded in Fig. 5.37. Find the centre of mass of the unshaded portion.

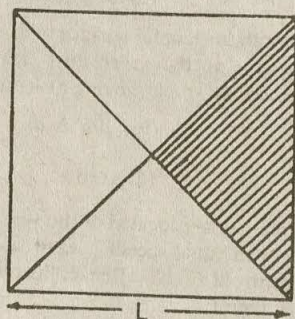


Fig. 5.37

- Show that the vertex angle of a cone must be $2 \sin^{-1} (1/3)$ if the centre of mass of the whole surface area of the cone (including the base) is to coincide with the centre of mass of its volume.
- A uniform wire PQR is bent such that $\angle PQR = 60^\circ$. The wire is suspended from a point P . What is the ratio PQ/QR so that QR is horizontal?
- A square hole is cut out from a circular uniform disc of radius R . The diagonal of the hole is equal to R . Show that the centre of mass of the remaining portion of the disc is at a distance

$$\frac{R}{8(2\pi - 1)}$$

from the centre of the disc.

- A uniform rod is bent into the arc of a circle such that the radius of the circle is equal to the length L of the rod. Locate the centre of mass of the bent rod.

A hemisphere of radius R is made up of a non-uniform material. If the density ρ at any point distant x from the centre is

$$\rho = \rho_0 x^{-2}$$

find the centre of mass of the hemisphere.

8. Figure 5.38 shows the structure of NH_3 molecule. The three hydrogen atoms are at the corners of an equilateral triangle of side 1.628×10^{-10} m. The centre of mass of the triangular base is 0.939×10^{-10} m away from each hydrogen atom. The nitrogen atom is located above the centre of mass of the base and at the apex of a pyramid. The nitrogen-hydrogen atom distance is 10.14×10^{-11} m. Locate the centre of mass of the ammonia molecule with respect to the nitrogen atom.

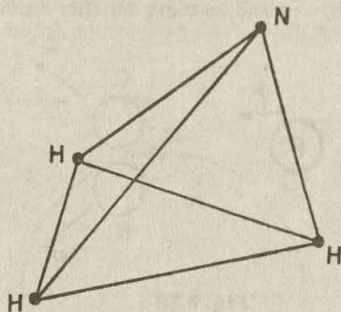


Fig. 5.38

9. A proton of mass 1.67×10^{-27} kg undergoes a head-on collision with an alpha particle initially at rest. After the collision, the alpha particle moves with a speed of 8×10^6 m s $^{-1}$. Calculate the velocity of the proton before and after the collision. Given, mass of alpha particle = 6.68×10^{-27} kg.
10. An unoccupied car of mass 10^3 kg has a speed of 15 m s $^{-1}$ on a level road. A truck of mass 5×10^3 kg moving in the opposite direction undergoes a head-on collision with the car. Calculate the speed the truck must have so that both vehicles come to rest after the collision.
11. A ball P of mass 10 kg moving with a speed of 8 m s $^{-1}$ collides with another ball Q of mass 20 kg initially at rest. After the collision, P and Q move in directions making angles of 30° and 45° respectively with the initial direction of motion of P . Calculate the speeds of P and Q after the collision.
12. Two cars A and B collide and stick together. Calculate the speed and the direction of motion of the wreckage. Given that initially A and B are moving with a speed of 20 m s $^{-1}$ north and 10 m s $^{-1}$ west respectively and masses of A and B are 1.2×10^3 kg and 1.6×10^3 kg respectively.
13. Two identical balls P and Q moving in the x - y plane undergo a collision at the origin of the co-ordinate system. The velocity components of the two balls before the collision are

$$\text{Ball } P \quad v_x = 0.4 \text{ m s}^{-1}, v_y = 0$$

$$\text{Ball } Q \quad v_x = -0.3 \text{ m s}^{-1}, v_y = 0.2 \text{ m s}^{-1}$$

After the collision, the ball P comes to rest. Find the velocity of ball Q after the collision.

14. A particle of mass m undergoes an elastic collision with a stationary particle of mass M . What is the fraction of kinetic energy lost by the striking particle, if,
- it moves at right angles to its original direction after the collision
 - the collision is a head-on one?

15. A particle of mass m collides with a stationary particle of mass M . After the collision, the particle of mass m moves in a direction at right angles and the particle of mass M recoils in a direction at 30° to the original direction of motion of particle of mass m . Obtain the fractional change in the kinetic energy of the system.
16. Three identical particles P , Q and R rest on a smooth horizontal table. Particle P starts moving with a velocity v and undergoes an elastic collision with particles Q and R simultaneously, as shown in Fig. 5.39. The distance between the centres of Q and R , before the collision, is x times greater than the diameter of each particle. Obtain the velocity of particle P after the collision. For what value of x will the particle P (a) recoil, (b) stop and (c) move on, after the collision?

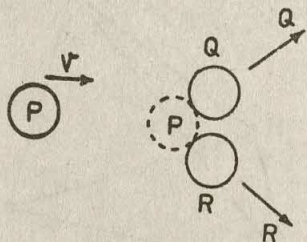


Fig. 5.39

17. An electron undergoes a one-dimensional collision with a hydrogen atom at rest. All motions before and after the collision are in a straight line. Calculate the fraction of electron's kinetic energy transferred to the hydrogen atom due to collision.
18. A constant force of 100 N is applied tangentially to the rim of a wheel 0.5 m in radius. The moment of inertia of the wheel is $10^2 \text{ kg}\cdot\text{m}^2$. Calculate
- the angular acceleration
 - the angular speed, 5 s from rest.
 - the number of revolutions made in 5 s
 - the kinetic energy of the wheel.
19. Figure 5.40 shows a pulley system with masses suspended as shown. Obtain the tension in the two strings and the acceleration of the two masses. The moment of inertia of the pulley system is $1.70 \text{ kg}\cdot\text{m}^2$.

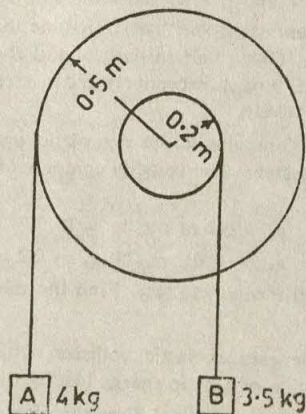


Fig. 5.40

20. A wheel and axle having a total moment of inertia of $2 \times 10^{-3} \text{ kg-m}^2$ is made to rotate about its own axis by means of a mass of 1 kg attached to a cord wrapped round the axle. The axle has a radius of 15 mm. How far must the weight descend so that the flywheel acquires a speed of 5 revolutions per second?
21. A torque of magnitude 2.1 N-m is required to make a flywheel rotate from rest. The torque is removed after the flywheel has completed 10 revolutions. Before coming to rest, the flywheel completes 60 revolutions in 20 seconds. What is the moment of inertia of the flywheel?
22. Figure 5.41 shows a circular disc of mass M and radius R having the centre at O . From this disc a small circular portion of radius r is removed. Show that the moment of inertia of the shaded portion about an axis through O and perpendicular to the plane of the disc is

$$\frac{M}{2R^2}(R^4 - r^4 - 2r^2 x^2)$$

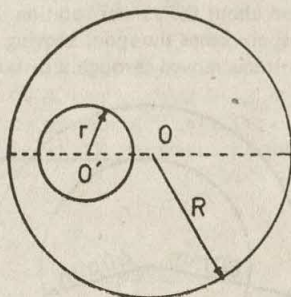


Fig. 5.41

23. Two particles of masses m and M are a distance d apart. Calculate the moment of inertia of the system about an axis passing through the centre of mass and perpendicular to the line joining the two masses. If ν is the frequency of revolution, show that the rotational kinetic energy is

$$\frac{2\pi^2 m M d^2}{(m + M)}$$

24. A cylinder of mass M is made to rotate about a horizontal axis, as shown in Fig. 5.42, with the help of a mass m hanging from a string wound around it. Obtain the tension in the string and the acceleration of mass m .

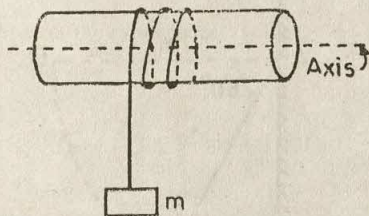


Fig. 5.42

25. AB and BC are two smooth planes inclined at an angle of 45° as shown in Fig. 5.43. A solid sphere P is rolling on AB with a speed of 30 m s^{-1} . It rolls up the incline BC and rises upto a height H . Obtain the value of H .

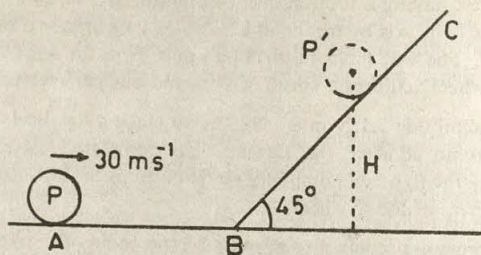


Fig. 5.43

26. A spool rests vertically on a horizontal table. A force of 5 N is applied to it as shown in Fig. 5.44. A string is wound around it. The mass of the spool is 10 kg and its radius of gyration about the axis of rotation is 10 mm. The friction between the table and the spool keeps the spool moving without slipping. Obtain the speed of the spool after it has moved through a distance of 2 m.

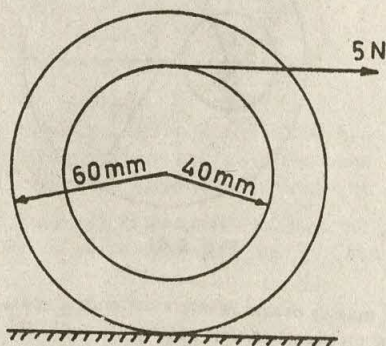


Fig. 5.44

27. A rod AB is 1 m long. It is hinged at A so as to rotate in a vertical plane. Initially the rod is horizontal and is released from rest (Fig. 5.45). Obtain the angular speed of the rod and the linear speed of its free end as it passes through the position shown in the diagram.

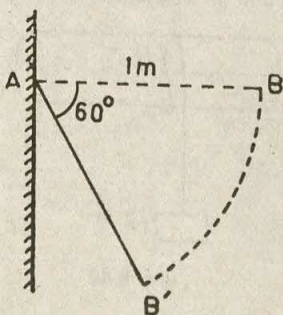


Fig. 5.45

28. A uniform cylinder of mass M and radius R is suspended from a rigid support by strings as shown in Fig. 5.46. It starts descending from rest. Neglecting the mass of the rod, find
- the tension in each string
 - the angular acceleration of the cylinder.

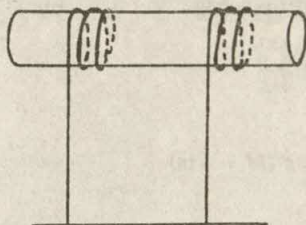


Fig. 5.46

29. A turntable rotates about a fixed vertical axis passing through its centre. Its angular speed is $1.33 \pi \text{ rad s}^{-1}$. A man weighing 80 kg, initially standing at the centre of the turntable runs along the radius. What is the angular speed of the turntable when the man is at a distance of 0.2 m from the centre? The moment of inertia of the turntable is $2.8 \text{ kg}\cdot\text{m}^2$.

ANSWERS

MULTIPLE CHOICE QUESTIONS

- | | | | | |
|---------|---------|----------|---------|---------|
| 1. (c) | 2. (a) | 3. (c) | 4. (b) | 5. (a) |
| 6. (c) | 7. (c) | 8. (d) | 9. (c) | 10. (c) |
| 11. (a) | 12. (d) | 13. (c) | 14. (d) | 15. (c) |
| 16. (a) | 17. (b) | 18. (a). | | |

NUMERICAL EXERCISES

- $9 L/14$
- $x = 7/18 L$, $y = L/2$
- 0.414
- $1.5 L/\pi$ from the centre on the symmetrical radius
- $R/3$ from the centre of the base
- $0.67 \times 10^{-11} \text{ m}$
- Before: $2 \times 10^6 \text{ m s}^{-1}$; After: $-1.2 \times 10^6 \text{ m s}^{-1}$
- 3 m s^{-1}
- $V_A = 5.85 \text{ m s}^{-1}$; $V_B = 2.07 \text{ m s}^{-1}$
- 10.3 m s^{-1} at 33.5° west of north
- $v'_x = 0.1 \text{ m s}^{-1}$, $v'_y = 0.2 \text{ m s}^{-1}$
- $\frac{2M}{M+m}$, $\frac{4Mm}{(M+m)^2}$

15. $\frac{2}{3} \left(1 - \frac{m}{M} \right)$
16. $\frac{-v(2-x^2)}{(6-x^2)}$; x small; $x = \frac{1}{\sqrt{2}}$; $x > \frac{1}{\sqrt{2}}$
17. 2.2×10^{-3}
18. (a) 0.5 rad s^{-1} , (b) 2.5 rad s^{-1} , (c) 4, (d) 312.5 J
19. 16.8 N , 18.6 N , 2.76 rad s^{-2}
20. 0.112 m
21. $1.17 \times 10^{-1} \text{ kg} - \text{m}^2$
24. $m M g / (M + 2 m)$; $2 m g / (M + 2 m)$
25. 6.43 m
26. 1.71 m s^{-1}
27. $\omega = 5.05 \text{ rad s}^{-1}$, $v = 5.05 \text{ m s}^{-2}$
28. $1/6 M g$; $2/3 g/R$
29. 0.621 rad s^{-1} .

Fluid Pressure and Buoyancy

6.1 REVIEW OF BASIC CONCEPTS

6.1.1 Pressure If a force F acts perpendicular to the surface of a fluid (liquid or gas) of area A , the pressure P is given by

$$P = \frac{F}{A} \quad (6.1)$$

In SI units, pressure is expressed in newton per square metre (N m^{-2}), which is also known as pascal (Pa). Thus,

$$1 \text{ Pa} = 1 \text{ N m}^{-2}$$

In CGS units, P is expressed in dyne cm^{-2} . At a given point in a fluid, the pressure can act in any direction.

6.1.2 Density and Relative Density The density of a substance is defined as the mass per unit volume of the substance. The SI unit of density is kilogram per cubic metre (kg m^{-3}). Density is also expressed in gram per cubic centimetre (g cm^{-3})

$$1 \text{ g cm}^{-3} = 1000 \text{ kg m}^{-3}$$

The relative density (RD) or specific gravity (SG) of a substance is defined as the ratio of its density to that of water, i.e.

$$\text{Relative density} = \frac{\text{density of substance}}{\text{density of water}} \quad (6.2)$$

Relative density, being a ratio of similar quantities, has no units; it is just a number. In CGS units, the density of water is 1000 g cm^{-3} . Hence the relative density of a substance is equal to the numerical value of its density expressed in g cm^{-3} .

6.1.3 Archimedes' Principle According to Archimedes' principle, if a body is completely or partially immersed in a fluid, it experiences an upward force or thrust which is equal to the weight of the fluid it displaces and acts in a vertically upward direction through the centre of gravity of the displaced fluid.

The upward force or thrust is known as the buoyant force. The force acts because the pressure on the lower surface of the body exceeds that on the upper surface of the body immersed in the fluid. If the body is sufficiently heavy, the force of buoyancy may not be sufficient to keep it afloat, but it would suffer an apparent loss of weight equal to the upward force exerted by the fluid on the body. The buoyant force always acts vertically upward through the centre of gravity of the displaced fluid.

6.1.4 Density Measurement and Archimedes' Principle The principle of Archimedes is very useful in finding the density of a solid or liquid. The object (which sinks in water) is suspended by a string so that it is completely immersed in water. The weight of the solid immersed in water is measured by a spring balance or a weighing machine. By subtracting the balance reading from the known weight of the solid in air, we get the weight of the displaced water from which the volume of the displaced water is obtained (since the density of water is known). This is obviously the volume of the solid. Knowing the mass of the solid, its density is obtained from the definition:

$$\text{Density } (\rho) = \frac{\text{mass}}{\text{volume}} = \frac{m}{V} \quad (6.3)$$

6.1.5 Floatation When an object is placed in a fluid, it will sink if the buoyant force is less than the weight of the object. The object floats in equilibrium at any depth in a fluid if the buoyant force equals the weight of the body. The object floats with a portion of its volume above the fluid surface if the weight of the object is less than the buoyant force. The principle of Archimedes still applies.

When an object floats in a fluid, a portion of its volume rises above the surface in order to establish equilibrium. In this case, the weight of the object is completely supported by the upward buoyant force of the fluid. The weight of the object equals the weight of the fluid displaced by the object. This is called the *law of floatation*.

6.1.6 Pascal's Principle According to this principle, if pressure is applied to an incompressible fluid, it is transmitted equally throughout the entire fluid. Every substance in the atmosphere experiences the atmospheric pressure. The force exerted on an open surface of a vessel of area A is $P_0 \times A$ where P_0 is the atmospheric pressure. According to Pascal's principle, the pressure P_0 is transmitted unchanged throughout the liquid. This means that, in the case of a liquid in communicating vessels a force applied to a surface of the liquid, where the area of cross-section is small, will be multiplied at some other place (at the same level as the first) where the area of cross-section is large. This is illustrated in Figure 6.1. Hydraulic press is one of the applications of this principle. Similar action takes place in brakes and other devices in which small forces overcome large forces.

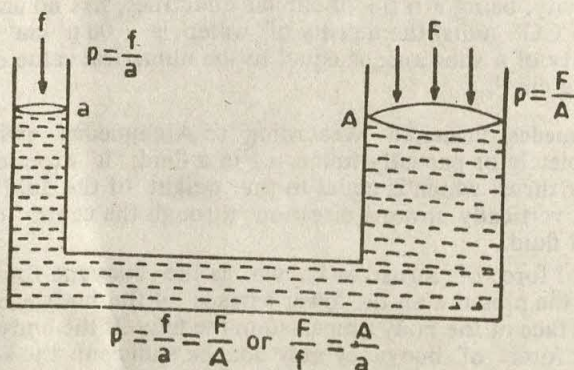


Fig. 6.1

6.1.7 Facts About Liquid Pressure

For a liquid of uniform density and at rest we find that

1. There exists pressure everywhere within the liquid.
2. The pressure due to the weight of the liquid increases with depth, being directly proportional to the distance below the free liquid surface. The hydrostatic pressure P at a depth h from the surface is given by

$$P = \rho g h \quad (6.4)$$

where ρ is the density of the liquid and g , the acceleration due to gravity.

3. The pressure is the same at all points at the same horizontal level.
4. The pressure at a point in a liquid is independent of the shape or the area of cross-section of the vessel.
5. The compressibility of all liquids is very small and for most practical purposes, liquids may be considered incompressible. Hence, the density is constant throughout the liquid.
6. The pressure at any point in a liquid is equal to the atmospheric pressure P_0 , acting on the surface of the liquid plus the pressure due to the liquid above the point (called the gauge pressure)

$$P = P_0 + \rho g h \quad (6.5)$$

7. To find the total force on the side of a vessel full of liquid we must allow for the fact that the pressure varies with depth. Since it varies uniformly, the pressure in the middle of the vessel is an average value. We can assume this pressure to exist uniformly over the wall if it is plane. The total force is then this average pressure multiplied by the area of the wall. Similarly,

Total force or thrust on a horizontal surface immersed in a liquid = pressure on the surface $\times$ area of the surface. Total force or thrust on a vertical surface immersed in a liquid = pressure at the centre of gravity of the area $\times$ area of the surface.

8. Atmospheric Pressure

Due to the fact that the gaseous atmosphere is a fluid, there is atmospheric pressure at its base i.e. at the earth's surface. Its normal value is $1.013 \times 10^5 \text{ N m}^{-2}$. This is the value at sea level at 0°C . This corresponds to the pressure at the base of a column of mercury (density 13.6 g cm^{-3}) 76 cm high.

Here SI units are used for all the quantities.

$$\begin{aligned} 1 \text{ atmosphere} &= 1.013 \times 10^5 \text{ N m}^{-2} = 1.013 \times 10^5 \text{ Pa} \\ &= 0.1013 \text{ M Pa (mega Pascal)} \end{aligned}$$

Another unit for pressure used in meteorology is a bar which is equal to 10^5 N m^{-2} or 0.1 M Pa and is thus very nearly equal to one atmosphere.

$$1 \text{ millibar} = 100 \text{ Pa}$$

$$\text{Also} \quad 1 \text{ bar} = 10^6 \text{ dyne cm}^{-2}$$

The density of air decreases appreciably as we go up in the atmosphere.

6.2 SI UNITS

Table 6.1

<i>Physical Quantity</i>	<i>SI Unit</i>	<i>Symbol</i>
Pressure	newton per square metre or Pascal	N m^{-2} or Pa
Thrust	newton	N
Density	kilogram per cubic metre	kg m^{-3}
Relative density or Specific gravity	dimensionless	no units

6.3 FORMULAE

1. Pressure $P = \frac{\text{Force}}{\text{Area}} = \frac{F}{A}$

2. Density $= \frac{\text{Mass}}{\text{Volume}} = \frac{m}{V}$

3. Relative density (RD) $= \frac{\text{Density of substance}}{\text{Density of water}}$

4. Hydrostatic pressure in a liquid at a depth h

$$P = h \rho g \quad (\text{gauge pressure})$$

Total pressure $= P_0 + \rho h g$

where P_0 is the atmospheric pressure.

5. When two liquids are used in a manometer to measure the same pressure

$$h_1 \rho_1 g = h_2 \rho_2 g$$

or $\frac{h_1}{h_2} = \frac{\rho_2}{\rho_1}$

6.4 SOME USEFUL DATA

1. Density of water $= 1 \text{ g cm}^{-3} = 1000 \text{ kg m}^{-3}$

2. Atmospheric pressure at S T P $= 1.013 \times 10^{-5} \text{ N m}^{-2}$ or Pa
 $= 1.013 \times 10^6 \text{ dyne cm}^{-2}$
 $= 1.013 \text{ bar}$
 $= 76 \text{ cm of mercury}$

3. Some Typical Values of Density

Table 6.2

Solids		Liquids		Gases at STP	
Solid	kg m ⁻³	Liquid	kg m ⁻³	Gas	kg m ⁻³
Aluminium	2700	Petrol	660-690	Air	1.293
Copper	8930	Glycerine	1260	Hydrogen	0.0899
Brass	8440	Mercury	13600	Oxygen	1.429
Iron	7860	Alcohol	791	Helium	0.1785
Ice	917	Sea-water	1025	Carbon dioxide	1.977
Cork	220-260	Kerosene	800	Methane	0.717
Wood	350-900	Blood	1100	(All values are at 0°C and one atmosphere)	
Diamond	3500	Benzene	880		

6.5 NUMERICAL EXAMPLES

Example 1: What is the force due to the oil acting on a circular plate 5 cm in diameter which covers a hole in the bottom of a tank of oil 1.25 m high? The RD of the oil is 0.5.

Solution: The pressure at the bottom of the tank is given by

$$P = \frac{F}{A}$$

The pressure due to the liquid only is

$$P - P_0 = \rho h g$$

Hence,

$$F = (P - P_0) A = \rho h g A$$

Since RD of the oil is 0.5, its density is

$$\rho = 500 \text{ kg m}^{-3}$$

Now, area

$$\begin{aligned} A &= \pi r^2 = \pi (2.5 \times 10^{-2})^2 \\ &= 6.25 \times 10^{-4} \pi \text{ m}^2 \end{aligned}$$

Therefore, the force is

$$\begin{aligned} F &= 500 \times 1.25 \times 9.8 \times 6.25 \times 10^{-4} \pi \\ &= 12.0 \text{ N} \end{aligned}$$

Example 2: A rectangular tank 2m × 3m is filled to a depth of 0.60 m with water. On top of the water is a layer of oil 1 m deep. The specific gravity of the oil is 0.6. What is the absolute pressure at the bottom, and what is the total thrust exerted on the bottom of the tank? Atmospheric pressure = $1.013 \times 10^5 \text{ N m}^{-2}$.

Solution: As shown in Fig. 6.2 there are two columns of the liquids, water at the bottom and oil above it. Total pressure P_2 at the bottom is given by

$$P_2 = h_2 \rho_2 g + P_1 + P_0$$

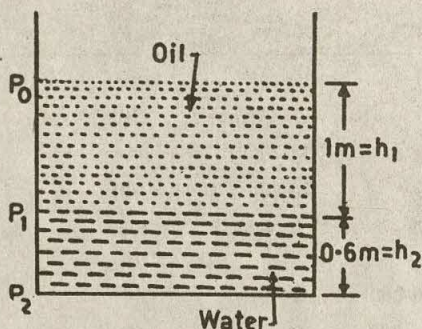


Fig. 6.2

where P_0 is atmospheric pressure and ρ_2 is the density of water.

$$P_1 = h_1 \rho_1 g$$

where ρ_1 denotes the density of oil.

$$\rho_2 = 1000 \text{ kg m}^{-3}, \quad \rho_1 = 0.6 \text{ g cm}^{-3} = 600 \text{ kg m}^{-3}$$

$$P_0 = 1.013 \times 10^5 \text{ N m}^{-2}$$

$$\begin{aligned} P_2 &= 0.6 \times 1000 \times 9.8 + 1 \times 600 \times 9.8 + 1.013 \times 10^5 \\ &= 5880 + 5880 + 1.013 \times 10^5 = 1.1306 \times 10^5 \text{ N m}^{-2} \end{aligned}$$

The total thrust on the bottom is due to the liquids and the atmosphere.

$$\text{Thrust } F = P_2 \times A$$

$$= 1.1306 \times 10^5 \times 2 \times 3 = 6.7836 \times 10^5 \text{ N}$$

Example 3: A piece of copper having an internal cavity weighs 264 g in air and 221 g in water. Find the volume of the cavity. The density of copper = 8.8 g cm^{-3} . (IIT 1963)

Solution: Mass of copper piece in air = 264 g

Mass of copper piece in water = 221 g

Apparent loss of mass = $264 - 221 = 43 \text{ g}$

This is the mass of water displaced by the copper piece when immersed in water.

Volume of copper piece with cavity = 43 cm^3

Volume of copper only = $\frac{m}{\rho} = \frac{264}{8.8} = 30 \text{ cm}^3$

Volume of the cavity = $43 - 30 = 13 \text{ cm}^3$

Example 4: A piece of brass (an alloy of copper and zinc) weighs 12.9 g in air. When completely immersed in water it weighs 11.3 g. What is the volume of copper contained in the alloy? RD of copper and zinc are 8.9 and 7.1 respectively. (IIT 1966)

Solution: Let m be the mass of copper in the alloy. Then the mass of zinc in the alloy = $(12.9 - m)$ g.

$$\text{Volume of copper in the alloy} = \frac{m}{8.9} \text{ cm}^3$$

$$\text{Volume of zinc in the alloy} = \frac{(12.9 - m)}{7.1} \text{ cm}^3$$

$$\text{Apparent loss of weight of brass} = 12.9 - 11.3 = 1.6 \text{ g}$$

$$\text{Volume of the alloy} = 1.6 \text{ cm}^3 \text{ (density of water} = 1 \text{ g cm}^{-3}\text{)}$$

Hence
$$\frac{m}{8.9} + \frac{12.9 - m}{7.1} = 1.6$$

Solving this equation, we get

$$m = 7.61 \text{ g}$$

Example 5: A stone of density 2.5 g cm^{-3} completely immersed in sea-water is allowed to sink from rest. Calculate the depth to which the stone would sink in two seconds. Neglect the effect of friction. Relative density of sea-water is 1.025 and $g = 980 \text{ cm s}^{-2}$. (IIT 1969)

Solution Let m be the mass of the stone.

$$\begin{aligned} \text{Weight of the sea-water displaced} &= \rho V g \\ &= 1.025 \times \frac{m}{2.5} \times g \end{aligned}$$

This must be equal to the buoyant force in the upward direction whereas the weight mg of the stone acts vertically downwards. Both these forces act through the centre of gravity of the stone. Hence, the resultant force is

$$mg - \frac{1.025}{2.5} mg$$

in a vertically downward direction. According to Newton's second law of motion this force = $m \times a$ where a is the acceleration produced. Therefore,

$$mg \left(1 - \frac{1.025}{2.5} \right) = ma$$

or
$$a = \frac{1.475}{2.5} \times 980$$

or
$$a = 578.2 \text{ cm s}^{-2} = 5.782 \text{ m s}^{-2}$$

The distance travelled is given by

$$s = ut + \frac{1}{2} at^2$$

Here
$$u = 0 \quad t = 2 \text{ s} \quad \text{and} \quad a = 5.782 \text{ m s}^{-2}$$

$$s = \frac{1}{2} \times 5.782 \times 4 = 11.564 \text{ m}$$

Hence the stone sinks to a depth of 11.6 m in 2 s.

Example 6: A cubical block of steel of density 7.8 g cm^{-3} floats on mercury (density 13.6 g cm^{-3}) with its sides vertical. Assume the side of the cube to be 10 cm.

- (a) What length of the block is above the mercury surface?
 (b) If water is poured on the mercury surface, what will be the height of the water column, when the water surface just covers the top of the steel block? (IIT 1971)

Solution:

(a) Volume of the steel block $= 10 \times 10 \times 10 = 1000 \text{ cm}^3$

weight of the steel block $= 1000 \times 7.8 \text{ g}$

Volume of the block below the surface is $(10 - l_1) \times 100$

where l_1 is the length of the block above the surface of mercury.

The weight of mercury displaced by the block

$$= (10 - l_1) \times 100 \times 13.6 \text{ g}$$

According to Archimedes' principle, this must be equal to the weight of the steel block. Therefore,

$$(10 - l_1) \times 100 \times 13.6 = 7800$$

or $l_1 = 4.26 \text{ cm}$

(b) Let l_2 be the height of the water column.

Weight of the block = weight of water displaced

+ weight of mercury displaced

$$7800 = l_2 \times 100 \times 1 + (10 - l_2) \times 100 \times 13.6$$

which gives $l_2 = 4.6 \text{ cm}$

Example 7: A vertical U-tube of uniform inner cross-section contains mercury in both sides of its arms. A glycerine (density $= 1.3 \text{ g cm}^{-3}$) column of length 10 cm is introduced into one of its arms. Oil of density 0.8 g cm^{-3} is poured into the other arm until the upper surface of the oil and glycerine are in the same horizontal level. Find the length of the oil column. Density of mercury $= 13.6 \text{ g cm}^{-3}$. (IIT 1972)

Solution: Let $h (=DE)$ be the height of the oil column when the upper surface of oil and glycerine are at the same horizontal level as shown in Fig 6.3 E and B indicate the mercury levels in the two arms. A and B are two points in mercury at the same horizontal level.

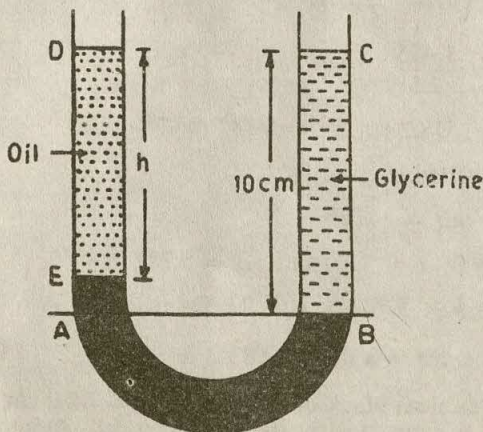


Fig. 6.3

$$\begin{aligned}
 \text{Pressure at } A &= \text{pressure due to oil column of height } h \\
 &+ \text{pressure due to the mercury column of height } EA (= 10 - h) \\
 &= h \times \rho (\text{oil}) \times g + (10 - h) \times \rho (\text{Hg}) \times g \\
 &= 0.8 h g + 13.6 (10 - h) g
 \end{aligned}$$

$$\text{Pressure at } B = 10 \times 1.3 \times g = 13 g$$

Since A and B are at the same horizontal level,

$$\text{Pressure at } A = \text{Pressure at } B$$

$$\text{i.e. } 0.8 h g + 13.6 (10 - h) g = 13 g$$

$$\text{or } h = 9.6 \text{ cm}$$

Example 8: A rod of length 6 m has a mass of 12 kg. It is hinged at one end at a distance of 3 m below the water surface.

- What weight must be attached to the other end of the rod so that 5 m of the rod is submerged?
- Find the magnitude and direction of the force exerted by the hinge on the rod. The SG of the material of the rod is 0.5. (IIT 1976)

Solution: As shown in Fig. 6.4, G is the centre of gravity (CG) of the rod and G' is the centre of gravity of the part of the rod which is under water.

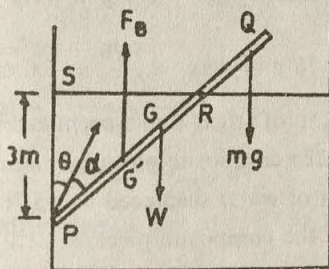


Fig. 6.4

The force of buoyancy F_B acts through G' in a vertically upward direction such that

$$PG' = 2.5 \text{ m}, \quad PG = 3.0 \text{ m}$$

$$\text{The density of the rod} = 0.5 \text{ g cm}^{-3} = 500 \text{ kg m}^{-3}$$

$$\text{Volume of the submerged part, } V = \text{mass/density}$$

$$\text{Mass of 5m of rod} = \frac{12}{6} \times 5 = 10 \text{ kg}$$

$$V = \frac{10}{500} = 0.02 \text{ m}^3$$

The force of buoyancy $F_B = \text{weight of water displaced}$

$$= V \rho g = 0.02 \times 1000 \times 9.8 = 196 \text{ N}$$

(a) For equilibrium of the rod, the algebraic sum of moments of all forces about P must be zero. Taking the anti-clockwise moments to be positive and the clock-wise moments to be negative, we have

$$F_B PG' \sin \alpha - mg PQ \sin \alpha - W \times PG \sin \alpha = 0$$

$$\begin{aligned} \text{or } mg PQ &= F_B PG' - W \times PG \\ &= 196 \times 2.5 - 12 \times 9.8 \times 3 \\ &= 137.2 \end{aligned}$$

$$\therefore m = \frac{137.2}{PQ g} = \frac{137.2}{6 \times 9.8} = 2.33 \text{ kg}$$

(b) For translational equilibrium of the rod, we must have

$$R \sin \theta = 0 \quad \text{or} \quad \theta = 0$$

where R is the reaction of the hinge, The vertical component of reaction R is $R \cos \theta$ for which

$$R \cos \theta + F_B = mg + W$$

$$\begin{aligned} \text{or } R &= mg + W - F_B \\ &= 2.33 \times 9.8 + 12 \times 9.8 - 196 \\ &= -55.6 \text{ N} \end{aligned}$$

Thus the force exerted by the hinge on the rod is 55.6 N.

Example 9: A piece of sugar weighing 40 g is coated with 5.76 g of wax of SG 0.96. If the coated piece weighs 14.76 g in water, find the SG of sugar. (IIT 1969)

$$\text{Solution: Volume of 5.76 g of wax} = \frac{5.76}{0.96} = 6.0 \text{ cm}^3$$

$$\text{Total weight of sugar and wax in air} = 40 + 5.76 = 45.76 \text{ g}$$

$$\text{Weight of the composite piece in water} = 14.76 \text{ g}$$

$$\text{The weight of water displaced} = 45.76 - 14.76 = 31.0 \text{ g}$$

$$\text{Volume of the composite piece} = 31.0 \text{ cm}^3$$

$$\text{Volume of sugar alone} = 31.0 - 6.0 = 25.0 \text{ cm}^3$$

$$\text{Density of sugar} = \frac{40}{25} = 1.6 \text{ cm}^{-3}$$

Example 10: A cubical block of iron 5 cm on each side is floating on mercury in a vessel.

(a) What is the height of the block above mercury level?

(b) Water is poured into the vessel until it just covers the iron block.

What is the height of the water column?

$$\text{Density of mercury} = 13.6 \text{ g cm}^{-3}$$

$$\text{Density of iron} = 7.2 \text{ g cm}^{-3} \quad (\text{IIT 1973})$$

$$\text{Solution: Volume of the iron block} = 5 \times 5 \times 5 = 125 \text{ cm}^3$$

(a) Let h_1 be the height of the block above the mercury surface. Then the weight of mercury displaced

$$= (5 - h_1) \times 25 \times 13.6 \text{ g}$$

This must be equal to the weight of the block

$$= 125 \times 7.2 = 900 \text{ g}$$

Therefore $(5 - h_1) \times 25 \times 13.6 = 900$

or $(5 - h_1) = \frac{900}{340} = 2.65$

or $h_1 = 2.35 \text{ cm}$

(b) Let h_2 be the height of the water column required to just submerge the iron block.

Weight of the block = weight of the water displaced + weight of the mercury displaced

$$900 = h_2 \times 25 \times 1 + (5 - h_2) \times 25 \times 13.6$$

which gives

$$h_2 = 2.54 \text{ cm}$$

Example 11: Compute the volume in m^3 of a life preserver of SG 0.20, which, when worn by a boy weighing 60 kg and having SG equal to 0.9, will just support him, if $3/4$ of his body is submerged in fresh water of density 1000 kg m^{-3} . Assume that the life preserver is completely submerged.

Solution: The weight of the boy W_b and the weight of the preserver W_p acting downward are just balanced by the upward buoyant force of the preserver B_p and the buoyant force of the boy B_b . Therefore,

$$W_b + W_p = B_b + B_p \quad (i)$$

But $B_b = \frac{3}{4} V_b \rho_w g$, $W_b = V_b \rho_b g = 60 \times g$

and $B_p = V_p \rho_w g$, $W_p = V_p \rho_p g$

where g is the acceleration due to gravity. V and ρ denote volume and density respectively.

$$\rho_b = 0.9 \times 1000 = 900 \text{ kg m}^{-3}$$

$$\rho_p = 0.20 \times 1000 = 200 \text{ kg m}^{-3}$$

and $V_b = \frac{W_b}{g \rho_b}$

From Eq. (2), we have

$$\frac{3}{4} V_b \rho_w g + V_p \rho_w g = W_b + V_p \rho_p g$$

or $\frac{3}{4} \frac{W_b}{g \rho_b} \rho_w g + V_p \rho_w g = 60 \times g + V_p \rho_p g$

or $\frac{3}{4} \times \frac{60 \times g}{\rho_b} \rho_w + V_p \rho_w g = 60 g + V_p \rho_p g$

or $45 \frac{\rho_w}{\rho_b} + V_p \rho_w = 60 + V_p \rho_p$

$$\begin{aligned}
 \text{or } V_p (\rho_w - \rho_p) &= 60 - 45 \frac{\rho_w}{\rho_p} \\
 &= 60 - \frac{45 \times 1000}{900} \\
 &= 10
 \end{aligned}$$

$$\therefore V_p = \frac{10}{\rho_w - \rho_p} = \frac{10}{800} = 1.25 \times 10^{-2} \text{ m}^3$$

Volume of the life preserver = 0.0125 m^3

Example 12: A U-tube is partially filled with water. Another liquid which is not miscible with water, is poured into one arm until it stands at a height d above the water level in the other arm, which has meanwhile risen a distance l (see Fig 6.5). Find the relative density of the liquid.

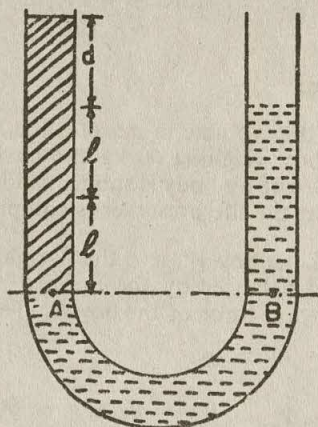


Fig. 6.5

Solution: The point A and B are at the same pressure, hence the pressure drop from A and B to the open surface in each limb is the same, for each surface is at the atmospheric pressure.

The pressure drop on the water side = $\rho \text{ (water)} \times g \times 2l$

The factor $2l$ enters in the expression because the water column has risen a distance l on one side and fallen a distance l on the other side, from its initial position.

The pressure drop on the other side = $\rho \times g \times (d + 2l)$

where ρ is the density of the unknown liquid.

Therefore $\rho \text{ (water)} \times g \times 2l = \rho \times g \times (d + 2l)$

Hence the RD of the liquid is given by

$$\text{RD} = \frac{\rho}{\rho \text{ (water)}} = \frac{2l}{2l + d}$$

Example 13: Two identical cylindrical vessels with their base at the same level each contain a liquid of density ρ . The area of either base is A , but in one vessel the liquid height is h_1 and in the other h_2 . Find the work done by gravity in equalizing the levels when the two vessels are connected.
(IIT 1981)

Solution: After the levels in the two vessels become equal, the increase in height of the liquid in one vessel is $\frac{1}{2} (h_2 - h_1)$ with the same decrease in height in the other. Thus, effectively a slab of liquid $\frac{1}{2} (h_2 - h_1)$ in thickness falls a vertical distance equal to its thickness under the action of gravity. Therefore,

Work done by the gravity is

$$W = m g \times \frac{1}{2} (h_2 - h_1)$$

where mass of the slab m is given by

$$m = \rho \times V = \rho \times \frac{1}{2} (h_2 - h_1) \times A$$

Therefore
$$W = \frac{1}{2} \rho (h_2 - h_1) \times A \times g \times \frac{1}{2} (h_2 - h_1)$$
$$= \frac{1}{4} \rho A g (h_2 - h_1)^2$$

Example 14: The density of ice is 0.92 g cm^{-3} and that of marine water is 1.03 g cm^{-3} . What percentage of the volume of an iceberg is submerged?

Solution: Let V be the volume of the iceberg, then the weight of the iceberg $= \rho (\text{ice}) \times V \times g$

Let V_{sub} be the volume of the iceberg that is submerged. The weight of the water displaced equals

$$\rho (\text{water}) \times V_{\text{sub}} \times g$$

For the iceberg to float, its weight must be balanced by the buoyant force on it, which is equal to the weight of the water displaced. Hence,

$$\rho (\text{ice}) \times V \times g = \rho (\text{water}) \times V_{\text{sub}} \times g$$

$$\frac{V_{\text{sub}}}{V} = \frac{\rho (\text{ice})}{\rho (\text{water})} = \frac{0.92}{1.03} = 0.89$$

Thus 89% of the volume of the iceberg is under water.

Example 15: A hydraulic press consists of a cylinder 12 cm in diameter, and a smaller cylinder 2 cm in diameter, filled with oil. A closely fitting piston is free to move in contact with the oil in the larger cylinder while another piston fits in the smaller cylinder. In order for the larger piston to overcome a force exerted by 5000 kg load, how much pressure must be developed in the oil? How much force must be applied to the smaller piston? If this is done by the use of a lever 0.5 m long whose fulcrum is 3 m from the piston end, what force is required at the end of the arm?

Solution: As illustrated in Fig. 6.6 (a), the pressure developed in the oil is given by

$$p = \frac{f}{a} = \frac{F}{A} \quad (\text{using Pascal's principle})$$

$$= \frac{m g}{\pi r^2}$$

$$= \frac{5000 \times 9.8}{\pi (6 \times 10^{-2})^2} = 4.33 \times 10^6 \text{ N m}^{-2}$$

The force applied to the smaller piston is

$$f = \frac{Fa}{A} = \frac{5000 \times 9.8 \times \pi \times (1 \times 10^{-2})^2}{\pi \times (6 \times 10^{-2})^2}$$

$$f = \frac{5000 \times 9.8}{36} = 1361.1 \text{ N}$$

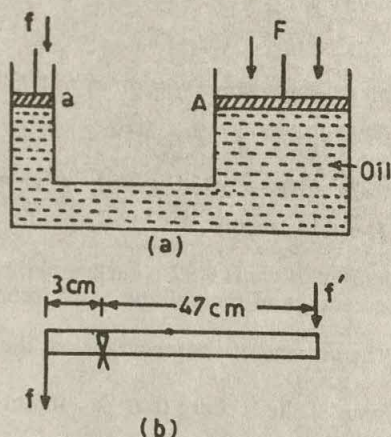


Fig. 6.6

Taking the moments about the fulcrum, we have

$$f' \times 0.47 = f \times 0.03$$

$$f' = \frac{1361.1 \times 0.03}{0.47} = 86.9 \text{ N}$$

Example 16: A wooden plank of length 1 m and uniform cross-section is hinged at one end to the bottom of a tank. The tank is filled with water upto a height of 0.5 m. The SG of the plank is 0.5. Find the angle θ that the plank makes with the vertical in the equilibrium position. (Exclude the case $\theta = 0$). (IIT 1984)

Solution: Let l be the length and a the area of cross-section of the plank (see Fig. 6.7).

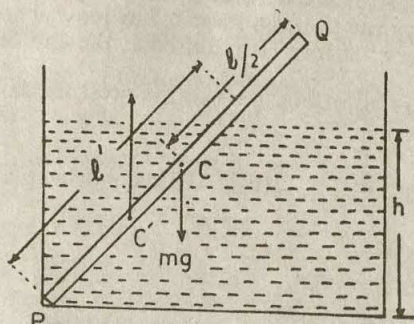


Fig. 6.7

If its density is ρ , the weight of the plank is

$$W = mg = l a \rho g$$

This is the gravitational force acting vertically downwards at C , the centre of gravity of the plank. If l' is the length of the plank inside water, the buoyant force acting vertically upwards at C' (distant $l'/2$ from the hinge) is given by

$$F_B = a l' \rho' g$$

where ρ' is the density of water. The couple acting on the plank due to the gravitational force

$$= (a l \rho g) \times \frac{l}{2} \sin \theta$$

The couple acting on the plank due to the buoyant force tending to rotate it in the opposite direction

$$= (a l' \rho' g) \times \frac{l'}{2} \sin \theta$$

For the plank to be in equilibrium, these two couples must cancel. Hence

$$a l \rho g \frac{l}{2} \sin \theta = a l' \rho' g \frac{l'}{2} \sin \theta$$

$$\text{or } l^2 \rho = l'^2 \rho' = l'^2 \quad (\because \rho' = 1 \text{ g cm}^{-3})$$

$$\therefore l' = l \sqrt{\rho} = 100 \times \sqrt{0.5} \text{ cm}$$

$$\cos \theta = \frac{h}{l'} = \frac{50\sqrt{2}}{100} = \frac{1}{\sqrt{2}}$$

$$\therefore \theta = 45^\circ$$

6.6 TRUE-FALSE STATEMENTS WITH REASONS

A few statements are given below. We have to decide whether they are true or false giving, in brief, reasons in support of our answers.

1. The buoyant force of water on a submerged wooden cube is greater than on a steel cube of equal volume.
2. By tying a stone to board, an object could be made which would float in a river and sink in the ocean.
3. The pressure on the bottom of tank (a) is greater than that at the bottom of (b).

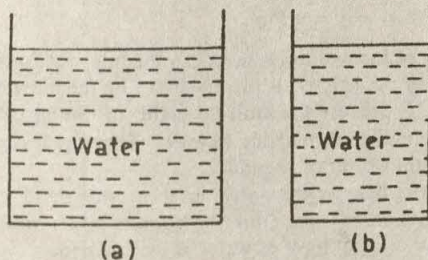


Fig. 6.8

4. A solid cube of ice is floating in a glass of water, with the water level just even with the top of the glass. When the ice melts, some water overflows.

5. An ice cube floats in a beaker of strong brine solution (density = 1.1 g cm^{-3}) such that the beaker is full up to the rim. When the ice melts, the water level will be the same.
6. If you float in water with just your nose out, the average density of your body must be the same as that of water.
7. A hydrogen filled balloon stops rising after it has obtained a certain height in the sky. This happens because the atmospheric pressure decreases with height and becomes zero when maximum height is attained.
8. A block of wood is floating in a tank containing water. The apparent weight of the floating block is equal to zero.
9. A man is sitting in a boat which is floating on a pond. If the man drinks some water from the pond, the level of the water in the pond decreases.
10. A block of ice with a lead shot embedded in it is floating on water contained in a vessel. The temperature of the system is maintained at 0°C as the ice melts. When the ice melts completely the level of water in the vessel rises. (IIT 1986)

ANSWERS

1. This is false because the upward force will be the same in both cases and equals the weight of the liquid displaced by the body. For a heavier body the buoyant force will not be sufficient to keep it afloat.
2. The statement is false because the density of marine water is greater than the density of river water and hence the buoyant force will be larger in sea-water to keep the body afloat.
3. The statement is false because pressure depends on the height of the liquid, which is same in the two tanks ($p = \rho h g$). It does not depend on the width or the shape of the containers.
4. The statement is false. If one gram of ice melts, the volume displaced is 1 cm^3 less. But the volume of water formed is 1 cm^3 . This means that the net change in the water level is zero. Hence the water level remains unchanged as the ice melts.
5. The statement is not true because the density of brine is more than 1 g cm^{-3} . This means that for x grams of ice, the volume of brine displaced is $x/\rho = x'$ (say). But on melting x gram of ice will produce $x \text{ cm}^3$ of water. Since $x > x'$, the solution will overflow. The volume of the brine displaced by the ice cube is smaller than the volume of the water produced when ice melts.
6. The statement is true because in this case, the weight of the water displaced (which is equal to the buoyant force acting vertically upwards) is just equal to the weight of the body which acts in the vertically downward direction.
7. The statement is false. As the height increases, the density of air decreases, resulting in a decrease in the buoyant force, which is equal to the weight of the volume of air displaced by the balloon.

$$F_B = mg \pm \rho V g$$

Initially the weight of the balloon is less than the buoyant force F_B . At maximum height, the weight of the balloon is just balanced by the buoyant force.

8. The statement is true because the entire weight of the block is supported by the buoyant force (the upward thrust) due to water. Thus the loss in weight in water is just equal to the total weight of the block.
9. The statement is false because the water level in the pond remains unchanged. When the man drinks some water from the pond, a larger volume of water is displaced and thus the original level of water is maintained.
10. The statement is false. The water level in the vessel will remain unchanged. The volume of (the melted ice + lead shot) is equal to the volume of the water displaced when the ice block with a lead shot is initially placed in water.

6.7 MULTIPLE-CHOICE QUESTIONS

Choose the correct answer from the given alternatives.

- The dimensions of pressure are
 (a) $[M L^2 T^{-2}]$ (b) $[M L T^{-1}]$
 (c) $[M L^{-1} T^{-2}]$ (d) $[M L^{-2} T^{-2}]$
- During a tornado, air pressure is $8 \times 10^4 \text{ N m}^{-2}$ inside a house and $1.0 \times 10^5 \text{ N m}^{-2}$ outdoors. The net force on a window of area 0.5 m^2 is
 (a) $1.0 \times 10^4 \text{ N}$ (b) $4.0 \times 10^4 \text{ N}$
 (c) $5.0 \times 10^4 \text{ N}$ (d) $6.0 \times 10^4 \text{ N}$
- A body weighs 5 N in air and 2 N when immersed in a liquid. The buoyant force is
 (a) 2 N (b) 3 N
 (c) 5 N (d) 7 N
- The pressure of a confined air is p and that of atmosphere is P . Then
 (a) P is equal to p
 (b) P is less than p
 (c) P is greater than p
 (d) P may be less than or greater than p depending on the mass of the confined air.

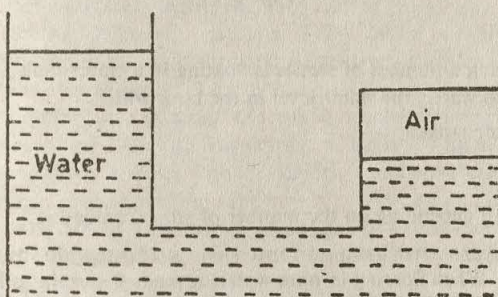


Fig. 6.9

- A body floats in a liquid contained in a beaker. The whole system, shown in Fig. 6.10 falls freely under gravity. The upthrust on the body due to the liquid is
 (a) zero
 (b) equal to the weight of the liquid displaced
 (c) equal to the weight of the body in air
 (d) equal to the weight of the immersed portion of the body.

(IIT 1982)

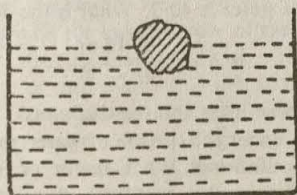


Fig. 6.10

6. A U-tube of uniform cross-section is partially filled with a liquid I (Fig. 6.11). Another liquid II which does not mix with liquid I is poured into one side. It is found that the liquid levels of the two sides of the tube are same, while the levels of liquid I has risen by 2 cm. If the SG of liquid I is 1.1, the specific gravity of liquid II must be

(a) 1.12 (b) 1.10
(c) 1.05 (d) 1.00 (IIT 1983)

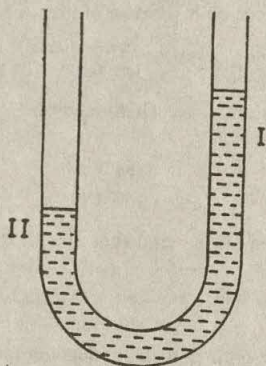


Fig. 6.11

7. A boat carrying a number of stones is floating in a water tank. If the stones are unloaded into water, the water level in the tank will
(a) remain unchanged
(b) rise
(c) fall
(d) rise or fall depending on the number of stones unloaded.
8. A common hydrometer has a uniform stem graduated downwards from 0, 1, 2, ... upto 10. When floating in pure water, it reads 0 and in a liquid of SG 1.5, it reads 10. What is the SG of a liquid in which it reads 5?
(a) 1.15 (b) 1.20
(c) 1.25 (d) 1.30
9. A wooden cube just floats inside water when a 200 g mass is placed on it. When the mass is removed, the cube is 2 cm above the water level. What is the size of the cube?
(a) 6 cm (b) 8 cm
(c) 10 cm (d) 12 cm
10. The volume of an air bubble is doubled as it rises from the bottom of a lake to its surface. The atmospheric pressure is 75 cm of mercury and the ratio of the density of mercury to that of lake water is 40/3. What is the depth of the lake?
(a) 10 m (b) 15 m
(c) 20 m (d) 25 m
11. A glass plate *X* (of negligible mass and the thickness) is held against the end of a tube *Y* and pushed 10 cm under the surface of water as shown in Fig. 6.12. When released, the plate does not fall off. What depth of kerosene (SG = 0.8) must be poured in tube *Y* so that the plate just falls off?
(a) 10 cm (b) 10.5 cm
(c) 12.0 cm (d) 12.5 cm

12. A man is carrying a block of a certain material (density 1000 kg m^{-3}) weighing 1 kg in his left hand and a bucket full of water and weighing 10 kg in his right hand. He drops the block in the bucket. How much load does he carry in his right hand now?
- (a) 9 kg (b) 10 kg
(c) 11 kg (d) 12 kg

6.8 NUMERICAL EXERCISES

(Assume $g = 9.8 \text{ m s}^{-2}$ and density of water $= 1000 \text{ kg m}^{-3}$, unless otherwise stated.)

- The interior of a submarine located at a depth of 60 m in sea-water is maintained at sea-level atmospheric pressure. Find the force acting on a window 40 cm^2 in area. The density of sea-water is 1030 kg m^{-3} .
- If the earth's atmosphere were everywhere of the same density ($= 1.290 \text{ kg m}^{-3}$) as at the surface of the earth, how high would the atmosphere have to extend in order to give rise to the observed atmospheric pressure of $1.013 \times 10^5 \text{ N m}^{-2}$?
- A hydraulic press has an input cylinder 3 cm in diameter and an output cylinder 15 cm in diameter. Assuming 100% efficiency, find the force exerted by the output piston when a force of 15 N is applied to the input piston.
- A block of brass has a mass of 1 kg and a density 8000 kg m^{-3} . It is suspended from a string. Find the tension in the string if the block is in air and if it is completely immersed in water. Neglect the buoyancy of the air.
- A balloon weighing 100 kg has a capacity of 1000 m^3 . If it is filled with hydrogen, how large a payload in kg can it support? At sea level, the density of hydrogen is 0.09 kg m^{-3} and that of air is 1.3 kg m^{-3} .
- What is the minimum area of a slab of ice 0.50 m thick floating on water that will hold up a car weighing 1000 kg? The density of ice is 920 kg m^{-3} .
- A block of wood floats in water with $2/3$ of its volume submerged. In oil it has 0.90 of its volume submerged. Find the densities of wood and oil.
- A metal sphere of density 10 g cm^{-3} is placed gently on the surface of water in a lake 450 m deep. Neglecting viscous effects, find the time the sphere will take to reach the bottom of the lake. Take $g = 10 \text{ m s}^{-2}$.
- A cubical block of wood 10 cm on a side and of density 0.5 g cm^{-3} floats in a jar of water. Oil of density 0.8 g cm^{-3} is poured on the water until the top of the oil layer is 4 cm below the top of the block. Calculate (a) the depth of the oil layer and (b) the gauge pressure at the lower face of the block.
- A hollow glass stopper just sinks in water. Compare the volume of the cavity to that of the stopper. The SG of glass is 2.5.
- An acid of mass 108 g and SG 1.8 is mixed with 108 g of water. If the SG of the mixture is 1.5, find the change in volume when the liquids are mixed.
- A cubical block of wood 10 cm along each side floats at the interface between an oil and water with its lower surface 2 cm below the interface. The heights of the oil and water columns are 10 cm each. The density of oil is 0.8 g cm^{-3} .
(a) What is the mass of the block?
(b) What is the pressure on the lower surface of the block? (IIT 1977)
- A flat-bottomed thin-walled glass tube has a diameter of 4 cm and it weighs 30 g. The centre of gravity of the empty tube is 10 cm above the bottom. Find the amount of water which must be poured in the tube so that, when it is floating vertically, in a tank of water, the centre of gravity of the tube and its contents is at the mid-point of the immersed length of the tube. (IIT 1968)
- A solid weighs 237.5 g in air and 12.5 g when totally immersed in a liquid of density 0.9 g cm^{-3} . Calculate (a) the density of the solid (b) the density of a liquid in which the solid would float with one fifth of its volume exposed above the liquid surface.

15. The volume of a hot air balloon is 600 m^3 and the density of the surrounding air is 1.25 kg m^{-3} . The balloon just hovers clear of the ground when the burner has heated the air inside to a temperature at which its density is 0.80 kg m^{-3} . (a) what is the total mass of the balloon including the hot air inside it? What is the total mass of the envelope of the balloon and its load? (c) Find the acceleration with which the balloon will start to rise when the density of the air inside is reduced to 0.75 kg m^{-3} . Take $g = 10 \text{ m s}^{-2}$.
16. A large wooden block of uniform cross-section sinks so its lower surface is 6 cm deep in a salt solution of SG = 1.15.
- (a) How deep will it sink in pure water?
- (b) If the vertical dimension of the block is 8 cm, what is its density?
- (c) A certain cork put on top of the block, makes it float 1 cm deeper in pure water. But if the cork is put under the bottom of the block, it floats 1 cm higher. When cork itself is floated on water, what fraction of its volume will be above the surface?

ANSWERS

MULTIPLE CHOICE QUESTIONS

- | | | | | |
|---------|---------|--------|--------|---------|
| 1. (c) | 2. (a) | 3. (b) | 4. (b) | 5. (a) |
| 6. (b) | 7. (c) | 8. (b) | 9. (c) | 10. (a) |
| 11. (d) | 12. (b) | | | |

NUMERICAL EXERCISES

- $9.69 \times 10^4 \text{ N}$
- 8.0 km
- 1250 N
- 9.8 N, 8.575 N
- 1110 kg
- 25 m^2
- 0.67 g cm^{-3} , 0.74 g cm^{-3}
- 10 s
- (a) 5 cm, (b) 490 Pa
- 3 : 5
- 24 cm^3
- (a) 840 g (b) $9800 \text{ dyne cm}^{-2}$
- 110.6 g
- (a) 0.95 g cm^{-3} (b) 1.19 g cm^{-3}
- (a) 750 kg (b) 270 kg (c) 0.42 m s^{-3}
- (a) 6.9 cm (b) 0.862 g cm^{-3} (c) 0.50

Elasticity

7.1 REVIEW OF BASIC CONCEPTS

7.1.1 Elastic Forces One can easily feel the force exerted by a rubber band when it is stretched and also see the distortion produced in it. What we feel is the reaction to the force that we apply to cause distortion. The change in the shape or size of a body when external forces act on it is determined by the forces between its atoms or molecules. These short range atomic forces are called *elastic forces*. A substance is said to be perfectly elastic if it recovers its original size and shape when the external forces are removed. Most substances such as copper or steel are perfectly elastic upto a point beyond which perfect or complete recovery does not occur. This point is called the *elastic limit*.

7.1.2 Stress Stress is defined as the ratio of the internal force F , produced when the substance is deformed, to the area A over which this force acts. In equilibrium, this force is equal in magnitude to the externally applied force. In other words,

$$\text{Stress} = \frac{F}{A} \quad (7.1)$$

The SI unit of stress is newtons per square metre (N m^{-2}). In CGS units, stress is measured in dyne cm^{-2} .

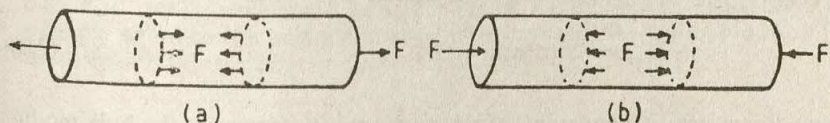


Fig. 7.1

If the ends of a rod of uniform cross-section are subjected to two equal and opposite forces as shown in Fig. 7.1 (a) or (b), the rod is said to be under tension. The stress produced is called *tensile stress*. Each portion of the rod pulls or pushes the other. Any arbitrary cross-section must be acted upon by equal and opposite internal forces of magnitude F distributed perpendicular to the area everywhere. This is called normal stress. If the external force is not at right angles to the area, we may resolve the force into normal and tangential components giving rise to normal stress and tangential stress. Tangential stress is the ratio of the tangential force to the area over which it acts.

7.1.3 Strain Strain is defined as the ratio of the change in size or shape to the original size or shape. It has no dimensions, it is just a number.

7.1.4 Hooke's Law Hooke's law states that, within elastic limits, the ratio of stress to the corresponding strain produced is a constant. This constant is called the modulus of elasticity. Thus

$$\text{Modulus of elasticity} = \frac{\text{stress}}{\text{strain}}$$

Since strain is a pure number, the units of this constant are the same as those of stress, i.e. Nm^{-2} .

7.1.5 Elastic Modulii There are three different kinds of strains and hence three kinds of modulii of elasticity are defined as follows.

(i) *Young's modulus* When a wire of length l is suspended from a rigid support and stretched by a weight Mg attached to its lower end, the increase in length, Δl , results in a strain $\Delta l/l$. This produces a tension T in the wire, which equals the force Mg when equilibrium is reached. The ratio of tensile stress to tensile strain is called Young's modulus of the material of the wire and is usually denoted by Y . Thus

$$Y = \frac{\text{Tensile stress}}{\text{Tensile strain}} = \frac{F/A}{\Delta l/l} = \frac{F}{A} \frac{l}{\Delta l} \quad (7.2)$$

Here $F = Mg$ and A is the area of cross-section of the wire.

The compressive strain of a bar in compression is defined as the ratio of decrease in length Δl to the original length l . Young's modulus for compression is identical in value to Young's modulus for tension in the same material. The value of Y for a given material is a constant and depends only on the nature of the material.

(ii) *Bulk modulus or Coefficient of volume elasticity* Application of pressure reduces the volume of matter. In this case, the applied pressure (force per unit area) is the stress. Suppose a sample of liquid or solid, already under pressure, undergoes a pressure change of magnitude ΔP . This causes the volume to change from V to $V + \Delta V$. In elastic materials, the volume change ΔV is proportional to V itself and to ΔP . Thus

$$\frac{\Delta P}{-\Delta V/V} = \text{constant} = B \quad (7.3)$$

where B is a positive constant of proportionality known as the bulk modulus or the coefficient of volume elasticity. The minus sign indicates that an increase in pressure results in a decrease in volume. Thus B is always positive. For solids and liquids, under pressure, the change in volume is so small that the value of B is independent of the initial volume V , provided that the pressure is not too large.

The reciprocal of B is called *compressibility*, β and has the units of reciprocal of pressure. Thus

$$\text{Compressibility } (\beta) = \frac{1}{B}$$

The volume change in a gas is relatively large. In the limit when ΔP , ΔV become very small, we can write

$$B = -V \frac{dP}{dV} \quad (7.4)$$

The bulk modulus of gases depends on the pressure. Under isothermal conditions (constant temperature) the bulk modulus of a gas is equal to its pressure. On the other hand, under adiabatic conditions (when no heat is allowed to enter or leave the system), the bulk modulus equals γP , where γ is the ratio C_p/C_v of the molar heat capacities at constant pressure and volume of the gas. Thus

Isothermal bulk modulus = P

Adiabatic bulk modulus = γP

(iii) *Shear modulus or modulus of rigidity*: The shape of a solid can undergo a change without a change in volume. Such a deformation is called a shear and may be produced by applying equal and opposite forces along the top and bottom surfaces of a rectangular block as shown in Fig. 7.2.

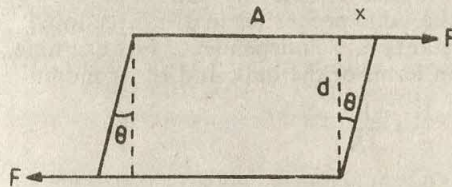


Fig. 7.2

The shearing stress set up in the material is the ratio of F to the area A of the surface over which the applied force F acts. Thus

$$\text{Shearing stress} = \frac{\text{Tangential force } (F)}{\text{Area } (A)}$$

Each layer of the material parallel to the top and bottom surfaces is acted upon by equal and opposite forces tending to shear the material. The shearing strain is given by (see Fig. 7.2)

$$\text{Shearing strain} = \frac{x}{d} = \tan \theta$$

Since θ is usually very small (x is always $\ll d$), $\tan \theta \approx \theta$. Thus, the strain is simply equal to the angle θ measured in radians. Applying Hooke's law, the shear modulus, denoted by n , is given by

$$n = \frac{\text{Shearing stress}}{\text{Shearing strain}} = \frac{\frac{F}{A}}{x/d} = \frac{F}{A\theta} \quad (7.5)$$

The shear modulus is also known as the modulus of rigidity. Since liquids and gases have no shape of their own (they take up the shape of the vessel in which they are contained), the modulus of rigidity has a significance only for solids.

If a cylinder (a wire or rod) is fixed at one end and a torque τ (co-axial with the cylinder) is applied to the other end, the cylinder is twisted about its axis. If the angle of twist is θ , the torque is given by

$$\tau = n \frac{\pi r^4}{2l} \theta \quad (7.6)$$

where n is the shear modulus, r the radius of the cylinder and l its length. The ratio τ/θ is known as the torsional torque per unit twist or the torsional rigidity of the cylinder.

7.1.6 Poisson's Ratio Our definitions of the various elastic constants have been limited to highly idealized cases of 'pure' stretch, 'pure' shear, etc. In actual practice, elastic deformations are more complex. Whenever a wire is stretched, there is a slight change in the volume of wire because the dimensions in a direction perpendicular to the direction of stress become shorter. Poisson's ratio σ is defined as the ratio of the fractional change in diameter to the fractional change in length. Thus

$$\sigma = \frac{\Delta r/r}{\Delta l/l} = \frac{\Delta r}{\Delta l} \frac{l}{r} \quad (7.7)$$

Here r is the radius of the wire. For a material, σ is a dimensionless constant. For most materials, σ has a value between 0.1 and 0.4.

7.1.7 Relations between Elastic Moduli For isotropic materials (i.e. materials having the same properties in all directions), only two of the three elastic constants are independent. For example, Young's modulus can be expressed in terms of the bulk and shear moduli.

$$\frac{3}{Y} = \frac{1}{n} + \frac{1}{3B} \quad (7.8)$$

Also

$$Y = 3B(1 - 2\sigma) = 2n(1 + \sigma) \quad (7.9)$$

7.1.8 Breaking Stress The ultimate tensile strength of a material is the stress required to break a wire or a rod by pulling on it. The breaking stress of the material is the maximum stress which a material can withstand. Beyond this point breakage occurs.

7.1.9 Strain Energy When a wire of original length l is stretched by a length Δl by the application of force F at one end, we have (see Eq. 7.2)

$$F = \frac{\Delta l \cdot AY}{l} \quad (7.10)$$

Since the force increases uniformly from 0 to F while the wire stretches, the average force applied is $F/2$. Hence the work done is given by

$$\begin{aligned} W &= \text{Force} \times \text{distance} = \text{Average force} \times \text{extension} \\ &= \frac{1}{2} F \Delta l \end{aligned} \quad (7.11)$$

This is the energy stored in the wire. This is the increase in the potential energy of the molecules due to their displacement from their average positions.

From Eqs (7.10) and (7.11) we have

$$W = \frac{1}{2} \frac{YA(\Delta l)^2}{l} \quad (7.12)$$

Since the volume of the wire is Al , the energy per unit volume stored in the wire is given by, using Eq. (7.11),

$$\frac{W}{lA} = \frac{1}{2} \left(\frac{F}{A} \right) \left(\frac{\Delta l}{l} \right)$$

$$\text{or} \quad \frac{W}{\text{volume}} = \frac{1}{2} \text{ stress} \times \text{strain} \quad (7.13)$$

When a weight Mg falls through a vertical distance to stretch a wire, the gravitational potential energy lost by the weight equals $Mg (\Delta l)$. Half of this energy is the potential energy gained by the wire; the other half is the energy which is lost in the form of heat when the weight comes to rest after making vibrations.

7.1.10 The Force Constant Any spring, wire or bar, that obeys Hooke's law, has a force constant. From Eq. (7.10), the tensile force is

$$F = \frac{YA \Delta l}{l} = k x \quad (7.14)$$

where $k = YA/l$, is a constant for the given spring or wire or rod and x represents the elongation produced by the force F . This is the original form of Hooke's law which tells us that the elongation produced in a body is directly proportional to the stretching force. For a spring, constant k depends on the shear modulus of the wire, its radius, the radius of the coils and the number of coils. The constant k is also known as the stiffness of the spring and is expressed in newton per metre (Nm^{-1}).

7.1.11 Vibrations of a Spring A simple harmonic motion is defined as the motion for which the restoring force is directly proportional to the displacement from the equilibrium position. Hooke's law expressed in Eq. (7.14) gives the simplest way of obtaining simple harmonic motion; by attaching a mass m on the end of a spring. The time period of the oscillations produced in this way is given by

$$T = 2\pi \sqrt{\frac{m}{k}} \quad (7.15)$$

where $k = \frac{F}{x} = \frac{mg}{x}$

7.1.12 Thermal Stresses If a bar is fixed at its ends, a change in the temperature of the bar, results in forces being exerted at the ends of the bar. Thus, due to expansion or contraction, tensile or compressive stresses are set up in the bar. These are known as thermal stresses. If the bar were free to contract or expand, the change in length Δl due to a change Δt in temperature is given by

$$\Delta l = \alpha l \Delta t \quad (7.16)$$

where α is the coefficient of linear expansion of the bar. Combining Eq. (7.16) with Eq. (7.10) we obtain

$$F = \alpha A Y \Delta t \quad (7.17)$$

Thus, the tensile stress in the rod is

$$\frac{F}{A} = \alpha Y \Delta t \quad (7.18)$$

Similarly if a material is enclosed in a rigid container so that its volume can not change, then an increase in temperature Δt results in an increase in pressure ΔP . We then have following relation:

$$\Delta P = B\beta' \Delta t \quad (7.19)$$

where B is the bulk modulus and β' , the coefficient of volume expansion.

7.2 SI UNITS

Table 1.1

Physical Quantity	SI Unit	Symbol
Stress	newtons per square metre	N m^{-2}
Elastic moduli (Y, B, n)	newtons per square metre	N m^{-2}
Compressibility	square metres per newton	$\text{m}^2 \text{N}^{-1}$
Strain energy	joule	J
Force constant or stiffness constant	newtons per metre	N m^{-1}

7.3 FORMULAE

$$1. \text{ Stress} = \frac{\text{Force}}{\text{Area}} = \frac{F}{A}$$

$$2. \text{ Strain} = \frac{\text{change in dimension}}{\text{Original dimension}}$$

$$3. \text{ Young's modulus} = \frac{\text{Tensile or compressive stress}}{\text{Longitudinal strain}} = \frac{Fl}{A \Delta l}$$

$$4. \text{ Bulk modulus} = \frac{\text{Hydrostatic pressure}}{\text{Volume strain}} = - \frac{\Delta PV}{\Delta V}$$

$$5. \text{ Shear modulus or modulus of rigidity}$$

$$= \frac{\text{Tangential stress}}{\text{Shear strain}} = \frac{F}{A\theta}$$

$$6. \text{ Torsional torque } \tau = \frac{n\pi r^4 \theta}{2l}$$

$$7. \text{ Poisson's ratio} = \frac{\text{Lateral strain}}{\text{Longitudinal strain}} = \frac{\Delta r}{r \Delta l}$$

$$8. \text{ Relations between elastic constants}$$

$$Y = 3B(1 - 2\sigma)$$

$$Y = 2n(1 + \sigma)$$

$$\frac{3}{Y} = \frac{1}{n} + \frac{1}{3B}$$

$$9. \text{ Energy stored in a wire or rod}$$

$$(a) W = \text{average force} \times \text{distance} = \frac{1}{2} F \Delta l = \frac{1}{2} \frac{F^2 l}{AY}$$

$$(b) \text{ Energy per unit volume } \frac{W}{V} = \frac{1}{2} \text{ stress} \times \text{strain}$$

$$10. \text{ Force constant } k = F/x \text{ where } k = YA/l \text{ and } x \text{ represents elongation or compression.}$$

$$11. \text{ Time period of an oscillating spring } T = 2\pi \sqrt{\frac{m}{k}}$$

12. Thermal stress $F/A = \alpha Y \Delta t$ (linear expansion)

$$\Delta P = B\beta' \Delta t \text{ (volume expansion)}$$

7.4 SOME TYPICAL VALUES OF ELASTIC CONSTANTS

Table 7.2

Material	Young's modulus (Nm ⁻²)	Shear modulus (Nm ⁻²)	Bulk modulus (Nm ⁻²)	Poisson's ratio
Aluminium	7.0×10^{10}	3.0×10^{10}	7.0×10^{10}	0.16
Brass	9.1×10^{10}	3.6×10^{10}	6.1×10^{10}	0.26
Copper	11.0×10^{10}	4.2×10^{10}	14×10^{10}	0.32
Glass	5.5×10^{10}	2.3×10^{10}	3.7×10^{10}	0.19
Steel	20×10^{10}	8.4×10^{10}	16×10^{10}	0.19

Compressibility of mercury = $3.7 \times 10^{-11} \text{ m}^2 \text{ N}^{-1}$, $B = 2.7 \times 10^{10} \text{ Nm}^{-2}$

Compressibility of water = $45.8 \times 10^{-11} \text{ m}^2 \text{ N}^{-1}$,

$$B = 0.22 \times 10^{10} \text{ Nm}^{-2}$$

Compressibility of air = $1 \times 10^{-5} \text{ m}^2 \text{ N}^{-1}$, $B = 1 \times 10^5 \text{ N m}^{-2}$

Thus, the pressure required to produce a small relative change in volume is about 70 times as much for steel as for water.

7.5 NUMERICAL EXAMPLES

Example 1: A wire cable 10 m long consists of 40 strands of steel each $5 \times 10^{-6} \text{ m}^2$ in cross-section. By how much does the cable stretch when it is used to lift a crate weighing 4000 N? Young's modulus of steel

$$= 20 \times 10^{10} \text{ Nm}^{-2}$$

Solution: Stress = $\frac{4000}{5 \times 10^{-6}} = 8.0 \times 10^8 \text{ Nm}^{-2}$

Young's modulus $Y = \text{Stress/Strain}$

$$\begin{aligned} \text{Strain} &= \frac{\text{Stress}}{Y} = \frac{8.0 \times 10^8}{20 \times 10^{10}} \\ &= 4.0 \times 10^{-3} \end{aligned}$$

Now, Strain = $\frac{\Delta l}{l} \times 40$, where Δl is the increase in length in each strand.

Therefore,

$$\Delta l = \text{Strain} \times \frac{l}{40} = \frac{10 \times 4.0 \times 10^{-3}}{40} = 1 \times 10^{-3} \text{ m} = 1 \text{ mm}$$

Since each strand stretches by 1 mm, the cable is stretched by 1 mm. Alternatively, one could divide the stress by 40 to obtain the stress for each strand, since the load must distribute equally over the entire cross-section of the cable.

Example 2: A boy's catapult is made of a rubber cord 42 cm long and 6 mm in diameter. The boy stretches the cord by 20 cm. Find the Young's modulus of the rubber if a stone weighing 0.02 kg when catapulted flies with a velocity of 20 ms^{-1} . Disregard the change in the cross-section of the cord in stretching.

Solution: Due to the extension produced in the cord, energy is stored in it which is converted into kinetic energy when the stone flies away. Assuming that there is no loss of energy in this process, the kinetic energy of the stone is given by

$$\begin{aligned} W &= \frac{1}{2} mv^2 \\ &= \frac{1}{2} \times 0.02 \times 20 \times 20 \\ &= 4 \text{ J} \end{aligned}$$

This must be equal to the work done in stretching the cord. Using Eq. (7.11) we have

$$W = \frac{1}{2} F \Delta l = 4 \text{ J}$$

where F is the stretching force. Since

$$\Delta l = 20 \text{ cm} = 0.2 \text{ m},$$

$$F = \frac{4 \times 2}{0.2} = 40 \text{ N}$$

$$\text{Stress} = \frac{F}{A} = \frac{40}{\pi r^2}$$

Now $r = 3 \text{ mm} = 3 \times 10^{-3} \text{ m}$

Hence,
$$\begin{aligned} \text{Stress} &= \frac{40}{\pi (3 \times 10^{-3})^2} \\ &= 1.415 \times 10^6 \text{ N m}^{-2} \end{aligned}$$

But
$$\text{Strain} = \frac{20}{42} = 0.476$$

$$\text{Young's modulus } Y = \frac{1.415 \times 10^6}{0.476} = 2.97 \times 10^6 \text{ N m}^{-2}$$

Example 3: A light rod of length 2 m is suspended from the ceiling horizontally by means of two vertical wires of equal length tied to its ends. One of the wires is made of steel and is of cross-section 0.1 cm^2 and the other is made of brass of cross-section 0.2 cm^2 . Find out the position along the rod at which a weight may be hung to produce

- equal stress in both wires
- equal strain in both wires.

Young's modulus of elasticity of brass and steel are $10 \times 10^{11} \text{ dyne cm}^{-2}$ and $20 \times 10^{11} \text{ dyne cm}^{-2}$ respectively. (IIT 1974)

Solution: Let the rod BD be suspended by the wires AB and CD as shown in Fig. 7.3. If W is the weight suspended at O at a distance x from the steel wire AB , the force acting at point O is in the vertically downward direction producing tensions T_1 and T_2 in the two strings.

(a) Let A_1 and A_2 be the areas of cross-section of the respective wires, i.e.

$$A_1 = 0.1 \text{ cm}^2 \quad \text{and} \quad A_2 = 0.2 \text{ cm}^2$$

$$\text{Stress in the steel wire} = \frac{T_1}{0.1}$$

$$\text{Stress in the brass wire} = \frac{T_2}{0.2}$$

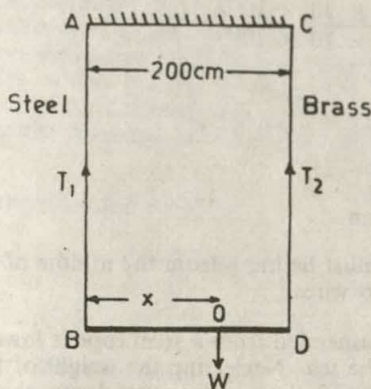


Fig. 7.3

For the stresses to be equal,

$$\frac{T_1}{0.1} = \frac{T_2}{0.2}$$

$$\text{or} \quad \frac{T_1}{T_2} = \frac{1}{2} \quad \text{(i)}$$

For the system to be in equilibrium, the moments of the forces T_1 and T_2 about O must be equal. Hence

$$x T_1 = (200 - x) T_2 \quad \text{(ii)}$$

Using Eq. (1) we have

$$\frac{T_1}{T_2} = \frac{200 - x}{x} = \frac{1}{2}$$

$$\text{or} \quad 400 - 2x = x$$

$$x = \frac{400}{3} = 133.3 \text{ cm}$$

(b) Young's modulus

$$Y = \frac{\text{Stress}}{\text{Strain}}$$

$$\text{Stress} = \frac{T}{A}$$

$$\text{Strain} = \frac{T}{(A Y)}$$

The strains in the two wires will be equal if

$$\frac{T_1}{A_1 Y_1} = \frac{T_2}{A_2 Y_2}$$

or
$$\frac{T_1}{T_2} = \frac{A_1 Y_1}{A_2 Y_2}$$

Now $Y_1 = 20 \times 10^{11} \text{ dyne cm}^{-2}$ and $Y_2 = 10 \times 10^{11} \text{ dyne cm}^{-2}$

Therefore
$$\frac{T_1}{T_2} = \frac{0.1 \times 20 \times 10^{11}}{0.2 \times 10 \times 10^{11}} = 1$$

or
$$T_1 = T_2$$

Using Eq. (ii) we have

$$x = (200 - x)$$

or
$$x = 100 \text{ cm}$$

Thus the weight W must be hung from the middle of the rod to produce equal strains in the two wires.

Example 4: A load suspended from a steel rope is lowered from a ship to measure the depth of the sea. Neglecting the weight of the load as compared with that of the rope, find the maximum depth that can be measured by this method. Assume the density of the sea water to be 1000 kg m^{-3} . The breaking stress of steel is $7.85 \times 10^8 \text{ N m}^{-2}$, the density of steel = 7700 kg m^{-3} and $g = 9.8 \text{ m s}^{-2}$.

Solution: Since the rope is immersed in sea water, there will be an upward thrust on the rope. This will decrease the stretching force which is the weight of the rope itself.

The net stretching force = weight of the rope — upthrust

or
$$F = Mg - (\rho' g d) A$$

where ρ' is the density of sea-water, d the maximum depth that can be measured and A , the area of cross-section of the rope.

The mass of the rope is given by

$$\begin{aligned} M &= \text{density} \times \text{volume} \\ &= \rho A d \end{aligned}$$

where ρ is the density of steel. Therefore,

$$F = A g (\rho - \rho') d \quad (i)$$

Now, the breaking stress = maximum stretching force divided by area of cross-section

The maximum depth that can be measured by this method cannot be greater than that corresponding to the breaking stress which is given by

$$\begin{aligned} \frac{F_{\max}}{A} &= g (\rho - \rho') d \\ &= 7.85 \times 10^{10} \text{ N m}^{-2} \end{aligned}$$

$$\begin{aligned}
 \text{Hence, } d &= \frac{7.85 \times 10^{10}}{g(\rho - \rho')} \\
 &= \frac{7.85 \times 10^{10}}{9.8(7700 - 1000)} \\
 &= 1.196 \times 10^4 \text{ m} = 11.96 \text{ km}
 \end{aligned}$$

This is the maximum depth that can be measured by this method.

Example 5: Artificial diamond crystals have been manufactured by subjecting carbon in the form of graphite to a pressure of $1.55 \times 10^{10} \text{ N m}^{-2}$ at a high temperature. Assuming that natural diamonds were formed at similar high pressures within the earth, what must have been the original volume of the Kohinoor diamond, whose mass before cutting was about 175 g? The density of the diamond = 3.5 g cm^{-3} and its bulk modulus = $62 \times 10^{10} \text{ N m}^{-2}$.

Solution: Mass of the diamond = 175 g

$$\text{Density} = 3.5 \text{ g cm}^{-3}$$

$$\text{Volume} = \frac{75}{3.5} = 50 \text{ cm}^3$$

If the original volume of the diamond were V , then

$$V = 50 + \Delta V$$

where ΔV is the decrease in volume under the pressure during its formation.

$$\text{Bulk modulus } B = - \frac{P V}{\Delta V}$$

Substituting $(V - 50)$ for ΔV and the values of P and B , we have

$$\frac{B}{P} = \frac{V}{V - 50} = \frac{62 \times 10^{10}}{1.55 \times 10^{10}} = 40$$

$$\text{or } V = 40 V - 2000$$

$$39 V = 2000$$

$$\text{or } V = 51.28 \text{ cm}^3$$

Alternatively, one could calculate $(-\Delta V)$ from the equation

$$(-\Delta V) = \frac{P V}{B} = \frac{1.55 \times 10^{10} \times 50}{62 \times 10^{10}} = 1.25 \text{ cm}^3$$

and add this value to the present value giving $V = 51.25 \text{ cm}^3$. The difference is only in the second decimal place, i.e. less than 0.06 %. Hence the original volume of the diamond must have been equal to 51.3 cm^3 .

Example 6: A rubber eraser $3.0 \text{ cm} \times 1.0 \text{ cm} \times 8.0 \text{ cm}$ is clamped at one end, with the 8 cm edge vertical. A horizontal force of 2.4 N is applied at the free end, and the top of the eraser is displaced horizontally. Calculate the displacement if the shear modulus of the rubber is $1.4 \times 10^5 \text{ N m}^{-2}$.

Solution: Figure 7.4 shows a rubber eraser $ABCD$ clamped at the end AB and a horizontal force F applied at the free end DC resulting in a displacement $DD' = CC'$ which is to be evaluated.

$$\text{The shearing strain} = \frac{DD'}{8}$$

The area (A) of the face on which the shearing stress is applied is

$$A = 3.0 \times 1.0 = 3.0 \text{ cm}^2 = 3.0 \times 10^{-4} \text{ m}^2$$

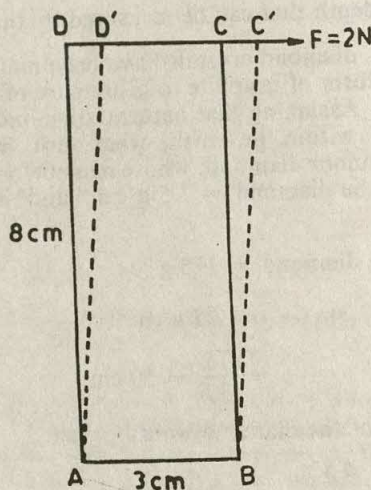


Fig. 7.4

$$\text{Shearing stress} = \frac{F}{A} = \frac{2.4}{3 \times 10^{-4}} = 0.8 \times 10^4 \text{ N m}^{-2}$$

$$\text{Shear modulus } n = \frac{\text{Shearing stress}}{\text{Shearing strain}}$$

$$= \frac{\frac{F}{A}}{DD'/8} = \frac{0.8 \times 10^4 \times 8}{DD'}$$

$$\begin{aligned} \text{Therefore, } DD' &= \frac{6.4 \times 10^4}{n} = \frac{6.4 \times 10^4}{1.4 \times 10^5} \\ &= 0.6 \text{ cm} = 6 \text{ mm} \end{aligned}$$

Hence the upper face of the eraser is displaced horizontally by 6 mm.

Example 7: Calculate the force required to punch a hole 2 cm square in a steel sheet 2 mm thick whose shearing strength is $3.5 \times 10^8 \text{ N m}^{-2}$.

Solution: The shearing stress is exerted on the rectangular surface ($2.0 \text{ cm} \times 2.0 \text{ cm} \times 0.2 \text{ cm}$) that is the boundary of the hole. The area of this surface is (see Fig. 7.5)

$$\begin{aligned} A &= 2.0 \times 10^{-2} \times 4 \times 0.2 \times 10^{-2} \text{ m}^2 \\ &= 1.6 \times 10^{-4} \text{ m}^2 \end{aligned}$$

Since the minimum shearing stress to rupture the steel is

$$\left(\frac{F}{A}\right)_{\min} = 3.5 \times 10^8 \text{ N m}^{-2}$$

the force required is given by

$$F = 3.5 \times 10^8 \times 1.6 \times 10^{-4} = 5.6 \times 10^4 \text{ N}$$

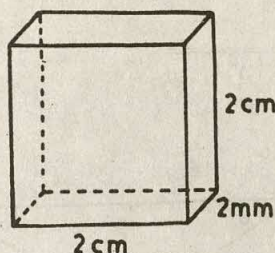


Fig. 7.5

Example 8: Find the potential energy of a wire 5 cm long and 4×10^{-3} cm in diameter twisted through an angle of 10° . The shear modulus of the wire material is 5.9×10^{11} dyne cm^{-2} .

Solution: To twist the wire through the angle $d\theta$, it is necessary to do the work

$$dW = \tau d\theta$$

where τ , the twisting torque is given by Eq. (7.6). Then

$$W = \int_0^\theta \tau d\theta = \int_0^\theta \frac{n \pi r^4 \theta d\theta}{2 l} = \frac{n \pi r^4 \theta^2}{4 l}$$

$$\theta = 10^\circ = \frac{10}{60} \times \frac{\pi}{180} \text{ rad}$$

$$= \frac{\pi}{1080} \text{ rad}$$

$$\text{Therefore, } W = \frac{5.9 \times 10^{11} \times \pi \times (2 \times 10^{-3})^2 \pi^2}{4 \times 5 \times (1080)^2}$$

$$= 1.255 \times 10^{-5} \text{ erg}$$

$$= 1.255 \times 10^{-12} \text{ J}$$

Example 9: A steel wire of diameter 1 mm is stretched horizontally between two clamps located at a distance of 2 m from each other. A load of 0.25 kg is suspended from the midpoint of the wire. What will be the vertical distance in cm through which the mid-point of the wire moves down? Young's modulus of steel = 20×10^{10} N m^{-2} and $g = 9.8 \text{ m s}^{-2}$

Solution: Let d ($= OP$) be the vertical distance through which the mid-point of the wire of length l moves down as shown in Fig. 7.6.

This produces tension T in the wires as shown. For equilibrium, the components of the tensions along OP must balance the weight W . Therefore,

$$2 T \sin \theta = W$$

or

$$T = \frac{W}{2 \sin \theta} \quad (i)$$

$$\text{Stress} = \frac{\text{Tension}}{\text{Area of cross-section}}$$

$$= \frac{T}{A} = \frac{W}{2 A \sin \theta}$$

$$\sin \theta = \frac{d}{\sqrt{\left(\frac{l}{2}\right)^2 + d^2}}$$

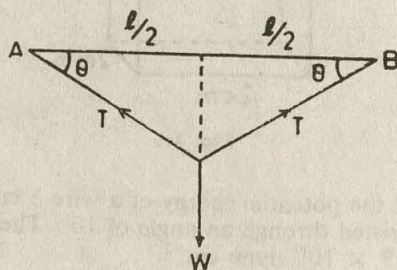


Fig. 7.2

$$\begin{aligned} \text{Strain} &= \frac{AP - OA}{OA} = \frac{\sqrt{\left(\frac{l}{2}\right)^2 + d^2} - \frac{l}{2}}{\frac{l}{2}} \\ &= \left[1 + \frac{d^2}{\left(\frac{l}{2}\right)^2} \right]^{1/2} - 1 \\ &= \left[1 + \frac{1}{2} \frac{d^2}{\left(\frac{l}{2}\right)^2} + \dots \right] - 1 \end{aligned}$$

where we have used Binomial expansion. Since d is much smaller than $l/2$, we can ignore the higher order terms in the binomial expansion. Hence,

$$\text{Strain} = \frac{1}{2} \frac{d^2}{\left(\frac{l}{2}\right)^2}$$

Also

$$\begin{aligned} \sin \theta &= \frac{d}{\sqrt{\left(\frac{l}{2}\right)^2 + d^2}} \\ &= \left(\frac{l}{2}\right) \left[1 + \frac{d^2}{\left(\frac{l}{2}\right)^2} \right]^{-1/2} \approx \left(\frac{l}{2}\right) \end{aligned}$$

Therefore, $Y = \frac{\text{Stress}}{\text{Strain}}$

$$= \frac{W}{2A} \frac{(l/2)}{d} \frac{2(l/2)^2}{d^2} = \frac{W}{A} \left(\frac{l}{2d} \right)^3$$

or $d^3 = \frac{W}{A Y} \left(\frac{l}{2} \right)^3$

or $d = \frac{l}{2} \left(\frac{W}{A Y} \right)^{1/3}$

Now $A = \pi r^2 = \pi (0.5 \times 10^{-3})^2$, $l/2 = 1 \text{ m}$

$$d = 1.0 \times \left(\frac{0.25 \times 9.8}{\pi \times 0.25 \times 10^{-6} \times 20 \times 10^{10}} \right)^{1/3}$$

$$= 2.5 \times 10^{-2} \text{ m} = 2.5 \text{ cm}$$

It may be noted that $\frac{d}{(l/2)} = 0.025$ and $\frac{d^2}{(l/2)^2} = 0.0006$, hence, our approximation of neglecting these quantities in the binomial expansions in the problem is justified. Thus, the midpoint of the wire moves down by 2.5 cm.

Example 10: A steel wire of cross-sectional area 0.5 mm^2 is held between two fixed supports. If the tension in the wire is negligible and it is just taut at a temperature of 20°C , determine the tension when the temperature falls to 0°C . Young's modulus of steel is $21 \times 10^{11} \text{ dyne cm}^{-2}$ and the coefficient of linear expansion of steel is $12 \times 10^{-6} \text{ per } ^\circ\text{C}$. Assume that the distance between the supports remains unchanged. (IIT 1973)

Solution: Let l be the length of the wire at 20°C and l_0 , the length at 0°C . Then

$$l - l_0 = \alpha l_0 \Delta T = 20 \alpha \cdot l_0$$

$$\text{Compressive strain} = \frac{l - l_0}{l_0} = 20 \alpha = 20 \times 12 \times 10^{-6} = 2.4 \times 10^{-4}$$

$$Y = \frac{\text{Stress}}{\text{Strain}}$$

$$\text{Stress} = Y \times \text{strain} = \frac{F}{A}$$

Hence,

$$\begin{aligned} \text{tension } T &= Y A \times \text{strain} \\ &= 21 \times 10^{11} \times 0.5 \times 10^{-2} \times 2.4 \times 10^{-4} \\ &= 2.52 \times 10^6 \text{ dyne} = 25.2 \text{ N} \end{aligned}$$

This is the tension in the wire when the temperature falls to 0°C .

Example 11: A composite rod is made by joining a copper rod end to end, with a second rod of a different material, but of the same cross-section. At 25°C , the composite rod is 1 m in length, of which the length of the copper rod is 30 cm. At 125°C the length of the composite rod increases by 1.91 mm.

When the composite rod is not allowed to expand by holding it between two rigid walls, it is found that the lengths of the two constituents do not change with rise in temperature. Find the Young's modulus and the coefficient of linear expansion of the second material.

cient of linear expansion of the second rod. Y for copper = 1.3×10^{11} dyne cm^{-2} , $\alpha = 1.7 \times 10^{-5} (\text{°C})^{-1}$ (IIT 1979)

Solution: Let α' be the thermal expansion coefficient of the second rod whose length is $100 - 30 = 70$ cm.

Rise in temperature = $125 - 25 = 10 \text{ °C}$

Total extension = $1.91 \text{ mm} = 0.191 \text{ cm}$

$$0.191 = 70 \times 100 \times \alpha' + 30 \times 100 \times 1.7 \times 10^{-5}$$

$$= 7000 \alpha' + 5.1 \times 10^{-2}$$

$$7000 \alpha' = 0.191 - 0.051 = 0.140$$

$$\therefore \alpha' = \frac{0.140}{7000} = 2 \times 10^{-5} (\text{°C})^{-1}$$

Since $Y = \frac{F/A}{\Delta l/l}$

the extension for the copper rod is given by

$$\Delta l = \frac{F}{A} \frac{l}{Y} = 30 \times 1.7 \times 10^{-5} \times 100$$

$$\text{Tensile stress} = \frac{F}{A} = \frac{30 \times 1.7 \times 10^{-5} \times 100 \times 1.3 \times 10^{11}}{30}$$

$$= 2.21 \times 10^8 \text{ N m}^{-2}$$

where F is the force required to maintain the copper rod at its initial length, when it is heated. The stress required to maintain the second rod at its initial length, when this rod is heated, is given by

$$\frac{F'}{A} = \frac{\Delta l' Y'}{l'}$$

where Y' is the Young's modulus of the second rod. The two stresses must be equal because the rods are not allowed to expand. Hence,

$$\frac{\Delta l'}{l'} Y' = 2.21 \times 10^8$$

$$\Delta l' = 70 \times 100 \times \alpha'$$

or $Y' = \frac{2.21 \times 10^8 \times 70}{70 \times 100 \times 2 \times 10^{-5}} = 1.105 \times 10^{11} \text{ dyne cm}^{-2}$

Example 12: A sphere of mass 0.25 kg is attached to one end of a steel wire, 1 m long having cross-sectional area of 2.5 mm^2 . The other end is attached to a vertical shaft which rotates about its axis to form a conical pendulum. Calculate the number of revolutions required to extend the wire by 2 mm . Young's modulus of steel is $20 \times 10^{10} \text{ N m}^{-2}$ and $g = 9.8 \text{ m s}^{-2}$.

Solution: Figure (7.7) shows the forces acting on the sphere, namely, tension T along the wire and the weight mg acting vertically downward.

The wire extends by 2 mm when the tension in the wire is T . Therefore,

$$Y = \frac{\frac{T}{A}}{\frac{\Delta l}{l}}$$

or
$$T = \frac{\Delta l}{l} Y A$$

Here $\Delta l = 2 \text{ mm} = 2 \times 10^{-3} \text{ m}$, $l = 1.0 \text{ m}$

$A = 2.5 \text{ mm}^2 = 2.5 \times 10^{-6} \text{ m}^2$

Substituting these values, we have

$$T = \frac{2 \times 10^{-3} \times 20 \times 10^{10} \times 2.5 \times 10^{-6}}{1}$$

$$= 1.0 \times 10^3 \text{ N}$$

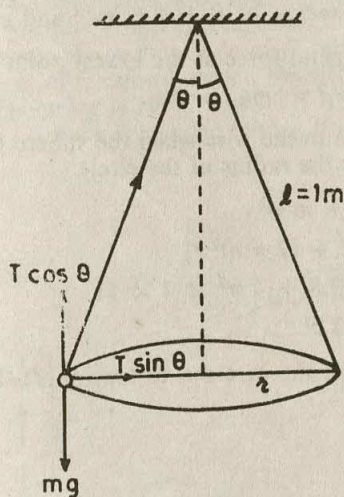


Fig. 7.7

Resolving T into vertical and horizontal components, we have

$$T \cos \theta = mg \quad \text{and} \quad T \sin \theta = \frac{m v^2}{r}$$

where r is the radius of the circle described and v is the velocity of the sphere. Now

$$r = l \sin \theta$$

$$\cos \theta = \frac{mg}{T} = \frac{0.25 \times 9.8}{1 \times 10^3} = 2.45 \times 10^{-3}$$

and
$$\sin \theta = \sqrt{1 - \cos^2 \theta}$$

$$= \sqrt{1 - (2.45 \times 10^{-3})^2} \approx 1.0$$

Therefore $r = l = 1.0 \text{ m}$

$$T \sin \theta = \frac{m v^2}{r} = T$$

or
$$v^2 = \frac{r T}{m} = \frac{1 \times 10^3}{0.25} = 4.0 \times 10^3$$

$$v = 63.25 \text{ m s}^{-1}$$

$$\text{Period of revolution} = \frac{2\pi r}{v} = \frac{2\pi}{63.25} = 0.0993 \text{ s}$$

$$\text{Frequency} = \frac{1}{\text{Time period}} = \frac{1}{0.0993} = 10.07 \text{ rev s}^{-1}$$

Hence the sphere must make about 10 revolutions per second to extend the wire by 2 mm.

Example 13: A steel wire 1 m long and of radius 1 mm has a sphere of mass 2.5 kg attached to one of its ends. It is whirled in a vertical circle and has an angular velocity of 2 rev s^{-1} at the bottom of the circle. Calculate the elongation of the wire when the weight is at the lowest point of its path. Young's modulus of steel is $20 \times 10^{10} \text{ N m}^{-2}$ and $g = 9.8 \text{ m s}^{-2}$.

Solution: The centripetal force at the lowest point is given by

$$m \omega^2 r = T - mg$$

where T is the tension in the wire when the sphere is at the bottom, ω the angular velocity and r the radius of the circle.

Therefore, $T = mg + m \omega^2 r$

$$\begin{aligned} &= m [g + (2\pi n)^2 r] \\ &= 2.5 (9.8 + 4\pi^2 \times 4 \times 1) \\ &= 419.3 \text{ N} \end{aligned}$$

If Δl is the elongation of the wire due to tension T , then

$$Y = \frac{T/A}{\Delta l/l}$$

$$\therefore \Delta l = \frac{T l}{A Y}$$

$$A = \pi (1 \times 10^{-3})^2 = \pi \times 10^{-6} \text{ m}^2$$

$$\begin{aligned} \Delta l &= \frac{419.3 \times 1.0}{\pi \times 10^{-6} \times 20 \times 10^{10}} \\ &= 6.67 \times 10^{-4} \text{ m} = 0.067 \text{ cm} \end{aligned}$$

Example 14: Find the value of Poisson's ratio at which the volume of a wire does not change when the wire is subjected to a tension.

Solution: Poisson's ratio $\sigma = \frac{\Delta r/r}{\Delta l/l}$

where r is the radius of the wire and l its length.

Volume of the wire before expansion $V_1 = \pi r^2 l$

Volume of the wire after expansion

$$V_2 = \pi (r - \Delta r)^2 (l + \Delta l)$$

If the volume is to remain unchanged during expansion, we require that $V_1 = V_2$, i.e.

$$\begin{aligned} \pi r^2 l &= \pi [r^2 - 2r\Delta r + (\Delta r)^2] (l + \Delta l) \\ &= \pi r^2 (l + \Delta l) - 2\pi r \Delta r l - 2r \Delta r \Delta l \pi \\ &\quad + (\Delta r)^2 (l + \Delta l) \pi \end{aligned}$$

or $2 \pi r \Delta r l = \pi r^2 \Delta l + \text{terms containing the product of } \Delta r$
and Δl , which can be neglected. Hence

$$\frac{(\Delta r/r)}{(\Delta l/l)} = \frac{1}{2}$$

or $\sigma = 0.5$

Thus the volume of the wire does not change if the Poisson's ratio of the material of the wire is 0.5

Example 15: Two blocks A and B are connected to each other by a string and a spring, the string passes over a frictionless pulley as shown in Fig. 7.8. Block B slides over the horizontal top surface of a stationary block C and the block A slides along the vertical side of C , both with the same uniform speed. The coefficient of friction between the surfaces of blocks is 0.2. The force constant of the spring is 1960 N m^{-1} . If the mass of block A is 2 kg, calculate the mass of block B and the energy stored in the spring. (IIT 1982)

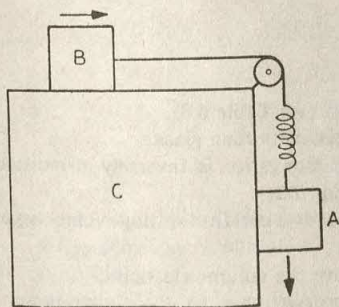


Fig. 7.8

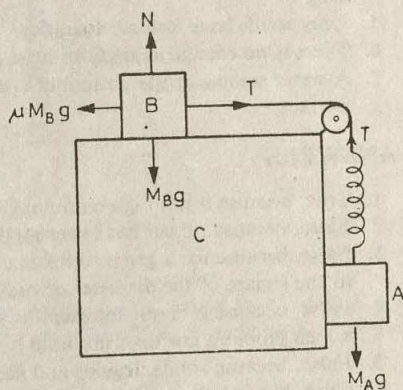


Fig. 7.9

Solution: Let M_A and M_B be the masses of blocks A and B respectively. The forces acting on the blocks are shown in Fig. 7.9. T is the tension and N , the normal reaction on block B .

Since block A is moving vertically downward with a uniform velocity, there is no normal reaction on A due to C . Therefore,

$$T - \mu M_B g = 0, \quad T = M_A g$$

$$\therefore \mu = \frac{M_A}{M_B} \quad \text{or} \quad M_B = \frac{M_A}{\mu} = \frac{2.0}{0.2} = 10 \text{ kg}$$

Force constant $k = 1960 \text{ N m}^{-1}$.

Stretching force on the spring

$$= M_A g = 2 \times 9.8 = 19.6 \text{ N}$$

Now, $F = kx$

where x is the extension in the spring.

$$x = \frac{F}{k} = \frac{19.6}{1960} = 10^{-2} \text{ m}$$

$$\begin{aligned}
 \text{Energy stored in the spring} &= \frac{1}{2} F \times \text{extension} \\
 &= \frac{1}{2} \times 19.6 \times 10^{-2} \\
 &= 9.8 \times 10^{-2} \text{ J}
 \end{aligned}$$

Alternatively,

$$PE = \frac{1}{2} k x^2 = \frac{1}{2} \frac{F^2}{k} = \frac{1}{2} \frac{19.6 \times 19.6}{1960} = 9.8 \times 10^{-2} \text{ J}$$

TRUE-FALSE STATEMENTS WITH REASONS

A few statements are given below. We have to decide whether they are true or false giving, in brief, the reasons in support of our answer.

1. Steel is a highly elastic material.
2. Rubber is more elastic than glass.
3. If two steel wires of equal length are stretched by equal forces, their elongations are inversely proportional to their diameters.
4. The strain produced in an elastic body is always proportional to the stress producing it.
5. Only solids have volume elasticity.
6. There is no change in angle of twist if the diameter of the cylinder is changed.
7. A water sample at the bottom of a lake 400 m deep has the same density as at the surface.

ANSWERS

1. True, because it has high modulus of elasticity (see Table 6.8).
2. False, because rubber has lower modulus of elasticity than glass.
3. False, because for a given stretching force, the elongation is inversely proportional to the square of the diameter or radius (see Eq. 7.2)
4. False, because it is not so when the stress exceeds a certain limiting value beyond which Hooke's law does not hold.
5. False, because solids, liquids and gases all have the volume elasticity.
6. False, because the angle of twist is inversely proportional to the fourth power of diameter (see Eq. 7.6).
7. False, because the volume of a given mass of water at the bottom of the lake will be smaller because of the pressure exerted by a column of water 400 m high and hence the density will be higher.

7.7 MULTIPLE-CHOICE QUESTIONS

Choose the correct answer from the given alternatives.

1. A wire stretches a certain amount under load. If the wire's length and diameter are both doubled, the stretch caused by the same load is
(a) greater (b) the same (c) less
2. Which of the following graphs of load against extension best represents the verification of Hooke's law.

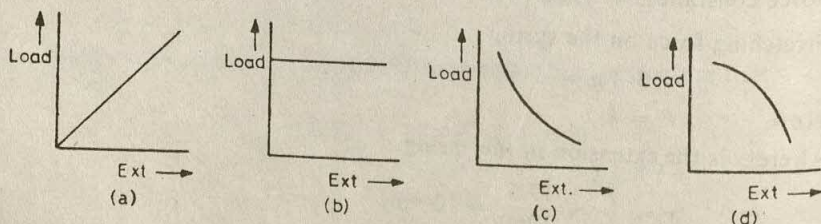


Fig. 7.10

3. The dimensions of force constant are
 (a) $[MT^{-2}]$ (b) $[MLT^{-1}]$ (c) $[MLT^{-2}]$
4. The potential energy of a stretched spring is proportional to
 (a) the square of the force constant
 (b) the square of the amount of stretch
 (c) both of these
5. A steel wire is 1 m long and 1 mm^2 in area of cross-section. If it takes 200 N to stretch this wire by 1 mm, how much force will be required to stretch a wire of the same material and diameter from its normal length of 10 m to a length of 1002 cm?
- (a) 100 N (b) 200 N
 (c) 400 N (d) 2000 N

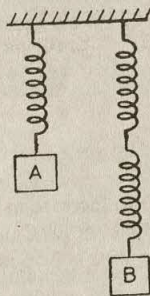


Fig. 7.11

6. Figure 7.11 shows three identical springs. When a 4 kg weight is hung on A, the spring shows an elongation of 1 cm. When a weight of 6 kg is hung on B, the hook descends
 (a) 1 cm (b) 2 cm
 (c) 3 cm (d) 4 cm
7. The following four wires are made of the same material. Which of these will have the largest extension when the same tension is applied.
 (a) length, $l = 0.5 \text{ m}$, diameter, $d = 0.5 \text{ mm}$
 (b) length, $l = 1 \text{ m}$, diameter, $d = 1 \text{ mm}$
 (c) length, $l = 2 \text{ m}$, diameter, $d = 2 \text{ mm}$
 (d) length, $l = 3 \text{ m}$, diameter, $d = 3 \text{ mm}$
8. A nylon rope 3 cm in diameter has a breaking strength of $1.5 \times 10^5 \text{ N}$. The breaking strength of a similar rope 1.5 cm in diameter is
 (a) $0.75 \times 10^5 \text{ N}$ (b) $3 \times 10^5 \text{ N}$
 (c) $0.375 \times 10^5 \text{ N}$ (d) $6 \times 10^5 \text{ N}$
9. Which of the following is not a unit of Young's modulus?
 (a) N m^{-2} (b) Mega Pascal (MPa)
 (c) dyne cm^{-2} (d) N m^{-1}
10. A suspended long metal wire is stretched a small distance x by a load W in newtons suspended at the other end. Select the best answer out of the following:
 (a) The loss in potential energy of the load W is equal to the gain in energy of the wire in stretching a length x .

- (b) The energy stored in the wire can be calculated from the area between the force-extension graph and the extension-axis.
- (c) The energy per unit volume stored in the wire is equal to half the product of stress and strain.
11. A wire suspended vertically from one end, is stretched by attaching a weight 200 N to the lower end. The weight stretches the wire by 1 mm. The energy gained by the wire is
- (a) 0.1 J (b) 0.2 J
(c) 0.4 J (d) 10 J
12. When a spiral spring is stretched by a force, the resultant strain is
- (a) tensile (b) shear
(c) volume (d) all these
13. A spring of force constant k is cut into two equal parts. The force constant of each part is
- (a) $k/2$ (b) k
(c) $2k$ (d) $4k$

NUMERICAL EXERCISES

- What stress would cause a wire to increase in length by 0.1 per cent if the Young's modulus of the wire is $12 \times 10^{10} \text{ N m}^{-2}$? Calculate the force required to produce this stress if the diameter of the wire is 0.56 mm.
- A spring is extended by 30 mm when a force of 1.5 N is applied to it. Calculate the energy stored in the spring when hanging vertically supporting a mass of 0.20 kg if the spring was unstretched before applying the load. Calculate the loss in potential energy of the mass.
Assume $g = 10 \text{ m s}^{-2}$
- Determine the length of a copper wire, which when suspended vertically breaks under its own weight. Density of copper $= 8600 \text{ kg m}^{-3}$, breaking stress for copper $= 2.45 \times 10^8 \text{ N m}^{-2}$ and $g = 9.8 \text{ m s}^{-2}$.
- The rubber cord of a catapult has a cross-sectional area 1 mm^2 and a total unstretched length 10 cm. It is stretched to 12 cm and then released to project a missile of mass 5 g. Calculate the velocity of projection assuming that the energy of the stretched cord is completely converted into kinetic energy. Young's modulus of rubber $= 5 \times 10^8 \text{ N m}^{-2}$
- Two wires each of length 1 m and radius 0.2 mm are tied together at both ends. A load of 25 kg hangs from the composite wire. If the Young's moduli are 12×10^{10} and $18 \times 10^{10} \text{ N m}^{-2}$, find tension in each wire. Also calculate the total extension.
 $g = 10 \text{ m s}^{-2}$.
- A uniform wire of unstretched length 2.49 m is attached to two points A and B which are 2 m apart and in the same horizontal line. When a 5 kg mass is attached to the midpoint C of the wire, the equilibrium position of C is 0.75 m below the line AB . Neglecting the weight of the wire and taking the Young's modulus for its material to be $20 \times 10^{10} \text{ N m}^{-2}$, find
 - the strain in the wire
 - the stress in the wire
 - the energy stored in the wire
- The density of water at the surface of a lake is 1000 kg m^{-3} . Find the density at a depth of 400 m. Bulk modulus for water is $2 \times 10^9 \text{ N m}^{-2}$ and $g = 9.8 \text{ m s}^{-2}$.

8. A steel cable whose cross-sectional area is 7.5 cm^2 is used to support an elevator cab weighing $2.5 \times 10^4 \text{ N}$. If the stress in the cable is not to exceed 20% of the cable's elastic limit of $2.5 \times 10^8 \text{ Pa}$, find the maximum permissible upward acceleration. $g = 9.8 \text{ m s}^{-2}$

9. Determine the length of a lead wire which when suspended vertically breaks under its own weight.

$$\text{Breaking stress for lead} = 2 \times 10^7 \text{ N m}^{-2}$$

$$\text{Density of lead} = 11300 \text{ kg m}^{-3}, g = 9.8 \text{ m s}^{-2}$$

10. A heavy rigid bar is supported horizontally from a fixed support by two vertical wires A and B , of the same initial length and which experience the same tension. If the ratio of the diameter of A to that of B is 2 and Young's modulus of A is twice that of B , calculate

(a) the ratio of the tension in A to that in B

(b) the distance of the wire A from the centre of gravity of the bar if the distance between the wires is D .

11. When a man whose mass is 60 kg gets into a car, the centre of gravity of the car is lowered by $3 \times 10^{-3} \text{ m}$.

(a) Find the elastic constant of the spring system of the car.

(b) If the mass of the car is 500 kg , what is the period of vibration when it is empty?

(c) Find the period of vibration when the man is sitting in the car.

12. A wire of length 1.20 m and diameter 2.6 mm is stretched between two rigid supports. If the temperature of the wire is decreased by 30°C , calculate the tension in the wire. Young's modulus of the material of the wire is $16 \times 10^{10} \text{ N m}^{-2}$ and its coefficient of linear expansion is $24 \times 10^{-6} \text{ per } ^\circ \text{C}$.

13. An elevator cable can carry a total weight of 1000 kg and accelerate it upward at 2.20 m s^{-2} . Calculate the diameter (in cm) the cable must have if the safe working stress for the material of the cable is $1.2 \times 10^8 \text{ N m}^{-2}$.

14. What pressure has to be applied to the ends of a steel cylinder to keep its length constant on raising its temperature by 100°C ? Young's modulus of steel is $20 \times 10^{10} \text{ N m}^{-2}$ and the coefficient of linear expansion of steel is $11 \times 10^{-6} \text{ per } ^\circ \text{C}$.

15. Find the elastic deformation energy of a steel rod whose one end is fixed and the other end is twisted through an angle of 6° . The length of the rod is 1 m and its radius is 1 cm . The shear modulus of steel is $8.1 \times 10^{10} \text{ N m}^{-2}$.

16. A copper wire is held at two ends by rigid supports. At 30°C , the wire is just taut, with negligible tension in it. Find the speed of transverse waves in the wire at 10°C . Given, density of copper = 9000 kg m^{-3} , Young's modulus = $11 \times 10^{10} \text{ N m}^{-2}$ and coefficient of linear expansion = $1.7 \times 10^{-5} \text{ per } ^\circ \text{C}$. (IIT 1979)

(Hint: The velocity of transverse waves is given by $\sqrt{T/m}$, where m is the mass per unit length of the wire and T is the tension in the wire.)

17. Find the maximum power which can be transmitted by means of a steel shaft rotating about its axis with an angular velocity $\omega = 120 \text{ rad s}^{-1}$, if its length $l = 2 \text{ m}$, radius $r = 1.5 \text{ cm}$ and the permissible torsion angle $\phi = 2.5^\circ$. The shear modulus of steel is $8.1 \times 10^{10} \text{ N m}^{-2}$.

(Hint: Power = torque $\times$ angular frequency)

18. A rope 50 m in length and 9 mm in diameter is used by a mountaineer. It stretches by 1.5 m under the weight of a climber of mass 80 kg . Find the Young's modulus of the material of the rope. If the Poisson's ratio of this material is 0.2 , calculate the change in its diameter under this stress.

19. A spring having a force constant of 250 N m^{-1} is loaded with a mass of 10 kg .
 (a) Find the period of vibration.
 (b) If the mass is displaced by 30 cm and then released, find the velocity with which it passes through the equilibrium position.
 (c) What is the total energy of the vibrating mass?
20. A copper wire 10 m in length has a 10 kg load suspended at its end such that it vibrates vertically in simple harmonic motion at a frequency of 10 Hz . Find the area of cross-section of the wire. Young's modulus of copper is $10 \times 10^{10} \text{ Nm}^{-2}$ and $g = 9.8 \text{ m s}^{-2}$.

ANSWERS

MULTIPLE CHOICE QUESTIONS

- | | | | | |
|---------|---------|---------|--------|---------|
| 1. (c) | 2. (a) | 3. (a) | 4. (b) | 5. (c) |
| 6. (c) | 7. (a) | 8. (c) | 9. (d) | 10. (c) |
| 11. (a) | 12. (b) | 13. (c) | | |

NUMERICAL EXERCISES

- $1.2 \times 10^8 \text{ Nm}^{-2}$, 29.6 N
- 0.04 J , 0.08 J
- 2.9 km
- 20 m s^{-1}
- $100, 150 \text{ N}$ and 6 mm
- (a) 4.02×10^{-3} , (b) $8.03 \times 10^8 \text{ N m}^{-2}$ (c) 2.08 J
- 1002 kg m^{-3}
- 4.9 m s^{-2}
- 180 m
- (a) $8 : 1$, (b) $D/9$
- (a) $1.95 \times 10^5 \text{ N m}^{-1}$, (b) $6.35 \times 10^{-4} \text{ s}$, (c) $6.00 \times 10^{-4} \text{ s}$
- 612 N
- 1.13 cm
- $2.23 \times 10^8 \text{ N m}^{-2}$
- 7 J
- $\sim 70 \text{ m s}^{-1}$
- 16.9 kW
- $4.11 \times 10^8 \text{ N m}^{-2}$, 0.054 mm
- (a) 1.256 s , (b) 1.50 m s^{-1} , (c) 11.3 J
- 3.95 mm^2

Simple Harmonic Motion (SHM)

8.1 REVIEW OF BASIC CONCEPTS

8.1.1 What is SHM? Simple harmonic motion (SHM) is the simplest kind of oscillatory motion in which a body, displaced from its stable equilibrium position, oscillates to and fro about that position when released. If the displacement (x) from the equilibrium position is small, the restoring force (F) acting on the body is given by

$$F = -kx$$

where k is a positive constant known as the force constant. In the SI system k is expressed in *newton per metre* (N m^{-1}). The acceleration (a) of the body is given by

$$a = \frac{F}{m} = -\frac{k}{m}x = -\omega^2 x$$

where m is the mass of the body and $\omega = \sqrt{\frac{k}{m}}$

Thus acceleration (a) = - constant $\times$ displacement (x).

The resulting motion is called simple harmonic motion. Thus, *a simple harmonic motion is a motion in which the acceleration (i) is proportional to the displacement from the equilibrium position and (ii) is directed towards the equilibrium position.*

8.2 CHARACTERISTICS OF SHM

The displacement x in SHM at time t is given by

$$x = A \sin(\omega t + \phi) \quad (8.1)$$

where the three constants A , ω and ϕ characterize the SHM, i.e. they distinguish one SHM from another. A SHM can also be described by a cosine function as

$$x = A \cos(\omega t + \delta)$$

We will use the sine function. A cosine function is equally valid.

8.2.1 Amplitude The amplitude of SHM is the maximum (positive or negative) value of the displacement from the equilibrium position. Quantity A (which is the coefficient of the sine or cosine function) is the amplitude of SHM.

8.2.2 Time Period The smallest time interval during which the motion repeats itself is called time period or simply period of SHM. The angular frequency ω is related to time period T as

$$\omega = \frac{2\pi}{T}$$

Frequency n of SHM is the number of complete oscillations completed in 1 second. Thus

$$n = \frac{1}{T} = \frac{\omega}{2\pi} \quad \text{or} \quad \omega = \frac{2\pi}{T} = 2\pi n \quad (8.2)$$

Since $\omega = \sqrt{\frac{k}{m}}$, we have

$$n = \frac{1}{T} = \frac{1}{2\pi} \sqrt{\frac{k}{m}} \quad (8.3)$$

Angular frequency ω is the coefficient of time t in the sine or cosine function.

8.2.3 Phase The quantity $(\omega t + \phi)$ is called the phase of SHM at time t ; it describes the state of motion at that instant. The quantity ϕ is the phase at time $t = 0$ and is called the phase constant or initial phase or epoch of the SHM. The phase constant is the time-independent term in the cosine or sine function.

8.3 VELOCITY AND ACCELERATION IN SHM

The velocity V of the particle in SHM is given by

$$V = \frac{dx}{dt} = A \omega \cos(\omega t + \phi)$$

But $\sin(\omega t + \phi) = \frac{x}{A}$

$$\therefore \cos(\omega t + \phi) = \left(1 - \frac{x^2}{A^2}\right)^{1/2}$$

Hence velocity $V = A \omega \left(1 - \frac{x^2}{A^2}\right)^{1/2} = \omega (A^2 - x^2)^{1/2} \quad (8.4)$

The acceleration a of the particle in SHM is given by

$$a = \frac{d^2x}{dt^2} = -A \omega^2 \sin(\omega t + \phi) = -\omega^2 x \quad (8.5)$$

Notice that, when the displacement is maximum, i.e. when $x = |A|$, the velocity is zero but the acceleration is maximum $= |\omega^2 A|$. But when the displacement is zero ($x = 0$), the velocity is maximum $= |\omega A|$ and the acceleration is zero.

8.4 ENERGY IN SHM

When a body executes SHM, its energy changes between kinetic and potential, but the total energy is always constant. At any displacement x from the equilibrium position,

$$\text{Kinetic energy (KE)} = \frac{1}{2} m V^2 = \frac{1}{2} m \omega^2 (A^2 - x^2) \quad (8.6)$$

$$\text{Potential energy (PE)} = \frac{1}{2} k x^2 = \frac{1}{2} m \omega^2 x^2 \quad (8.7)$$

$$\text{Total energy } E = \text{KE} + \text{PE} = \frac{1}{2} m \omega^2 A^2 \quad (8.8)$$

8.5 SI UNITS

Table 8.1

Quantity	SI unit	Symbol
Amplitude	metre	m
Period	second	s
Frequency	hertz	Hz
Angular frequency	radians per second	rad s ⁻¹
Energy	joule	J
Force constant	newtons per metre	N m ⁻¹

8.6 FORMULAE

1. Restoring force $F = -kx$, k = force constant
2. Displacement $x = A \sin(\omega t + \phi)$ or $x = A \cos(\omega t + \delta)$
3. $\omega = \frac{2\pi}{T} = 2\pi n = \sqrt{\frac{k}{m}}$, m = mass of the body in SHM
4. Velocity $V = \omega (A^2 - x^2)^{1/2}$. A = amplitude
5. Acceleration $a = -\omega^2 A$
6. Kinetic energy (KE) $= \frac{1}{2} m \omega^2 (A^2 - x^2)$
7. Potential energy (PE) $= \frac{1}{2} k x^2 = \frac{1}{2} m \omega^2 x^2$
8. Total energy (E) $= \text{KE} + \text{PE} = \frac{1}{2} m \omega^2 A^2$
9. The expressions for time period (T) of a few simple harmonic motions are given below. These expressions are valid only for small oscillations about the equilibrium position.

(a) Simple Pendulum: $T = 2\pi \sqrt{\frac{l}{g}}$

l = length of pendulum, g = acceleration due to gravity

(b) A light spring loaded with a mass: $T = 2\pi \sqrt{\frac{m}{k}}$

m = mass, k = spring constant

(c) Liquid column in a U-tube: $T = 2\pi \sqrt{\frac{L}{2g}}$

L = length of liquid column, g = acceleration due to gravity

(d) Floating pole: $T = 2\pi \sqrt{\frac{M}{\rho A g}}$

M = mass of pole, ρ = its density, A = its cross-sectional area

(e) *LC Circuit*: $T = 2\pi\sqrt{LC}$ L = inductance, C = capacitance

8.7 NUMERICAL EXAMPLES

Example 1: A body oscillates with SHM of amplitude 4 cm and a frequency of 5 Hz. At time $t = 0$, the body is at its equilibrium position ($x = 0$). Write down the equation of the SHM.

Solution: The equation of motion is

$$x = A \sin(\omega t + \phi)$$

where amplitude $A = 4$ cm, frequency $n = 5$ Hz. Hence

Angular frequency $\omega = 2\pi n = 2\pi \times 5 = 10\pi \text{ rad s}^{-1}$

Now, at $t = 0$, $x = 0$. Substituting in $x = A \sin(\omega t + \phi)$ we have

$$0 = A \sin \phi, \text{ giving } \phi = 0$$

Hence, the equation of motion is

$$\begin{aligned} x &= A \sin(\omega t + \phi) \\ &= 4 \sin(10\pi t + 0) \end{aligned}$$

or $x = 4 \sin 10\pi t$

Example 2: A particle is executing SHM according to the equation

$$x = 5 \sin \pi t$$

where x is in cm. How long will the particle take to move from the position of equilibrium to the position of maximum displacement?

Solution: The displacement of the particle varies with time according to the equation

$$x = 5 \sin \pi t$$

Maximum displacement = amplitude
= 5 cm

At time $t = 0$, $x = 0$ (equilibrium position).
Hence time t taken by the particle to move from $x = 0$ to $x = 5$ cm is given by

$$5 = 5 \sin \pi t$$

$$\text{or } 1 = \sin \pi t$$

$$\text{or } \pi t = \frac{\pi}{2} \text{ or } t = 0.5 \text{ s}$$

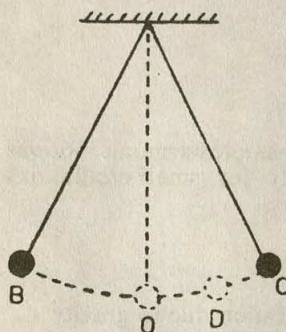


Fig. 8.1

Example 3: A simple pendulum is moving simple harmonically with a period of 6 s between two extreme positions B and C as shown in Fig. 8.1. If the angular distance between B and C is 10 cm, how long will the pendulum take to move from position C to a position D exactly midway between O and C

Solution: Given time period $T = 6$ s

$$\text{Amplitude} = OC = OB = \frac{1}{2} BC$$

$$= \frac{10}{2} = 5 \text{ cm}$$

$$\therefore OD = 2.5 \text{ cm}$$

Let the displacement of the pendulum be given by

$$x = A \sin (\omega t + \phi)$$

where $A = 5 \text{ cm}$ and $\omega = \frac{2\pi}{T} = \frac{2\pi}{6} = \frac{\pi}{3} \text{ rad s}^{-1}$

Let us suppose that at $t = 0$, the pendulum is at C , i.e. at $t = 0$, $x = A$, so that

$$A = A \sin (\omega \times 0 + \phi)$$

or $A = A \sin \phi$ or $\sin \phi = 1$ or $\phi = \frac{\pi}{2}$

Thus the motion of the pendulum is given by (putting $\phi = \frac{\pi}{2}$)

$$x = A \sin \left(\omega t + \frac{\pi}{2} \right) = A \cos \omega t = 5 \cos \omega t$$

The value of t for which $x = 2.5 \text{ cm}$ is given by

$$2.5 = 5 \cos \omega t$$

or $\cos \omega t = 1/2$ or $\omega t = \frac{\pi}{3}$

Since $\omega = \frac{\pi}{3}$, we have $t = 1 \text{ second}$

Example 4: The displacement x (in cm) of an oscillating particle varies with time t (in seconds) according to the equation

$$x = 2 \cos \left(0.5 \pi t + \frac{\pi}{3} \right)$$

Find

- amplitude of oscillation
- the time period of oscillation
- the maximum velocity of the particle
- the maximum acceleration of the particle.

Solution: The displacement of the particle is given by

$$x = 2 \cos \left(0.5 \pi t + \frac{\pi}{3} \right) \text{ cm}$$

To find the amplitude and time period of the oscillation, we compare this equation with

$$x = A \cos (\omega t + \delta)$$

- Amplitude $A = 2 \text{ cm}$
- Angular frequency $\omega = 0.5 \pi \text{ rad s}^{-1}$

or $T = \frac{2\pi}{\omega} = \frac{2\pi}{0.5\pi} = 4 \text{ s}$

$$\begin{aligned}
 \text{(c) Maximum velocity } V_{\max} &= |A \omega| \\
 &= 2 \times 0.5 \pi = \pi \text{ cm s}^{-1} \\
 &= 3.142 \text{ cm s}^{-1}
 \end{aligned}$$

$$\begin{aligned}
 \text{(d) Maximum acceleration } a_{\max} &= |-\omega^2 A| \\
 &= \omega^2 A \\
 &= (0.5 \pi)^2 \times 2 \\
 &= \frac{\pi^2}{2} \text{ cm s}^{-2} = 4.935 \text{ cm s}^{-2}
 \end{aligned}$$

Example 5: A body moves with SHM about a point $x = 0$. At time $t = 0$ it has a displacement of 5 cm and zero velocity. If the frequency of the motion is 0.5 Hz, find the magnitudes of the displacement, velocity and acceleration at $t = 1.25$ s.

Solution: Let the displacement of the body be given by

$$x = A \sin(\omega t + \phi)$$

Its velocity is given by

$$V = A \omega \cos(\omega t + \phi)$$

Since $V = 0$ at $t = 0$, we have $0 = A \omega \cos \phi$ which

$$\text{gives } \phi = \frac{\pi}{2}$$

Thus we have

$$x = A \sin\left(\omega t + \frac{\pi}{2}\right) = A \cos \omega t$$

At $t = 0$, $x = A \cos 0 = A = 5 \text{ cm}$ (Given). Also

$$\omega = 2 \pi n = 2 \pi \times 0.5 = \pi \text{ rad s}^{-1}. \text{ Thus we have}$$

$$x = 5 \cos \pi t$$

$$\therefore \text{ Displacement at } t = 1.25 \text{ s} = 5 \cos (1.25 \pi) = 5 \cos \left(\pi + \frac{\pi}{4}\right)$$

$$= 5 \cos \pi \cos \frac{\pi}{4} = -5 \times \frac{1}{\sqrt{2}} = -3.535 \text{ cm}$$

$$\therefore \text{ Magnitude of displacement} = 3.535 \text{ cm}$$

Velocity at $t = 1.25$ s is

$$\begin{aligned}
 V &= -A \omega \sin \omega t = -5 \pi \sin(\pi t) \\
 &= -5 \pi \sin(1.25 \pi) \\
 &= -5 \pi \sin\left(\pi + \frac{\pi}{4}\right) \\
 &= -5 \pi \cos \pi \sin \frac{\pi}{4} \\
 &= 5 \times 3.142 \times \sin 45^\circ \\
 &= 11.5 \text{ cm s}^{-1}
 \end{aligned}$$

Acceleration at $t = 1.25$ s is

$$\begin{aligned} a &= -\omega^2 x = -\omega^2 A \cos \omega t \\ &= -\pi^2 \times 5 \times \cos (1.25 \pi) \\ &= \pi^2 \times 5 \times \cos 45^\circ \\ &= 34.9 \text{ cm s}^{-2} \end{aligned}$$

Example 6: The displacement of an oscillating particle is given by

$$x = a \sin \omega t + b \cos \omega t$$

where a , b and ω are constants. (a) Show that the motion of the particle is simple harmonic. (b) Also show that the amplitude A is given by

$$A = (a^2 + b^2)^{1/2}$$

Solution: The displacement of the particle is given by

$$x = a \sin \omega t + b \cos \omega t \quad (i)$$

Differentiating this expression with respect to t we obtain the velocity V of the particle.

$$\begin{aligned} V &= \frac{dx}{dt} = \frac{d}{dt} (a \sin \omega t + b \cos \omega t) \\ &= \omega (a \cos \omega t - b \sin \omega t), \end{aligned}$$

($\because$ a, b, ω are constants)

The acceleration a of the particle is given by

$$\begin{aligned} a &= \frac{dV}{dt} = \frac{d}{dt} \{ \omega (a \cos \omega t - b \sin \omega t) \} \\ &= -\omega^2 (a \sin \omega t + b \cos \omega t) \end{aligned} \quad (ii)$$

From Eqs (i) and (ii) we get

$$a = -\omega^2 x$$

or acceleration = - constant $\times$ displacement

Since the acceleration is proportional to the displacement and acts in a direction opposite to displacement, the motion of the particle is simple harmonic.

(b) Since the motion of the particle is simple harmonic, we may write the displacement as

$$x = A \sin (\omega t + \phi) \quad (iii)$$

where A is the amplitude.

From Eqs (i) and (iii) we have

$$A \sin (\omega t + \phi) = a \sin \omega t + b \cos \omega t$$

$$\text{or } A (\sin \omega t \cos \phi + \cos \omega t \sin \phi) = a \sin \omega t + b \cos \omega t$$

$$\text{or } (A \cos \phi - a) \sin \omega t + (A \sin \phi - b) \cos \omega t = 0$$

This equation will hold for all values of t if the coefficients of $\sin \omega t$ and $\cos \omega t$ vanish separately, i.e. if

$$A \cos \phi - a = 0$$

and $A \sin \phi - b = 0$

or $A \cos \phi = a$ and $A \sin \phi = b$

which give $A^2 = a^2 + b^2$

or $A = (a^2 + b^2)^{1/2}$

Note: The phase constant ϕ of the motion described by Eq. (i) or Eq. (iii) can also be obtained. It is given by

$$\tan \phi = \frac{b}{a}$$

or $\phi = \tan^{-1} \left(\frac{b}{a} \right)$

Example 7: A body executing linear SHM has a velocity of 3 cm s^{-1} when its displacement is 4 cm and a velocity of 4 cm s^{-1} when its displacement is 3 cm .

(a) Find the amplitude and period of the oscillation. (b) If the mass of the body is 50 g , calculate the total energy of oscillation.

Solution: (a) In SHM, the velocity V at a displacement x is given by

$$V = \omega (A^2 - x^2)^{1/2}$$

or $V^2 = \omega^2 (A^2 - x^2)$

Now $V = 3 \text{ cm s}^{-1}$ when $x = 4 \text{ cm}$. Therefore

$$9 = \omega^2 (A^2 - 16) \quad \text{(i)}$$

Also $V = 4 \text{ cm s}^{-1}$ when $x = 3 \text{ cm}$. Therefore

$$16 = \omega^2 (A^2 - 9) \quad \text{(ii)}$$

Simultaneous solution of Eqs (i) and (ii) gives

$$\text{Amplitude } A = 5 \text{ cm}$$

and angular frequency $\omega = 1 \text{ rad s}^{-1}$

$$\text{Hence time period } T = \frac{2\pi}{\omega} = 2\pi \text{ seconds}$$

$$\approx 6.28 \text{ s}$$

(b) $m = 50 \text{ g} = 50 \times 10^{-3} \text{ kg}$

$$A = 5 \text{ cm} = 5 \times 10^{-2} \text{ m}$$

$$\omega = 1 \text{ rad s}^{-1}$$

$$\text{Total energy} = \frac{1}{2} m A^2 \omega^2$$

$$= \frac{1}{2} \times (50 \times 10^{-3}) \times (5 \times 10^{-2})^2 (1)^2$$

$$= 6.25 \times 10^{-5} \text{ J}$$

Example 8: A particle is executing linear SHM of amplitude A .

(a) What fraction of the total energy is kinetic when the displacement is half the amplitude? (b) At what displacement is the energy half kinetic and half potential?

Solution:

$$\text{Kinetic energy (KE)} = \frac{1}{2} m \omega^2 (A^2 - x^2)$$

$$\text{Potential energy (PE)} = \frac{1}{2} m \omega^2 x^2$$

$$\text{Total energy (E)} = \frac{1}{2} m \omega^2 A^2$$

(a) When $x = A/2$

$$\text{KE} = \frac{1}{2} m \omega^2 \left(A^2 - \frac{A^2}{4} \right) = \frac{3}{8} m \omega^2 A^2$$

$$E = \frac{1}{2} m \omega^2 A^2$$

$$\therefore \frac{\text{KE}}{E} = \frac{3}{4}$$

(b) The energy will be half kinetic and half potential at a value of x when $\text{KE} = \text{PE}$, i.e.

$$\frac{1}{2} m \omega^2 (A^2 - x^2) = \frac{1}{2} m \omega^2 x^2$$

$$\text{or } A^2 - x^2 = x^2$$

$$\text{or } x = \frac{A}{\sqrt{2}}$$

Example 9: A horizontal platform with an object placed on it is executing SHM in the vertical direction. The amplitude of oscillation is 2.5 cm. What must be the least period of these oscillations so that the object is not detached from the platform? Take $g = 10 \text{ m s}^{-2}$.

Solution: The object will not detach from the platform, if the angular frequency ω is such that, during the downward motion, the maximum acceleration equals the acceleration due to gravity, i.e.

$$\omega_{\max}^2 A = g$$

$$\text{or } \omega_{\max} = \sqrt{\frac{g}{A}}$$

$$\text{or } T_{\min} = \frac{2\pi}{\omega_{\max}} = 2\pi \sqrt{\frac{A}{g}}$$

$$\text{Now } A = 2.5 \text{ cm} = 2.5 \times 10^{-2} \text{ m and } g = 10 \text{ m s}^{-2}.$$

Substituting these values we get

$$T_{\min} = \frac{\pi}{10} = 0.314 \text{ s}$$

Example 10: A spring with no mass attached to it hangs from a rigid support. A mass m is now hung on the lower end of the spring. The mass is supported on a platform so that the spring remains relaxed. The supporting platform is then suddenly removed and the mass begins to oscillate. The lowest position of the mass during the oscillation is 5 cm below the

place where it was resting on the platform. (a). What is the frequency of oscillation? (b) What is the velocity when the mass is 2.5 cm below its original resting place? Take $g = 10 \text{ m s}^{-2}$.

Solution: (a) It is clear from Fig. 8.2 that the separation between the two extreme positions of the oscillating mass is 5 cm. Therefore, the equilibrium position O is 2.5 cm below the supporting platform. In other words, force mg produces an extension $y=2.5 \text{ cm}$ in the spring. If k is the force constant of the spring we have

$$mg = ky$$

$$\text{or } \frac{k}{m} = \frac{g}{y} = \frac{1000 \text{ cm s}^{-2}}{2.5 \text{ cm}} = 400 \text{ s}^{-2}$$

The angular frequency ω of oscillation is

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{400} = 20 \text{ rad s}^{-1}$$

$$\text{Hence frequency } n = \frac{\omega}{2\pi} = \frac{20}{2 \times 3.142} = 3.18 \text{ Hz}$$

(b) Amplitude of oscillation is

$$A = 2.5 \text{ cm.}$$

When the mass is 2.5 cm below the platform, it is passing through the equilibrium position O and hence has the maximum velocity given by

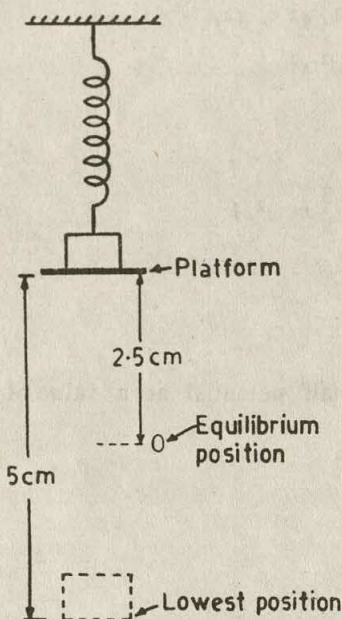


Fig. 8.2

$$\begin{aligned} V_{\max} &= \omega A \\ &= 20 \times 2.5 = 50 \text{ cm s}^{-1} \end{aligned}$$

Example 11: When a mass is hung from the lower end of a spring of negligible mass, an extension of 10 cm is produced in the spring. The mass is set into vertical oscillations. What is the period of oscillations?

Solution: Let the mass be m and the force constant of the spring be k . When the mass is hung from the spring, the force mg produces an extension $y = 10 \text{ cm.} = 0.1 \text{ m}$. Thus

$$mg = ky$$

$$\text{or } \frac{m}{k} = \frac{y}{g}$$

Period of vertical oscillations of the mass is given by

$$\begin{aligned} T &= \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{k}} = 2\pi \sqrt{\frac{y}{g}} \\ &= 2 \times 3.142 \times \sqrt{\frac{0.1}{9.8}} \\ &= 0.635 \text{ s} \end{aligned}$$

Example 12: A small spherical steel ball is placed a little away from the centre of a large concave mirror whose radius of curvature $R = 2.5$ m. When the ball is released, it begins to oscillate about the centre. Show that the motion of the ball is simple harmonic and find the period of the motion. Neglect friction and take $g = 10 \text{ m s}^{-2}$.

Solution: Let the steel ball be placed at A , a little away from the centre O of a concave mirror of radius of curvature R ($= OC = AC$), as in Fig. 8.3. Let $\angle ACO = \theta$. If m is the mass of the ball, its weight mg acts vertically downwards at A . This force is resolved into two rectangular components: $mg \cos \theta$ (which is balanced by the normal reaction N of the mirror) and $mg \sin \theta$ (which provides the restoring force F). Thus

$$F = -mg \sin \theta$$

$$= -mg \theta \quad (\because \theta \text{ is small; } x/R \text{ being very small})$$

$$= -mg x/R \quad (\because x = R \theta; x \text{ being the arc } OA)$$

or

$$F = -kx$$

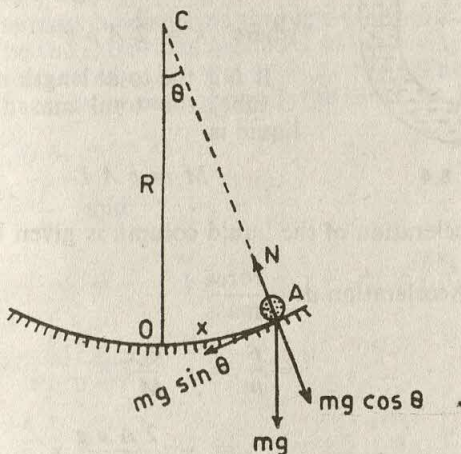


Fig. 8.3

where force constant $k = \frac{mg}{R}$ The acceleration a is given by

$$a = \frac{F}{m} = -\frac{k}{m}x = -\omega^2 x$$

where $\omega = \sqrt{\frac{k}{m}}$

Hence the motion is simple harmonic. The angular frequency ω is given by

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{mg}{mR}} = \sqrt{\frac{g}{R}}$$

$$\text{Time period } T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{R}{g}}$$

$$= 2 \times 3.142 \times \sqrt{\frac{2.5}{10}}$$

$$= 3.142 \text{ s}$$

Example 13: A vertical U-tube of uniform cross-sectional area A contains a liquid of density ρ . The liquid column is disturbed by gently blowing into the tube. Show that the resulting oscillation of the liquid is simple harmonic and find the period of oscillation. Neglect viscous effects.

Solution: Let the liquid level in the left arm be depressed by, say x . The liquid level in the right arm is raised by an equal amount x . When left, the liquid levels will oscillate in each arm about their respective equilibrium positions A, A' (see Fig. 8.4). If viscous effects are neglected, the liquid will never come to rest.

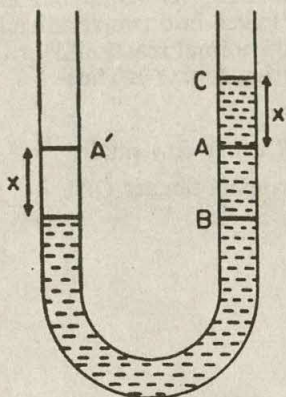


Fig. 8.4

Since the column in the right arm is higher by $2x$ than the column in the left arm, the mass of this column of liquid is $m = 2A\rho x$. The gravitational force mg of this column provides the restoring force F . Thus

$$F = -mg = -2A\rho g x = -kx$$

$$\text{where } k = 2A\rho g$$

If L is the total length of the liquid in the U-tube, the total mass M of the oscillating liquid is

$$M = \rho AL$$

Hence the acceleration of the liquid column is given by

$$\begin{aligned} \text{Acceleration } a &= \frac{\text{force}}{\text{mass}} \\ &= \frac{F}{m} = -\frac{kx}{M} \\ &= -\frac{2A\rho g}{\rho AL} x \\ &= -\frac{2g}{L} x \end{aligned}$$

$$\text{or } a = -\omega^2 x; \text{ with } \omega = \sqrt{\frac{2g}{L}}$$

Hence the motion is simple harmonic. The time period of the motion is given by

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{L}{2g}}$$

Note: The period of oscillation is independent of the density ρ of the liquid and the cross-sectional area A of the U-tube.

Example 14: Two springs of force constants k_1 and k_2 are connected to a mass m placed on a horizontal frictionless surface as shown in Fig. 8.5 (a) and (b). Obtain the expression for the time period of horizontal oscillation in each case.

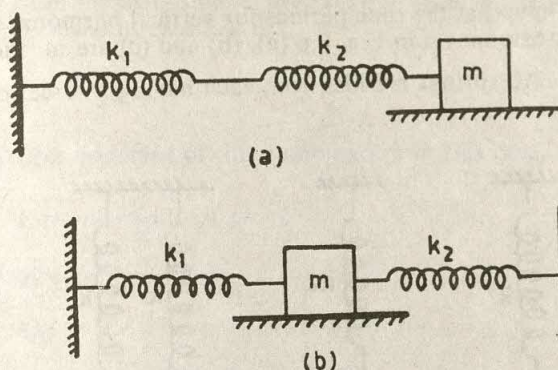


Fig. 8.5

Solution: (a) Let the mass be displaced through a distance x , say to the right. Let x_1 and x_2 be the extensions produced in the springs of force constants k_1 and k_2 respectively, so that $x = x_1 + x_2$. Since the restoring force F exerted on the mass by each spring is the same, we have

$$F = -k_1 x_1 = -k_2 x_2$$

giving $x_1 = -\frac{F}{k_1}$ and $x_2 = -\frac{F}{k_2}$

$$\therefore x = x_1 + x_2 = -F \left(\frac{1}{k_1} + \frac{1}{k_2} \right) = -F \left(\frac{k_1 + k_2}{k_1 k_2} \right)$$

Thus $F = -\frac{k_1 k_2}{(k_1 + k_2)} x$

or $F = -k x$

where $k = \frac{k_1 k_2}{(k_1 + k_2)}$ is the effective force constant of the combination.

Hence the time period of oscillation is

$$T = 2\pi \sqrt{\frac{m}{k}} = 2\pi \sqrt{\frac{m(k_1 + k_2)}{k_1 k_2}}$$

(b) In this case, if the mass is displaced through a distance x , say to the right, the spring k_1 is extended by x and spring k_2 is compressed by x ; so that the restoring force exerted by each spring on mass m is in the same direction, tending to bring it to its original equilibrium position. If F_1 and F_2 are the restoring forces due to k_1 and k_2 respectively, we have

$$F_1 = -k_1 x \quad \text{and} \quad F_2 = -k_2 x$$

$$\therefore \text{Total restoring force } F = -k_1 x - k_2 x = -(k_1 + k_2) x$$

$$F = -k x$$

where $k = (k_1 + k_2)$ is the effective force constant of the combination.

Hence

$$T = 2\pi \sqrt{\frac{m}{k}} = 2\pi \sqrt{\frac{m}{(k_1 + k_2)}}$$

Example 15: Show that the time periods for vertical harmonic oscillations of the three systems shown in Fig. 8.6 (a), (b) and (c) are in the ratio of $1 : \sqrt{2} : \frac{1}{\sqrt{2}}$. All springs are identical, each having a force constant k .

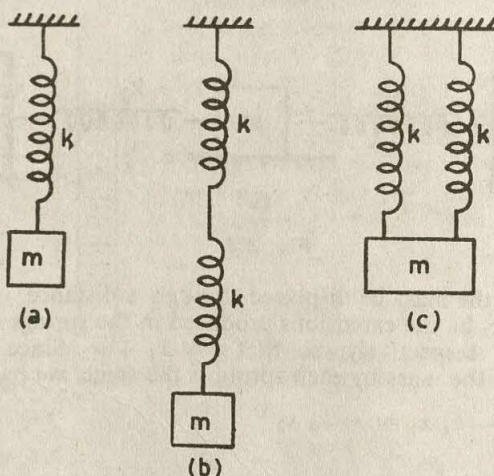


Fig. 8.6

Solution: Let us suppose that an extension x is produced in the spring when a force mg is applied to it. The equilibrium position in case (a) is given by

$$F = mg = kx \quad (i)$$

The time period in this case is given by

$$T_a = 2\pi \sqrt{\frac{\text{mass}}{\text{spring constant}}} = 2\pi \sqrt{\frac{m}{k}} \quad (ii)$$

(b) In this case, the length of the spring is doubled. Hence a given force mg will double the extension. Let x' be the extension produced and k_{eff} be the force constant of the combination. Thus

$$\begin{aligned} F = mg &= k_{\text{eff}} x' \\ &= 2 k_{\text{eff}} x \quad (\because x' = 2x) \end{aligned} \quad (iii)$$

Comparing (i) and (iii) we get

$$F = kx = 2 k_{\text{eff}} x$$

or
$$k_{\text{eff}} = \frac{k}{2}$$

Thus the time period in this case is given by

$$T_b = 2\pi \sqrt{\frac{m}{k_{\text{eff}}}} = 2\pi \sqrt{\frac{2m}{k}} = \sqrt{2} T_a \quad (iv)$$

(c) In this case, the extension x'' produced in each spring by a force mg is half that produced in case (a), i.e.

$$x'' = \frac{x}{2}$$

If k_{eff} is the force constant of the combination in this case, we have

$$F = mg = k_{\text{eff}} x'' = \frac{k_{\text{eff}}}{2} x \quad (\text{v})$$

Comparing (v) with (i) we have

$$k_{\text{eff}} = 2k$$

$$\text{Hence, } T_c = 2\pi \sqrt{\frac{m}{k_{\text{eff}}}} = 2\pi \sqrt{\frac{m}{2k}} = \frac{T_a}{\sqrt{2}} \quad (\text{vi})$$

From (ii), (iv) and (vi) we have

$$T_a : T_b : T_c = 1 : \sqrt{2} : \frac{1}{\sqrt{2}}$$

Example 16: A simple pendulum is displaced from its equilibrium position O to a position P until the height of P above O is 0.05 m (see Fig. 8.7). It is then released. (a) Calculate the velocity V of the bob when it passes the mean position O . Take $g = 10 \text{ m s}^{-2}$ and neglect friction. (b) What is the tension in the supporting string at this instant? The mass of the bob is 50 g and the time period is 2 s .

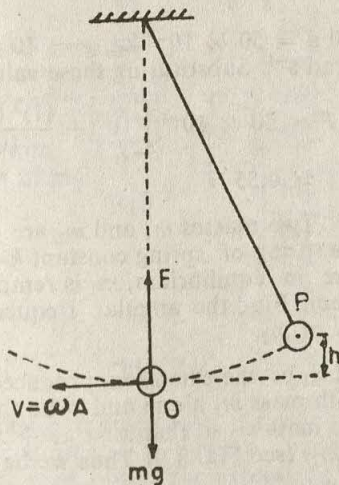


Fig. 8.7

Solution: (a) The potential energy of the pendulum at P
 $=$ work done (against gravity) in raising the bob of mass m
to a vertical height h
 $= mgh$

When the bob is released, it oscillates. The energy when it passes through O is entirely kinetic and is equal to $\frac{1}{2} mV^2$ where V is the velocity at point

O . Since friction is neglected, the entire potential energy at P is converted into kinetic energy at O , giving

$$\frac{1}{2} m V^2 = mgh$$

or

$$V = \sqrt{2gh}$$

$$= \sqrt{2 \times 10 \times 0.05} = 1 \text{ ms}^{-1}$$

(b) Time period $T = 2 \text{ s}$

$$\therefore \text{Angular frequency } \omega = \frac{2\pi}{T} = \pi \text{ rad s}^{-1}$$

At position O , the tension F in the supporting string is given by

$$(F - mg) = \text{centripetal force} = \frac{m V^2}{l}$$

where

l = length of the string. We know that

$$T = 2\pi \sqrt{\frac{l}{g}} \text{ or } \omega = \frac{2\pi}{T} = \sqrt{\frac{g}{l}}$$

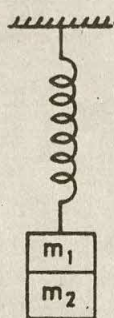
Thus

$$l = \frac{g}{\omega^2}$$

Hence

$$F = mg + \frac{m V^2 \omega^2}{g}$$

$$= m \left(g + \frac{V^2 \omega^2}{g} \right)$$



Here $m = 50 \text{ g} = 50 \times 10^{-3} \text{ kg}$, $g = 10 \text{ ms}^{-2}$, $V = 1 \text{ ms}^{-1}$ and $\omega = \pi \text{ rad s}^{-1}$. Substituting these values we get

$$F = 50 \times 10^{-3} \left[10 + \frac{(1)^2 (\pi)^2}{10} \right]$$

$$\approx 0.55 \text{ N}$$

Example 17: Two masses m_1 and m_2 are suspended together by a massless spring of spring constant k (see Fig. 8.8). When the masses are in equilibrium, m_1 is removed without disturbing the system. Find the angular frequency and amplitude of oscillation of m_2 . (IIT 1981)

Fig. 8.8 **Solution:** Let x_1 be the extension produced in the spring when it is loaded with mass m_2 alone and x_2 be the further extension when mass m_1 is added to mass m_2 so that $x = x_1 + x_2$ is the total extension produced by $(m_1 + m_2)$ (see Fig. 8.9) Thus we have,

For equilibrium state of m_2

$$m_2 g = k x_1 \quad (i)$$

For equilibrium state of $(m_1 + m_2)$

$$(m_1 + m_2) g = k (x) = k (x_1 + x_2) \quad (ii)$$

When the mass m_1 is removed, the mass m_2 will move upwards under the unbalanced force $= m_1 g$. Hence

$$\text{Restoring force (F) on } m_2 = - m_1 g$$

Subtracting (i) and (ii) we have

$$m_1 g = k x_2 \quad \text{(iii)}$$

Hence, Restoring force on $m_2 = -k x_2$

$$\therefore \text{accelerating of } m_2 = \frac{F}{m_2} = -\frac{k}{m_2} x_2$$

i.e. acceleration $\propto$ - displacement

$$\therefore \text{Angular frequency is } \omega = \sqrt{\frac{k}{m_2}}$$

$$\therefore \text{Frequency of oscillation is } n = \frac{\omega}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{k}{m_2}}$$

It is clear that A is the equilibrium position of m_2 and B its maximum displacement position. Hence $AB = x_2$ is the amplitude of oscillation of m_2 which from Eq. (iii) is given by

$$\text{Amplitude} = x_2 = \frac{m_1 g}{k}$$

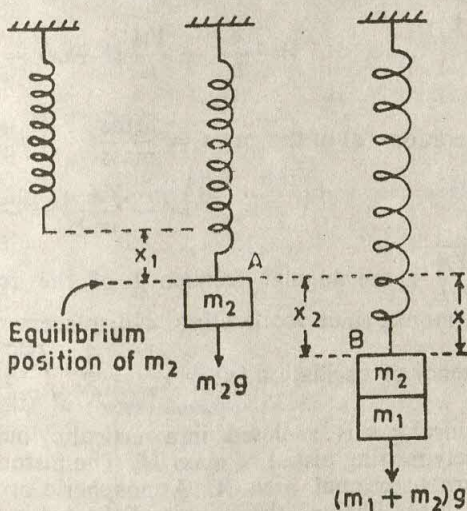


Fig. 8.9

Example 18: A point mass m is suspended at the end of a massless wire of length L and cross-sectional area A . If Y is the Young's modulus of elasticity of the material of the wire, obtain the expression for the frequency of the simple harmonic motion along the vertical line. (IIT 1978)

Solution: Let l be the increase in the length of the wire due to the force $F = mg$ (see Fig. 8.10). Then

$$\text{Stress} = \frac{\text{Force}}{\text{Area}} = \frac{F}{A}$$

$$\text{Strain} = \frac{\text{Change in length}}{\text{Original length}} = \frac{l}{L}$$

By definition, Young's modulus is

$$Y = \frac{\text{Stress}}{\text{Strain}} = \frac{F/A}{l/L} = \frac{mg L}{l A}$$

or $F = mg = \frac{YA l}{L}$ (i)

This is the force acting upwards in the equilibrium state.

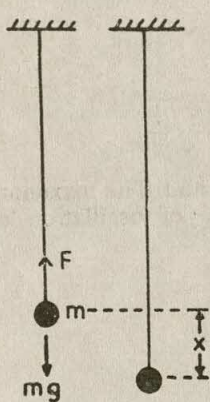


Fig. 8.10

If the mass is pulled down a little through a distance x , so that the total extension in the string is $(l + x)$, then the force in the wire acting upwards will be

$$= \frac{YA}{L}(l + x)$$

and downward force is $F = mg$. The restoring force is the net downward force. Hence, from Eq. (i)

$$\begin{aligned} \text{Restoring force} &= mg - \frac{YA}{L}(l + x) \\ &= \frac{YA l}{L} - \frac{YA}{L}(l + x) = - \frac{YA x}{L} \end{aligned}$$

$\therefore$ Acceleration (a) of the mass $= \frac{\text{force}}{\text{mass}}$

$$= - \frac{YA x}{m L} = - \omega^2 x$$

where $\omega = \sqrt{\frac{YA}{mL}}$ is the angular frequency of the resulting motion which is simple harmonic, since acceleration $\propto$ -displacement x . Thus,

$$\text{Frequency of oscillation } (n) = \frac{\omega}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{YA}{mL}}$$

Example 19: An ideal gas is enclosed in a vertical cylindrical container and supports a freely moving piston of mass M . The piston and the cylinder have equal cross-sectional area A . Atmospheric pressure is P_0 ; and when the piston is in equilibrium, the volume of the gas is V_0 . The piston is now displaced slightly from the equilibrium position. Assuming that the system is completely isolated from its surroundings, show that the piston executes simple harmonic motion and find the frequency of oscillation. (IIT 1981)

Solution: Since the system is completely isolated from its surroundings, it cannot give heat to or take heat from the surroundings. Hence the pressure and volume of the gas are related to each other by the adiabatic relation

$$PV^\gamma = \text{constant} \quad (i)$$

where γ is the ratio of the specific heats at constant pressure and constant volume.

In the equilibrium state, the total pressure P is given by

$$P = \text{atmospheric pressure} + \text{force per unit area exerted by the piston on the enclosed gas}$$

$$\text{or} \quad P = P_0 + \frac{Mg}{A} \quad (\text{ii})$$

Initial volume $V = V_0$

When the piston is given a small displacement x , let ΔP be the change in pressure and $\Delta V (= Ax)$ be the change in volume. Then Eq. (i) gives

$$\begin{aligned} P V_0^\gamma &= (P + \Delta P) (V_0 + \Delta V)^\gamma \\ &= (P + \Delta P) V_0^\gamma \left(1 + \frac{\Delta V}{V_0}\right)^\gamma \end{aligned}$$

Since $\frac{\Delta V}{V_0}$ is a small fraction, we could Binomially expand $\left(1 + \frac{\Delta V}{V_0}\right)^\gamma$ and retain only the first order terms in $\frac{\Delta V}{V_0}$, as

$$\left(1 + \frac{\Delta V}{V_0}\right)^\gamma \simeq 1 + \gamma \frac{\Delta V}{V_0}$$

$$\begin{aligned} \text{Thus} \quad P V_0^\gamma &= (P + \Delta P) V_0^\gamma \left(1 + \gamma \frac{\Delta V}{V_0}\right) \\ &= P V_0^\gamma \left(1 + \frac{\Delta P}{P}\right) \left(1 + \gamma \frac{\Delta V}{V_0}\right) \\ &\simeq P V_0^\gamma \left(1 + \frac{\Delta P}{P} + \gamma \frac{\Delta V}{V_0}\right) \end{aligned}$$

where the second order term $\left(\frac{\Delta P}{P} \frac{\Delta V}{V_0}\right)$ has been neglected. Thus

$$1 = 1 + \frac{\Delta P}{P} + \gamma \frac{\Delta V}{V_0}$$

$$\text{or} \quad \Delta P = -\gamma P \frac{\Delta V}{V_0}$$

$$\begin{aligned} \text{Restoring force } F &= \text{excess pressure} \times \text{area} \\ &= \Delta P \times A \end{aligned}$$

$$\begin{aligned} \text{or} \quad F &= -\gamma P A \frac{\Delta V}{V_0} \\ &= -\gamma \frac{A x}{V_0} \left(P_0 + \frac{Mg}{A}\right) A \end{aligned}$$

where we have used Eq. (ii) and $\Delta V = Ax$. Thus

$$F = -kx$$

$$\text{where} \quad k = \frac{\gamma A^2}{V_0} \left(P_0 + \frac{Mg}{A}\right)$$

Since restoring force $\propto$ displacement, the motion of the piston is simple harmonic with the force constant k . The angular frequency of oscillation is

$$\omega = \sqrt{\frac{k}{M}}$$

∴ Frequency of oscillation is given by

$$n = \frac{\omega}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{k}{M}}$$

or
$$n = \frac{1}{2\pi} \sqrt{\frac{\gamma A^2 \left(P_0 + \frac{Mg}{A} \right)}{MV_0}}$$

Example 20: A thin fixed ring of radius 1 m has a positive charge of 1×10^{-5} coulomb uniformly distributed over it. A particle of mass 0.9 g and having a negative charge of 1×10^{-6} coulomb is placed on the axis at a distance of 1 cm from the centre of the ring. Show that the motion of the negatively charged particle is approximately simple harmonic. Calculate the time period of oscillations. (IIT 1982)

Solution: Let r be the distance of the negative charge q from the centre O of the ring of radius R . If Q is the charge on the ring, the electric field E at point P is given by

$$E = \frac{Qr}{4\pi\epsilon_0(R^2 + r^2)^{3/2}}$$

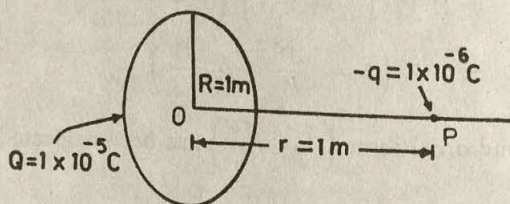


Fig. 8.11

where ϵ_0 is the electrical permittivity of free space for which

$$\frac{1}{4\pi\epsilon_0} = 9 \times 10^9 \text{ N m}^2 \text{ C}^{-2}$$

Since $r = 1 \text{ cm}$ and $R = 1 \text{ m} = 100 \text{ cm}$, we may write

$$(R^2 + r^2)^{3/2} = R^3 \left(1 + \frac{r^2}{R^2} \right)^{3/2} \approx R^3$$

Thus for $r \ll R$, we have

$$E = \frac{Qr}{4\pi\epsilon_0 R^3}$$

The force experienced by the negative charge q is

$$\begin{aligned} F &= -qE = -\frac{qQr}{4\pi\epsilon_0 R^3} \\ &= -kr \end{aligned}$$

where
$$k = \frac{qQ}{4\pi\epsilon_0 R^3}$$

The acceleration of the negatively charged particle is

$$a = \frac{F}{m} = -\frac{k r}{m} = -\omega^2 r$$

where m is the mass of the particle and $\omega = \sqrt{\frac{k}{m}}$. Thus, acceleration is proportional to displacement r along the axis and is directed towards the centre of the ring, if $r \ll R$. Hence the particle executes simple harmonic motion whose time period T is given by

$$\begin{aligned} T &= \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{k}} \\ &= 2\pi \sqrt{\frac{4\pi\epsilon_0 m R^3}{q Q}} \\ &= 2\pi \sqrt{\frac{(0.9 \times 10^{-3}) \times (1)^3}{(9 \times 10^9) \times (10^{-5}) \times (10^{-6})}} \\ &= \frac{\pi}{5} = 0.63 \text{ s} \end{aligned}$$

8.8 TRUE-FALSE STATEMENTS WITH REASONS

A few statements are given below. We have to decide whether they are true or false giving, in brief, the reasons in support of our answers.

1. Any physical system, when disturbed, will execute simple harmonic motion.
2. The greater the mass of a pendulum bob, the shorter is its period of oscillation.
3. The phase of an oscillator is determined by its displacement and velocity at time $t = 0$.
4. The amplitude of a pendulum, oscillating in air, decreases with time.
5. All simple harmonic motions are periodic, but all periodic motions are not necessarily simple harmonic.

ANSWERS

1. **False** In order to execute simple harmonic motion the system must have elasticity and inertia. Plastic systems cannot oscillate.
2. **False** The time period of a simple pendulum is independent of the mass of its bob.
3. **True** The phase of an oscillator is determined by how it is set into oscillation initially, i.e. by its initial displacement and velocity.
4. **True** Due to the friction offered by air, the energy (and hence its amplitude) will decrease with time. The loss in energy appears as heat in the system.
5. **True** The motion of the blades of a fan or the hands of a clock is periodic but not simple harmonic. The necessary condition for SHM is that the restoring force must be proportional to the displacement and must be directed towards the mean position.

8.9 MULTIPLE CHOICE QUESTIONS

Choose the correct answer from the given alternatives.

1. Which of the following equations represent a simple harmonic motion? The displacement of the system from the equilibrium position is x at time t and α is a positive constant.

- (a) $\frac{d^2x}{dt^2} - \alpha x = 0$ (b) $\frac{d^2x}{dt^2} + \alpha x = 0$
 (c) $\frac{d^2x}{dt^2} + \alpha x^2 = 0$ (d) $\frac{d^2x}{dt^2} + \frac{\alpha}{x} = 0$
2. The time period of the motion represented by the correct equation in Q. 1 is
 (a) $2\pi \alpha$ (b) $2\pi \sqrt{\alpha}$
 (c) $2\pi/\alpha$ (d) $2\pi/\sqrt{\alpha}$
3. Figure 8.12 shows the displacement-time graphs of two simple harmonic motions I and II. From the graphs it follows that the two motions I and II have
 (a) equal amplitudes, equal frequencies and zero phase difference
 (b) unequal amplitudes, unequal frequencies and a finite phase difference
 (c) equal amplitudes, unequal frequencies and zero phase difference
 (d) equal amplitudes, equal frequencies and finite phase difference
4. What is the phase difference between the two simple harmonic motions whose displacement-time graphs are shown in Fig. 8.12?
 (a) zero (b) $\pi/4$
 (c) $\pi/2$ (d) π

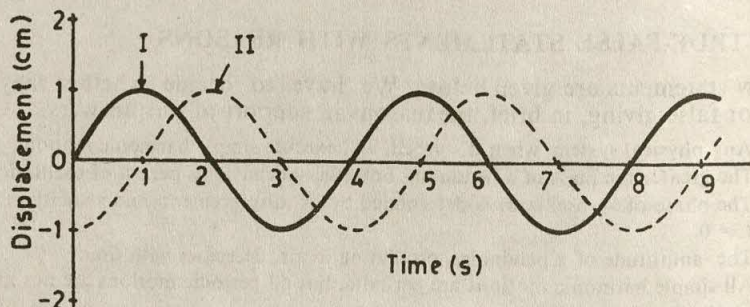


Fig. 8.12

5. The displacement x (in metres) of a particle in simple harmonic motion is related to time t (in seconds) as

$$x = 0.01 \cos \left(\pi t + \frac{\pi}{4} \right)$$

What is the frequency of the motion?

- (a) 0.5 Hz (b) 1.0 Hz
 (c) $\frac{\pi}{2}$ Hz (d) π Hz.
6. A simple pendulum of frequency n is taken upto a certain height above the ground and then dropped along with its support so that it falls freely under gravity. The frequency of oscillations of the falling pendulum will
 (a) remain equal to n (b) become greater than n
 (c) become less than n (d) become zero
7. A person measures the time period of a simple pendulum inside a stationary lift and finds it to be T . If the lift starts accelerating upwards with an acceleration of $g/3$, the time period of the pendulum will be
 (a) $\sqrt{3} T$ (b) $\sqrt{\frac{3}{2}} T$
 (c) $T/\sqrt{3}$ (d) $T/3$

8. The length of a simple pendulum is increased by 44%. What is the percentage increase in its time period?

(a) 10% (b) 20%
(c) 40% (d) 44%

9. A hollow metallic sphere filled with water is hung from a support by a long thread. A small hole is made at the bottom of the sphere and it is oscillated. How will the time period of oscillations be affected as water slowly flows out of the hole?

(a) The time period will remain unchanged as water is flowing out
(b) The time period will keep increasing until the sphere is empty
(c) The time period will keep decreasing until the sphere is empty
(d) The time period will increase at first, then decrease until the sphere is empty.

Finally the period will be the same as that when the sphere was full of water.

10. A simple pendulum with a brass bob has a time period T . The bob is now immersed in a non-viscous liquid and oscillated. If the density of the liquid is $1/8$ th that of brass, the time period of the same pendulum will be

(a) $\sqrt{\frac{8}{7}} T$ (b) $\frac{8}{7} T$
(c) $\frac{8^2}{7^2} T$ (d) T

11. A simple pendulum of frequency f has a metal bob. If the bob is charged negatively and is allowed to oscillate with a positively charged plate placed under it, the frequency will

(a) remain equal to f (b) become less than f
(c) become more than f (d) become zero

12. A wooden block is suspended from a spring. Its period for vertical oscillations is T . In a second experiment the same apparatus is laid on a horizontal frictionless table with the free end of the spring fixed to a nail on the table. The block is pulled a little so as to stretch the spring and then released. The mass will

(a) execute simple harmonic motion with period T
(b) execute simple harmonic motion with period $T/2$
(c) execute simple harmonic motion with period $2T$
(d) not execute simple harmonic motion

13. A uniform spring has a certain mass suspended from it and its period for vertical oscillations is T_1 . The spring is now cut into two equal halves and the same mass is suspended from one of the halves. The period of vertical oscillations now is T_2 . The ratio T_2/T_1 is

(a) $1/2$ (b) $1/\sqrt{2}$
(c) 2 (d) $\sqrt{2}$

14. A uniform spring of force constant k is cut into two pieces whose lengths are in the ratio of $1 : 2$. What is the force constant of the longer piece of spring?

(a) $k/2$ (b) k
(c) $3k/2$ (d) $2k$

15. A solid brass sphere is hung from a spring whose mass is negligible compared to that of the sphere. The sphere is set into vertical oscillations and its time period is found to be T . The sphere is now completely immersed in a liquid whose density is $1/8$ th that of brass. The time period of vertical oscillations with the sphere always remaining immersed in the liquid will be equal to

(a) T (b) $\sqrt{\frac{7}{8}} T$
(c) $\frac{7}{8} T$ (d) $\left(\frac{7}{8}\right)^2 T$

16. A heavy metal sphere is suspended from a spring. When pulled down a little and released, it oscillates up and down with a frequency f . If it is taken to the moon (where the acceleration due to gravity is one-sixth that on earth) the frequency of vertical oscillation will
- (a) remain unchanged (b) become $6f$
(c) become $f/6$ (d) become $\sqrt{6}f$
17. A test tube of cross-sectional area a has some lead shots in it. The total mass is m . It floats upright in a liquid of density d . When pushed down a little and released, it oscillates up and down with a period T . Use dimensional considerations and choose the correct relationship from the following.
- (a) $T = 2\pi \sqrt{\frac{ma}{gd}}$ (b) $T = 2\pi \sqrt{\frac{mg}{ad}}$
(c) $T = 2\pi \sqrt{\frac{md}{ag}}$ (d) $T = 2\pi \sqrt{\frac{m}{adg}}$
18. A mass M attached to a light spring oscillates with a period of 2 seconds. If the mass is increased by 2 kg the period increases by 1 second. What is the value of M ?
- (a) 1.0 kg (b) 1.6 kg
(c) 2.0 kg (d) 2.4 kg
19. Two equal negative charges $-q$ are fixed at points $(0, a)$ and $(0, -a)$ on the y -axis. A positive charge Q is released from rest at point $(2a, 0)$ on the x -axis. The charge Q will
- (a) execute simple harmonic motion about the origin
(b) move to the origin and remain at rest
(c) move to infinity
(d) execute oscillatory but not simple harmonic motion (IIT 1983)
20. Two simple harmonic motions are represented by the equations
- $$y_1 = 10 \sin(4\pi t + \pi/4)$$
- $$y_2 = 5(\sin 3\pi t + \sqrt{3} \cos 3\pi t)$$
- Their amplitudes are in the ratio of
- (a) 1 : 1 (b) 2 : 1
(c) $2 : \sqrt{3}$ (d) $\sqrt{3} : 2$

8.10 NUMERICAL EXERCISES

- A particle is executing SHM of amplitude 5 cm and a period of 2 s. Find the speed of the particle at a point where its acceleration is half the maximum value.
- A body is moving with SHM of frequency 0.5 Hz and amplitude 4 cm. Starting at a time when the displacement is 4 cm, find the values of displacement, velocity and acceleration 1.25 s later.
- A particle is oscillating with SHM of amplitude 5 cm and period 6 s. How long will it take to move from one end of its path on one side of the equilibrium position to a position 2.5 cm on the other side of the equilibrium position? What is the magnitude of its velocity at this point?
- The equation of a simple harmonic motion is given by

$$x = 3 \sin 5\pi t + 4 \cos 5\pi t$$

where displacement x is in cm and time t in seconds. Find the amplitude; period and phase constant of the motion.

5. A particle in a linear SHM has a maximum velocity of 20 cm s^{-1} and a maximum acceleration of 80 cm s^{-2} . Find the amplitude and angular frequency of oscillation.
6. The amplitude of the SHM of a particle is 5 cm . When its displacement is 3 cm , its velocity has a magnitude of 0.5 m s^{-1} . Find the period of the motion. If the mass of the particle is 10 g , what is its total energy?
7. A load of mass 0.5 kg hangs from a spring of force constant 10 N m^{-1} . The mass is pulled down 0.05 m from its equilibrium position and then released. (a) What is the distance between two most widely separated positions of the mass? (b) How long does it take to traverse this distance?
8. A heavy object, placed on a shock absorber such as a rubber pad, compresses it by 1 cm . If the object is given a vertical tap, it will oscillate. Find the frequency of oscillation. Neglect damping.

Hint : Assume that the pad acts like an elastic spring.

9. A pole of cross-sectional area A and mass M is floating vertically in a liquid of density ρ . The centre of gravity of the pole is below the liquid surface. If the pole is pressed down a little and released, show that the resulting motion is simple harmonic. Obtain an expression for the period of the motion.
10. A uniform spring has a certain mass suspended from it. The spring is cut into two equal halves and the same mass is suspended from one of the halves. Will the frequency of vertical oscillations be the same as before? If not, how is the frequency in the second case related to that in the first?
11. A uniform spring of force constant k is cut into two pieces whose lengths are in the ratio of $1 : 2$. What is the force constant of each piece in terms of k ?
12. A pendulum clock shows accurate time. If the length of its pendulum is increased by 0.1% , what will be the error in time per day?
13. By what percentage should the length of the pendulum of a clock be changed so that it keeps correct time when moved from a place where $g = 9.80 \text{ m s}^{-2}$ to a place where $g = 9.82 \text{ m s}^{-2}$?
14. A person normally weighing 60 kg stands on a platform which is oscillating up and down harmonically with a period of 1.0 s and an amplitude of 10 cm . If a weighing machine on the platform gives the person's weight against time, what will be the maximum and minimum readings shown by it. Take $g = 9.8 \text{ m s}^{-2}$.
15. A platform is executing SHM in a vertical direction with an amplitude of 5 cm and an angular frequency of 20 rad s^{-1} . A block is placed on the platform when it is at the lowest point of its path. (a) At what point will the block leave the platform? (b) How far will the block rise above the highest point reached by the platform? Take $g = 10 \text{ m s}^{-2}$.
16. Suppose a tunnel is dug through the earth (assumed to be a homogeneous sphere) from one side to the other along a diameter. A body dropped into the tunnel takes time t_1 to reach the centre of the earth. If the acceleration due to gravity had remained unchanged, the body would have taken time t_2 to traverse this distance. What is the ratio $t_1 : t_2$? Neglect friction offered to the motion.

(Hint : In the first case, the motion of the body is simple harmonic with a period $T = 2\pi \sqrt{\frac{R}{g}}$, where R is the radius of the earth and g is the acceleration due to gravity at its surface).

ANSWERS

MULTIPLE CHOICE QUESTIONS

- | | | | | |
|---------|---------|---------|---------|---------|
| 1. (b) | 2. (d) | 3. (d) | 4. (c) | 5. (a) |
| 6. (d) | 7. (b) | 8. (b) | 9. (d) | 10. (a) |
| 11. (b) | 12. (a) | 13. (b) | 14. (c) | 15. (a) |
| 16. (a) | 17. (d) | 18. (b) | 19. (d) | 20. (a) |

NUMERICAL EXERCISES

- 13.6 cm s⁻¹
- 2.83 cm, 8.89 cm s⁻¹, 28 cm s⁻²
- 2.5 s, 4.53 cm s⁻¹
- 5 cm, 0.4 s, 53.1°
- 5 cm, 4 rad s⁻¹
- 0.5 s, 1.95 × 10⁻³ J
- (a) 0.1 m, (b) 0.7 s
- 5 Hz approximately
- $T = 2\pi \sqrt{\frac{M}{\rho Ag}}$
- No, the frequency in the second case is $\sqrt{2}$ times that in the first case.
- 3 k, $\frac{3}{2}k$
- The clock slows down by 43.2 seconds per day.
- 0.2 %
- 83.7 kg wt, 36.3 kg wt
- (a) 2.5 cm, (b) 1.25 cm
- $t_1 : t_2 = \pi : 2\sqrt{2}$

Wave Motion and Sound

9.1 REVIEW OF BASIC CONCEPTS

9.1.1 Wave Motion Wave motion involves the transport of energy without any transport of matter. The key word in wave motion is 'disturbance'. In case of mechanical waves, the disturbance is the physical displacement of particles of a medium. In the case of electromagnetic waves, the disturbance is a change in electric and magnetic fields.

9.1.2 Types of Waves There are two types of wave motions: (1) transverse and (2) longitudinal.

(1) *Transverse Waves* In transverse waves the particles of the medium vibrate at right angles to the direction in which the wave propagates. Waves on strings, surface water waves and electromagnetic waves are transverse waves. In electromagnetic waves (which include light waves) the disturbance that travels is not a result of vibrations of particles but it is the oscillation of electric and magnetic fields which takes place at right angles to the direction in which the wave travels.

(2) *Longitudinal Waves* In longitudinal waves the particles of the medium vibrate along the direction of wave propagation. Sound waves are longitudinal.

9.1.3 Characteristics of a Harmonic Wave The characteristics of a harmonic travelling wave are defined as follows.

(1) *Amplitude* The amplitude of a wave is the maximum displacement of the particles of the medium from their mean position.

(2) *Period* The time period of a wave is the period of harmonic oscillations of particles of the medium. The frequency of a wave is the reciprocal of the time period, i.e.

$$n = \frac{1}{T}$$

(3) *Wave Velocity* Wave velocity is the distance travelled by the wave in one second.

(4) *Wavelength* The wavelength is defined as the distance (measured along the direction of propagation of wave) between two nearest particles which are in the same phase of vibration.

9.1.4 Interference of Waves The term 'interference' describes the physical effects of superposition of two or more waves travelling through the

same region of space. The actual displacement of any particle in that region is the resultant of the separate displacements that the individual waves will give it if each wave were present alone. The process of vector addition of displacements of a given particle is called *superposition*. The resultant displacement of a particle is equal to the algebraic (or vectorial) sum of individual displacements given to it by two or more waves. This is called the *principle of superposition*.

Beats It is a special case of interference in which the frequencies of the two interfering waves are slightly different and the two waves travel almost in the same direction. As a result of interference the intensity of the resulting wave periodically rises to a maximum value and falls to a minimum value. One maximum of intensity followed by a minimum is technically called a *beat*. The beat period is the time interval between two successive maxima or minima of intensity. The beat frequency is the number of beats per second.

Standing (or Stationary) Waves The interference between two identical waves travelling in opposite directions in a medium gives rise to standing or stationary waves. In a standing wave, the amplitudes of particle oscillations are different at different points in the medium. At certain points (called *nodes*) the amplitude is permanently zero. At other points (called *antinodes*) the amplitude is maximum. The distance between two consecutive nodes or antinodes is equal to half the wavelength of either of the interfering waves. The distance between a node and the next antinode is $1/4$ th the wavelength.

The most striking feature of a standing wave is that energy is not transported along the medium. The reason is that energy cannot flow past the nodes of the medium which are permanently at rest.

9.1.5 The Doppler Effect When the source emitting waves and the observer receiving them are in relative motion with respect to the medium in which the waves propagate, the frequency of the waves received by the observer is different from the frequency of the vibrating source. This phenomenon was first noticed in sound waves by CJ Doppler in 1842 and is known as the Doppler effect.

9.2 FORMULAE

1. Equation of a harmonic wave travelling along the positive x direction is

$$\begin{aligned} y &= A \sin 2\pi \left(\frac{t}{T} - \frac{x}{\lambda} \right) \\ &= A \sin 2\pi n \left(t - \frac{x}{v} \right) \\ &= A \sin \frac{2\pi}{\lambda} (vt - x) \end{aligned} \quad (9.1)$$

where y = displacement of the particle of the medium at a location x at time t .

A = amplitude of the wave

λ = wavelength of the wave in the medium

T = time period

n = frequency

v = wave velocity in the medium

For a wave travelling along the negative x direction, the equation is

$$y = A \sin 2\pi \left(\frac{t}{T} + \frac{x}{\lambda} \right) \quad (9.2)$$

2. The wave velocity is related to frequency and wavelength as

$$v = n \lambda \quad (9.3)$$

3. Velocity of sound in an elastic medium is given by

$$v = \sqrt{\frac{E}{\rho}}$$

where E = modulus of elasticity of the medium and

ρ = density of the medium.

(a) For gases

$$E = \gamma P$$

where $\gamma = C_P/C_V$ is the ratio of the specific heat of the gas at constant pressure and that at constant volume and P is the pressure of the gas. Thus

$$v = \sqrt{\frac{\gamma P}{\rho}} \quad (\text{Laplace's formula}) \quad (9.4)$$

(b) For solids: $E = Y$, the Young's modulus of the solid. Thus

$$v = \sqrt{\frac{Y}{\rho}} \quad (9.5)$$

(c) For liquids: $E = K$, the bulk modulus of the liquid. Thus

$$v = \sqrt{\frac{K}{\rho}} \quad (9.6)$$

9.2.4 The velocity of sound in a gas is independent of the pressure but is directly proportional to the square root of the absolute temperature.

$$\frac{v_t}{v_0} = \sqrt{\frac{T}{T_0}} = \sqrt{\frac{t + 273}{273}} \quad (9.7)$$

where t is the temperature in $^{\circ}\text{C}$.

5. Beat frequency

$$n_b = (n_1 - n_2)$$

where n_1 and n_2 are the frequencies of the two interfering waves; n_1 being greater than n_2 .

6. Equation of a standing wave

$$\begin{aligned} y &= y_1 + y_2 \\ &= a \sin \frac{2\pi}{\lambda} (v t - x) + a \sin \frac{2\pi}{\lambda} (v t + x) \\ &= A \sin \frac{2\pi v t}{\lambda} \end{aligned}$$

where $A = 2a \cos \frac{2\pi x}{\lambda}$

7. Velocity of a transverse wave on a stretched string is given by

$$v = \sqrt{\frac{T}{m}} \quad (9.8)$$

where T = tension in the string and m = mass per unit length of the string. For a string of diameter d and density ρ , we have

$$m = \frac{\pi d^2 \rho}{4}$$

Thus,
$$v = \frac{2}{d} \sqrt{\frac{T}{\pi \rho}} \quad (9.9)$$

9.2.8 Modes of Vibration of a String A string of length L can vibrate in a number of modes depending on the number of segments in which it is vibrating (see Fig. 9.1).

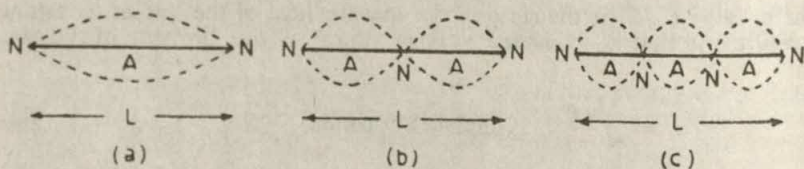


Fig. 9.1

The frequency of a string vibrating in p segments is given by

$$n_p = \frac{p}{2L} \sqrt{\frac{T}{m}} \quad (9.10)$$

(1) *Fundamental Mode (or First Harmonic)* In the fundamental mode, the string vibrates in one segment ($p = 1$); the ends are the two nodes with an antinode in the middle. This is shown in Fig. 9.1 (a). Since the distance between consecutive nodes is half the wavelength, we have

$$L = \frac{\lambda_1}{2} \quad \text{or} \quad \lambda_1 = 2L$$

and
$$n_1 = \frac{1}{2L} \sqrt{\frac{T}{m}}$$

Here n_1 is the frequency of the fundamental mode.

(ii) *Second Harmonic:* In this mode, the string vibrates in two segments same ($p = 2$) as shown in Fig. 9.1 (b). For the second harmonic

$$\lambda_2 = L$$

$$n_2 = \frac{1}{L} \sqrt{\frac{T}{m}} = 2n_1$$

(iii) *Third Harmonic:* The same string can also vibrate in three segments ($p = 3$). In this mode, shown in Fig. 9.1 (c), we have

$$\lambda_3 = \frac{2}{3} L$$

$$n_3 = \frac{3}{2L} \sqrt{\frac{T}{m}} = 3 n_1$$

Thus, there are various modes of vibrations of a stretched string fixed at both ends. The frequencies $n_1, n_2, n_3, \dots$ etc. of these modes are in the ratio of $1 : 2 : 3 \dots$ etc.

9.2.9 Modes of Vibration of Air Columns: When a sound wave travels in an air column in a pipe, the air column is set into longitudinal oscillations. The wave is reflected at the end of the pipe. The direct and reflected waves interfere to give rise to standing longitudinal waves in the air column. The open end of the pipe is an anti-node and the closed end is a node.

A. A Cylindrical Pipe, Closed at One End and Open at the Other End
Figure 9.2 illustrates the first three modes of vibration of the air column in a pipe of length L , closed at one end and open at the other.

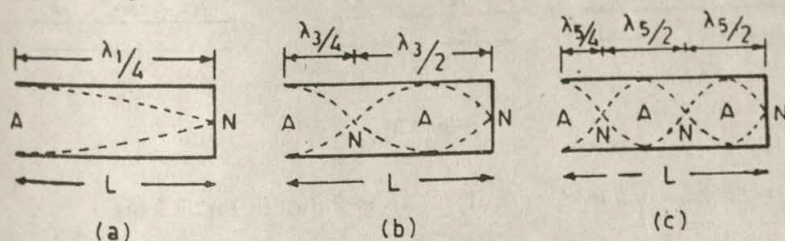


Fig. 9.2

The frequency of vibration is given by

$$n_p = \frac{v}{\lambda_p} = \frac{1}{\lambda_p} \sqrt{\frac{\gamma P}{\rho}} \quad (9.11)$$

where $p = 1, 3, 5, \dots$ etc.

(i) *Fundamental mode ($p = 1$)* Figure 9.2 (a) represents the fundamental mode or the first harmonic.

$$L = \frac{\lambda_1}{4} \quad \text{or} \quad \lambda_1 = 4L$$

If n_1 is the fundamental frequency, then the frequency of the sound emitted by the pipe is given by

$$n_1 = \frac{v}{\lambda_1} = \frac{v}{4L} = \frac{1}{4L} \sqrt{\frac{\gamma P}{\rho}}$$

(ii) *Third harmonic (or first overtone) $p = 3$* This is shown in Fig. 9.2 (b).

$$L = \frac{\lambda_3}{4} + \frac{\lambda_3}{2} + \frac{\lambda_3}{4} = \frac{3\lambda_3}{4} \quad \text{or} \quad \lambda_3 = \frac{4L}{3}$$

and

$$n_3 = \frac{v}{\lambda_3} = \frac{3v}{4L} = 3n_1$$

(iii) *Fifth harmonic (or second overtone)* $p = 5$ This is shown in Fig. 9.2 (c)

$$L = \frac{\lambda_5}{4} + \frac{\lambda_5}{2} + \frac{\lambda_5}{2} = \frac{5\lambda_5}{4} \quad \text{or} \quad \lambda_5 = \frac{4L}{5}$$

and
$$n_5 = \frac{v}{\lambda_5} = \frac{5v}{4L} = 5n_1$$

Thus $n_1 : n_3 : n_5 \dots = 1 : 3 : 5 \dots$

For a cylindrical pipe, open at one end and closed at the other, only those harmonics which are *odd integral multiples* of the fundamental can be excited; all the even harmonics ($p = 2, 4, 6, \dots$ etc.) are absent.

B. A Cylindrical Tube Open at Both Ends In an open tube both even and odd harmonics can be excited as shown in Fig. 9.3.

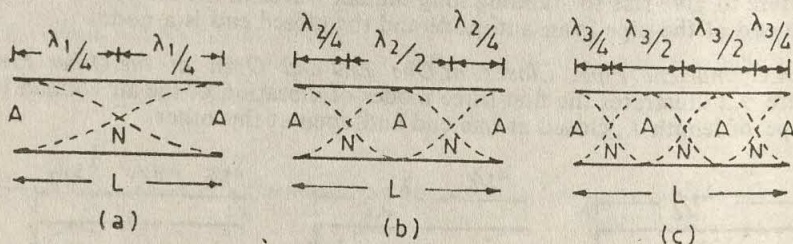


Fig. 9.3

(i) *Fundamental mode* ($p = 1$) This is shown in Fig. 9.3 (a)

$$L = \frac{\lambda_1}{4} + \frac{\lambda_1}{4} = \frac{\lambda_1}{2} \quad \text{or} \quad \lambda_1 = 2L$$

$$n_1 = \frac{v}{\lambda_1} = \frac{v}{2L} = \frac{1}{2L} \sqrt{\frac{\gamma P}{\rho}}$$

(ii) *Second harmonic (or first overtone)* $p = 2$ This is shown in Fig. 9.3 (b).

$$L = \frac{\lambda_2}{4} + \frac{\lambda_2}{2} + \frac{\lambda_2}{4} = \lambda_2$$

$$n_2 = \frac{v}{\lambda_2} = \frac{v}{L} = 2n_1$$

(iii) *Third harmonic (or second overtone)* $p = 3$ This is shown in Fig. 9.3 (c).

$$L = \frac{\lambda_3}{4} + \frac{\lambda_3}{2} + \frac{\lambda_3}{2} + \frac{\lambda_3}{4} = \frac{3\lambda_3}{2} \quad \text{or} \quad \lambda_3 = \frac{2L}{3}$$

$$n_3 = \frac{v}{\lambda_3} = \frac{3v}{2L} = 3n_1$$

Thus $n_1 : n_2 : n_3 \dots = 1 : 2 : 3 \dots$

End Correction We have assumed that the open end is an antinode. This is not strictly true because reflection does not occur exactly at the open

end. As a result of this, the antinode is located slightly away from the open end. The effective length of the pipe has to be corrected to take account of this. The end-correction e is given by

$$e = 0.3 D$$

where D is the diameter of the pipe. For a pipe closed at one end, the effective length is $(L + e)$ and for a pipe open at both ends, the effective length is $(L + 2e)$.

9.2.10 Doppler Effect: Expressions for Apparent Frequency

u_s = speed of the source emitting sound waves

u_0 = speed of the observer receiving the sound waves

n = actual frequency of the source of sound

v = speed of sound in the medium.

(i) *Source approaching a stationary observer*

$$n_1 = \frac{n}{\left(1 - \frac{u_s}{v}\right)} \quad (9.12)$$

(ii) *Source receding from a stationary observer*

$$n_2 = \frac{n}{\left(1 + \frac{u_s}{v}\right)} \quad (9.13)$$

(iii) *Observer approaching a stationary source of sound*

$$n_3 = n \left(1 + \frac{u_0}{v}\right) \quad (9.14)$$

(iv) *Observer receding from a stationary source of sound*

$$n_4 = n \left(1 - \frac{u_0}{v}\right) \quad (9.15)$$

(v) *Source approaching a receding observer*

$$n_5 = n \frac{(v - u_0)}{(v - u_s)} \quad (9.16)$$

(vi) *observer approaching a receding source*

$$n_6 = n \frac{(v + u_0)}{(v + u_s)} \quad (9.17)$$

(vii) *Both source and observer approaching each other*

$$n_7 = n \frac{(v + u_0)}{(v - u_s)} \quad (9.18)$$

(viii) *Both source and observer receding from each other*

$$n_8 = n \frac{(v - u_0)}{(v + u_s)} \quad (9.19)$$

Effect of the Motion of the Medium The velocity of material or mechanical waves is affected by the motion of the medium. If the medium is moving with a velocity u_m in the direction of propagation of sound, the effective velocity of sound is increased from v to $(v + u_m)$. In this case, v is replaced by $(v + u_m)$ in the above expressions. On the other hand, if the medium is moving with a velocity u_m in a direction opposite to the direction of wave propagation, v is replaced by $(v - u_m)$.

9.2.11 Intensity, Quality and Pitch of Sound

1. **Intensity** The intensity of a travelling wave is defined as the rate at which energy is transferred per unit area of a surface held perpendicular to the direction of propagation. In other words, intensity is the power transported by unit normal area. Intensity I of a wave in a medium of density ρ is given by

$$I = 2 \pi^2 \rho v A^2 n^2 \quad (9.20)$$

where n is the frequency of particle oscillation, A their amplitude and v is the velocity of the wave in that medium.

The SI unit of intensity is watt per square meter (W m^{-2}).

The intensity level of a sound wave is defined by an arbitrary scale. The zero of the scale is taken at the sound wave intensity,

$$I_0 = 1 \times 10^{-12} \text{ W m}^{-2}$$

which corresponds to the faintest audible sound. The unit of intensity level is called decibel (dB).

Intensity level in decibels of a sound of intensity

$$I = 10 \log \left(\frac{I}{I_0} \right)$$

2. **Quality** Wind and string instruments emit sounds in which in addition to the fundamental, many harmonics are also present at the same time, with different intensities. The *quality* (or timbre) of the resulting sound depends upon the number of overtones present and their intensity distribution. The greater the number of overtones, the better is the total quality of the sound. Thus two notes of the same frequency can be distinguished because they have different quality, i.e. different number of overtones present in them. Thus two pipes, one open at both ends and the other closed at one end, may emit sounds of the same frequency, but their quality will be different because the open pipe will have all the harmonics but a closed pipe has only the odd harmonics. The quality of a sound is determined by the number of overtones as well as their respective intensities in that sound.

3. **Pitch** Pitch is that characteristic of sound which enables us to distinguish and 'high' or 'shrill' sound from a 'low' or 'grave' sound. Pitch depends upon the frequency of the sound. Low or grave sounds have a low frequency; high or shrill sounds have a high frequency.

Frequency is a measurable physical quantity but pitch is a sensation which corresponds to and depends upon frequency. Pitch is subjective and cannot be measured.

9.3 NUMERICAL EXAMPLES

Example 1: When a plane wave traverses a medium, the displacement of particles is given by

$$y(x, t) = 0.01 \sin 2\pi(2t - 0.01x)$$

where y and x are expressed in metres and t in seconds. Calculate (a) the amplitude, wavelength, velocity and frequency of the wave; (b) the phase difference between two positions of the same particle at a time interval of 0.25 s; and (c) the phase difference, at a given instant of time, between two particles 50 m apart.

Solution: (a) The given equation can be rewritten in the form

$$y(x, t) = 0.01 \sin \left\{ \frac{2\pi}{100} (200t - x) \right\}$$

Comparing this with the wave equation,

$$y(x, t) = A \sin \left\{ \frac{2\pi}{\lambda} (vt - x) \right\}$$

we have,

$$\text{amplitude } A = 0.01 \text{ m}$$

$$\text{wavelength } \lambda = 100 \text{ m}$$

$$\text{wave velocity } v = 200 \text{ m s}^{-1}$$

$$\text{and frequency } n = \frac{v}{\lambda} = \frac{200}{100} = 2 \text{ Hz}$$

(b) Phase change in a time interval of Δt is

$$\Delta \phi = \frac{2\pi}{T} \Delta t = 2\pi n \Delta t$$

or phase difference $= 2\pi \times 2 \times 0.25 = \pi = 180^\circ$. In other words, the particle phase is reversed in a time 0.25 s which is obvious since its period

$$T = \frac{1}{n} = 0.5 \text{ s}$$

(c) Phase difference for path difference of Δx is

$$\Delta \phi = - \frac{2\pi}{\lambda} \Delta x = - \frac{2\pi}{100} \times 50 = -\pi = -180^\circ$$

In other words, the particle located 50 m (which is half the wavelength) ahead of another particle lags in phase by 180° .

Example 2: A transverse harmonic wave of amplitude 0.01 m is generated at one end ($x = 0$) of a long horizontal string by a tuning fork of frequency 500 Hz. At a given instant of time the displacement of the particle at $x = 0.1$ m is -0.005 m and that of the particle at $x = 0.2$ m is $+0.005$ m. Calculate the wavelength and the wave velocity. Obtain the equation of the wave assuming that the wave is travelling along the $+x$ direction and that the end $x = 0$ is at the equilibrium position at $t = 0$.

Solution: Since the wave is travelling along $+x$ direction and the displacement of the end $x = 0$ is zero at time $t = 0$, the general equation of this wave is

$$y(x, t) = A \sin \left\{ \frac{2\pi}{\lambda} (vt - x) \right\} \quad (i)$$

where

$$A = 0.01 \text{ m.}$$

When $x = 0.1 \text{ m}$, $y = -0.005 \text{ m}$

$$\therefore -0.005 = 0.01 \sin \left\{ \frac{2\pi}{\lambda} (vt - x_1) \right\},$$

where $x_1 = 0.1 \text{ m}$

$$\text{or} \quad \sin \left\{ \frac{2\pi}{\lambda} (vt - x_1) \right\} = -\frac{1}{2}$$

$$\therefore \text{phase } \phi_1 = \frac{2\pi}{\lambda} (vt - x_1) = \frac{7\pi}{6} \quad (\text{ii})$$

When $x = 0.2 \text{ m}$, $y = +0.005$. Therefore, we have

$$+0.005 = 0.01 \sin \left\{ \frac{2\pi}{\lambda} (vt - x_2) \right\}$$

where $x_2 = 0.2 \text{ m}$

$$\text{or} \quad \phi_2 = \frac{2\pi}{\lambda} (vt - x_2) = \frac{\pi}{6} \quad (\text{iii})$$

From Eqs (ii) and (iii)

$$\Delta \phi = \phi_1 - \phi_2 = \pi$$

$$\text{Now} \quad \Delta \phi = -\frac{2\pi}{\lambda} \Delta x$$

$$\text{Thus,} \quad \pi = -\frac{2\pi}{\lambda} (x_1 - x_2) = -\frac{2\pi}{\lambda} (0.1 - 0.2)$$

$$\text{or} \quad \lambda = 0.2 \text{ m}$$

$$\text{Now,} \quad \text{frequency } n \text{ of the wave} = \text{frequency of the tuning fork} \\ = 500 \text{ Hz}$$

$$\text{Hence,} \quad \text{wave velocity } v = n\lambda = 500 \times 0.2 = 100 \text{ m s}^{-1}$$

Substituting for A , λ and v in Eq. (i) we get

$$y(x, t) = 0.01 \sin \{10\pi(100t - x)\}$$

This is the equation of the wave where y and x are in metres and t in seconds.

Example 3: Transverse waves are generated in two uniform steel wires A and B of diameters 10^{-3} m and $0.5 \times 10^{-3} \text{ m}$ respectively, by attaching their free end to a vibrating source of frequency 500 Hz . Find the ratio of the wavelengths if they are stretched with the same tension.

Solution: The density ρ of a wire of mass M , length L and diameter d is given by

$$\rho = \frac{4M}{\pi d^2 L} = \frac{4m}{\pi d^2}$$

$$\text{Now} \quad v_A = \sqrt{\frac{T}{m_A}}$$

and
$$v_B = \sqrt{\frac{T}{m_B}}$$

$$\therefore \frac{v_A}{v_B} = \sqrt{\frac{m_B}{m_A}} = \frac{d_B}{d_A}$$

but $v_A = n \lambda_A$ and $v_B = n \lambda_B$, n being the frequency of the source.

Hence
$$\frac{\lambda_A}{\lambda_B} = \frac{v_A}{v_B} = \frac{d_B}{d_A} = \frac{0.5 \times 10^{-3}}{10^{-3}} = 0.5$$

Example 4: Compare the velocities of sound in hydrogen (H_2) and carbon dioxide (CO_2). The ratio (r) of specific heats of H_2 and CO_2 are respectively 1.4 and 1.3.

Solution:

$$v_1 = \sqrt{\frac{r_1 P}{\rho_1}} \quad \text{and} \quad v_2 = \sqrt{\frac{r_2 P}{\rho_2}}$$

$$\therefore \frac{v_1}{v_2} = \sqrt{\frac{r_1 \rho_2}{r_2 \rho_1}}$$

Since density of a gas is proportional to its molecular weight,

$$\frac{\rho_2}{\rho_1} = \frac{44.01}{2.016} = 21.83$$

$$\therefore \frac{v_1}{v_2} = \sqrt{\frac{1.4}{1.3} \times 21.83} = 4.85$$

Velocity of sound in hydrogen is 4.85 times that in carbon dioxide.

Example 5: The intensity of sound in a normal conversation at home is about $3 \times 10^{-6} \text{ W m}^{-2}$ and the frequency of normal human voice is about 1000 Hz. Find the amplitude of waves, assuming that the air is at standard conditions.

Solution: At standard conditions (STP)

density (ρ) of air = 1.29 kg m^{-3}

velocity of sound $v = 332.5 \text{ m s}^{-1}$

Now $I = 2\pi^2 \rho n^2 A^2 v$

where $n = 1000 \text{ Hz}$, $I = 3 \times 10^{-6} \text{ W m}^{-2}$

$$\begin{aligned} \therefore A &= \frac{1}{\pi n} \sqrt{\frac{I}{2 \rho v}} \\ &= \frac{1}{3.142 \times 1000} = \sqrt{\frac{3 \times 10^{-6}}{2 \times 1.29 \times 332.5}} \\ &= 1.88 \times 10^{-8} \text{ m} \end{aligned}$$

Note that the amplitude of sound waves in normal conversation is extremely small.

Example 6: Two sources S_1 and S_2 , separated by 2.0 m, vibrate according to the equation $y_1 = 0.03 \sin \pi t$ and $y_2 = 0.02 \sin \pi t$ where y_1 , y_2 and t

are in MKS units. They send out waves of velocity 1.5 m s^{-1} . Calculate the amplitude of the resultant motion of a particle co-linear with $S_1 S_2$ and located at a point (a) to the right of S_2 , (b) to the left of S_1 and (c) in the middle of S_1 and S_2 .

Solution: The oscillations $y_1 = 0.03 \sin \pi t$ and $y_2 = 0.02 \sin \pi t$, have amplitude $A_1 = 0.03 \text{ m}$ and $A_2 = 0.02 \text{ m}$ and equal period $T = 2 \text{ s}$ or frequency $n = 1/2 = 0.5 \text{ s}^{-1}$.

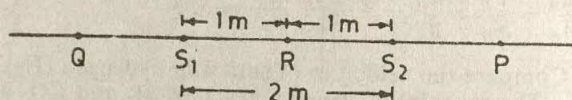


Fig. 9.4

Since velocity of each wave $= 1.5 \text{ m s}^{-1}$, the wavelength

$$\lambda = \frac{v}{n} = \frac{1.5}{0.5} = 3.0 \text{ m}$$

(a) The path difference for all points P to the right of S_2

$$\Delta = S_1 P - S_2 P = S_1 S_2 = 2.0 \text{ m}$$

$$\therefore \text{phase difference } \phi = \frac{2\pi}{\lambda} \Delta = \frac{2\pi}{3} \times 2.0 = \frac{4\pi}{3}$$

The resultant amplitude at such points is given by

$$\begin{aligned} A &= (A_1^2 + A_2^2 + 2A_1 A_2 \cos \phi)^{1/2} \\ &= \left[(0.03)^2 + (0.02)^2 + 2 \times (0.03)(0.02) \times \cos \frac{4\pi}{3} \right]^{1/2} \\ &= 0.0265 \text{ m} \end{aligned}$$

(b) The path difference for all points Q to the left of S_1

$$= S_2 Q - S_1 Q = S_1 S_2 = 2.0 \text{ m}$$

Hence, A for such points $= 0.0265 \text{ m}$

(c) For a point R at the centre of $S_1 S_2$, the path difference $= 0$.

$$\therefore \phi = 0$$

$$\begin{aligned} \text{Hence } A &= (A_1^2 + A_2^2 + 2A_1 A_2)^{1/2} = A_1 + A_2 \\ &= 0.03 + 0.02 = 0.05 \text{ m} \end{aligned}$$

Example 7: Two tuning forks A and B produce 10 beats per second when sounded together. On loading the fork A slightly it was observed that 15 beats are heard in one second. If the frequency of the fork B is 480 Hz, calculate the frequency of the fork A (a) before loading and (b) after loading.

Solution: Let n_1 be the frequency of A and n_2 that of fork B . There are two possibilities (a) $n_1 > n_2$ and (b) $n_1 < n_2$.

$$(a) \quad n_1 > n_2: \quad n_2 = 480 \text{ Hz}, \quad n_b = 10 \text{ Hz} \quad \text{Now } n_b = n_1 - n_2$$

$$\therefore n_1 = n_2 + n_b = 490 \text{ Hz}$$

On loading fork A its frequency decreases. In other words, n_1 decreases.

$\therefore n_1 - n_2 = n_b$ should also decrease. But n_b increases to 15 beats/s.

Hence n_1 is not greater than n_2

(b) $n_1 < n_2$: In this case $n_2 - n_1 = n_b$

$\therefore n_1 = n_2 - n_b = 470 \text{ Hz}$

On loading n_1 decreases, therefore $(n_2 - n_1) = n_b$ increases which is what is observed.

$\therefore n_1 = 470 \text{ Hz}$ before it is loaded.

and $n_1 = n_2 - n_b = 480 - 15 = 465 \text{ Hz}$ after it is loaded.

Example 8: Standing waves are produced by the superposition of two waves

$$y_1 = 0.05 \sin(3\pi t - 2x)$$

and $y_2 = 0.05 \sin(3\pi t + 2x)$

where y and x are measured in metres and t in seconds. Find the amplitude of a particle at $x = 0.5 \text{ m}$.

Solution: The resultant displacement is given by

$$y = y_1 + y_2 = 0.05 \{ \sin(3\pi t - 2x) + \sin(3\pi t + 2x) \}$$

Using trigonometric relation

$$\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2 \sin \alpha \cos \beta, \text{ we have}$$

$$y = 0.1 \cos 2x \sin 3\pi t$$

or $y = A \sin 3\pi t$

where A , the amplitude of standing waves, is given by $A = 0.1 \cos 2x$ with

$$x = 0.5 \text{ m},$$

$$\cos 2x = \cos(2 \times 0.5 \text{ rad})$$

$$= \cos(1 \text{ rad})$$

$$= \cos\left(\frac{\pi}{3.142}\right)$$

$$= \cos 57.3^\circ$$

$$= 0.54$$

$$\text{amplitude } A \text{ at } (x = 0.5 \text{ m}) = 0.1 \times 0.54$$

$$= 0.054 \text{ m}$$

Example 9: A wire of density 9 g cm^{-3} is stretched between two clamps 100 cm apart while subjected to an extension of 0.05 cm. What is the lowest frequency of transverse vibrations of the wire, assuming

$$Y = 9 \times 10^{11} \text{ dynes cm}^{-2}$$

(IIT 1975)

Solution: Young's modulus

$$Y = \frac{\text{stress}}{\text{strain}} = \frac{\frac{T}{A}}{\frac{l}{L}} = \frac{TL}{Al}$$

where T = tension, L = original length = 100 cm, l = extension = 0.05 cm and A = area of cross-section of the wire. Hence,

$$\frac{T}{A} = \frac{Yl}{L} = \frac{9 \times 10^{11} \times 0.05}{100} = 45 \times 10^7$$

$$\begin{aligned} \text{Now mass per unit length } m &= \frac{\text{mass}}{\text{length}} \\ &= \frac{\text{volume} \times \text{density}}{\text{length}} \\ &= \frac{\text{area} \times \text{length} \times \text{density}}{\text{length}} \\ &= \text{Area} \times \text{density} \\ &= A \rho \end{aligned}$$

Hence the lowest frequency (i.e. frequency of the fundamental mode) is given by

$$\begin{aligned} n &= \frac{1}{2L} \sqrt{\frac{T}{m}} = \frac{1}{2L} \sqrt{\frac{T}{A\rho}} \\ &= \frac{1}{2 \times 100} \sqrt{\frac{45 \times 10^7}{9}} = 35.3 \text{ Hz} \end{aligned}$$

Example 10: A wire having a linear density of 0.05 g cm^{-1} is stretched between two rigid supports with a tension of 4.5×10^7 dynes. It is observed that the wire resonates at a frequency of 420 Hz. The next higher frequency at which the wire resonates is 490 Hz. Find the length of the wire.

(IIT 1971)

Solution: Let 420 Hz be the p th harmonic, then 490 Hz is the $(p + 1)$ th harmonic. Therefore

$$420 = \frac{p}{2L} \sqrt{\frac{T}{m}} \quad (i)$$

$$\text{and} \quad 490 = \frac{p+1}{2L} \sqrt{\frac{T}{m}} \quad (ii)$$

Dividing (i) and (ii) we have

$$\frac{490}{420} = \frac{p+1}{p} \text{ giving } p = 6$$

Substituting this value of p in Eq. (i) we get

$$420 = \frac{6}{2L} \times \sqrt{\frac{4.5 \times 10^7}{0.05}}$$

which gives $L = 214.3 \text{ cm}$

Example 11: A light pointer fixed to one prong of a tuning fork touches a vertical plate. The fork is set vibrating and the plate is allowed to fall freely. Eight complete oscillations are counted when the plate falls through 10 cm. What is the frequency of the tuning fork? (IIT 1977)

Solution: The time taken (t) for the plate to fall freely through a distance $S = 10$ cm is given by

$$S = \frac{1}{2} g t^2$$

or
$$t = \sqrt{\frac{2S}{g}}$$

During this time the fork completes 8 oscillations. Hence the frequency of the fork is

$$\begin{aligned} n &= \frac{8}{t} = 8 \times \sqrt{\frac{g}{2S}} \\ &= 8 \times \sqrt{\frac{980}{2 \times 10}} = 56 \text{ Hz} \end{aligned}$$

Example 12: Two wires of radii r and $2r$ are welded together end to end. This combination is used as a sonometer wire kept under tension T . The welded point is midway between the two bridges. What would be the ratio of the number of loops formed in the wires such that the joint is a node when stationary waves are set up in the wires? (IIT 1976)

Solution: Let ρ_1 and ρ_2 be the densities of the wires and A_1 and A_2 their respective cross-sectional areas. The mass per unit length of the wires will be

$$m_1 = A_1 \rho_1 = \pi r^2 \rho_1$$

and

$$m_2 = A_2 \rho_2 = \pi (2r)^2 \rho_2 = 4\pi r^2 \rho_2$$

so that

$$\frac{m_1}{m_2} = \frac{\rho_1}{4\rho_2}$$

If p and q are the number of loops in the wires, their frequencies are

$$n_1 = \frac{p}{2L_1} \sqrt{\frac{T}{m_1}}$$

$$n_2 = \frac{q}{2L_2} \sqrt{\frac{T}{m_2}}$$

Since the joint is midway between the bridges $L_1 = L_2 = L$. Since the combination vibrates with the same frequency, $n_1 = n_2$. Hence

$$\frac{p}{2L} \sqrt{\frac{T}{m_1}} = \frac{q}{2L} \sqrt{\frac{T}{m_2}}$$

$$\therefore \frac{p}{q} = \sqrt{\frac{m_1}{m_2}} = \frac{1}{2} \sqrt{\frac{\rho_1}{\rho_2}}$$

If the wires are made of the same material, $\rho_1 = \rho_2$ in which case

$$\frac{p}{q} = \frac{1}{2}$$

Example 13: A sonometer wire of length 110 cm is stretched with a tension T and fixed at its ends. The wire is divided into three segments by placing two bridges below it. Where should the bridges be placed so that the fundamental frequencies of the segments are in the ratio 1 : 2 : 3?

Solution: Let L_1 , L_2 and L_3 be the lengths of the segments of wire AB (Fig. 9.5). Then

$$L_1 + L_2 + L_3 = 110 \text{ cm} \quad (i)$$

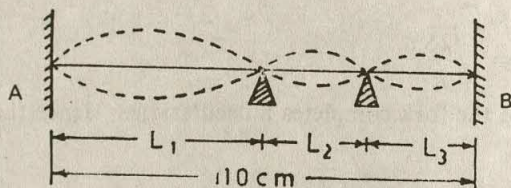


Fig. 9.5

Let n_1 , n_2 and n_3 be their respective fundamental frequencies. Thus

$$n_1 = \frac{1}{2L_1} \sqrt{\frac{T}{m}}$$

$$n_2 = \frac{1}{2L_2} \sqrt{\frac{T}{m}}$$

and

$$n_3 = \frac{1}{2L_3} \sqrt{\frac{T}{m}}$$

Hence

$$n_1 L_1 = n_2 L_2 = n_3 L_3 \quad (ii)$$

But

$$n_1 : n_2 : n_3 = 1 : 2 : 3$$

$\therefore$

$$n_2 = 2n_1 \quad \text{and} \quad n_3 = 3n_1 \quad (iii)$$

From (ii) and (iii) we have

$$L_1 = 2L_2 = 3L_3 \quad (iv)$$

Substituting (iv) in (i) we get

$$L_1 + \frac{1}{2} L_1 + \frac{1}{3} L_1 = 110$$

or

$$L_1 = 60 \text{ cm}$$

Hence

$$L_2 = 30 \text{ cm} \quad \text{and} \quad L_3 = 20 \text{ cm}$$

Thus, the bridges should be placed at distances of 60 cm and 90 cm from end A .

Example 14: An underwater swimmer sends a sound signal to the surface. It produces 5 beats per second when compared with the fundamental tone of a pipe 20 cm long closed at one end. What is the wavelength of sound in water? Given velocity of sound in air and water are 360 m s^{-1} and 1500 m s^{-1} respectively, (IIT 1973)

Solution: The frequency of the fundamental tone of the pipe is

$$n = \frac{v_a}{4L} = \frac{360}{4 \times 0.20} = 450 \text{ Hz}$$

$\therefore$ The frequency of sound signal = $450 \pm 5 = 445 \text{ Hz}$ or 455 Hz

Since the frequency remains unchanged when sound travels from water to air, the frequency of the sound signal in water is either 445 Hz or 455 Hz. Hence the wavelength in water is either

$$\frac{1500}{445} = 3.371 \text{ m} \quad \text{or} \quad \frac{1500}{455} = 3.297 \text{ m}$$

Example 15: AB is a cylinder of length 1.0 m, fitted with a thin flexible diaphragm C at the middle and two thin flexible diaphragms A and B at the ends (see Fig. 9.6). The portions AC and BC contain hydrogen and oxygen respectively. The diaphragms A and B are set into vibrations of the same frequency. What is the minimum frequency of these vibrations for which the diaphragm C is a node? Under the conditions of the experiment, the velocity of sound in hydrogen is 1100 m s^{-1} and in oxygen is 300 m s^{-1} . (IIT 1978)

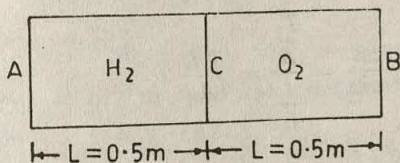


Fig. 9.6

Solution: Since C is a node, the two parts AC and BC behave as closed pipes with the closed end of each at C . Since diaphragms A and B are set into vibrations, the ends A and B must be anti-nodes.

The fundamental frequency of each pipe corresponds to just one node and one antinode. If n_1 and n_2 are the fundamental frequencies of gases in AC and BC respectively, we have

$$n_1 = \frac{v_1}{4L} = \frac{1100}{4 \times 0.5} = 550 \text{ Hz}$$

and
$$n_2 = \frac{v_2}{4L} = \frac{300}{4 \times 0.5} = 150 \text{ Hz}$$

where v_1 and v_2 are the sound speeds in hydrogen and oxygen respectively.

These frequencies are not the same. Hence the two gas columns are not vibrating in the fundamental mode. We know that a closed pipe has only odd harmonics with frequencies which are 3, 5, 7, 9, 11, etc. times the fundamental frequency. To find out which harmonics of n_1 and n_2 have the same frequency we notice that

$$\frac{n_1}{n_2} = \frac{550}{150} = \frac{11}{3} \quad \text{or} \quad 3n_1 = 11n_2$$

Thus, the third harmonic of n_1 and the eleventh harmonic of n_2 have equal frequencies. Similarly 6th harmonic of n_1 and 22nd harmonic of n_2 have equal frequencies and so on.

$$\begin{aligned} \therefore \text{Common minimum frequency} &= 3n_1 = 3 \times 550 = 1650 \text{ Hz} \\ &= 11n_2 = 11 \times 150 = 1650 \text{ Hz} \end{aligned}$$

Example 16: A copper wire is held at the two ends by rigid supports. At 30°C the wire is just taut, with negligible tension. Find the speed of transverse waves in the wire at 10°C . Given $\alpha = 1.7 \times 10^{-5} (\text{C})^{-1}$, $Y = 1.3 \times 10^{11} \text{ N m}^{-2}$, $\rho = 9 \times 10^3 \text{ kg m}^{-3}$. (IIT 1979)

Solution: Let L be the original length of the wire and l the contraction in length as the temperature falls from 30°C to 10°C . Now

$$L_t = L(1 + \alpha t)$$

$$\therefore L_{30} = L(1 + 30\alpha)$$

$$L_{10} = L(1 + 10\alpha)$$

$$\therefore l = L_{30} - L_{10} = 20\alpha L = 20 \times 1.7 \times 10^{-5} L = 34 \times 10^{-5} L$$

$$\text{Strain in the wire} = \frac{l}{L} = 34 \times 10^{-5}$$

Since the wire is rigidly fixed at its ends, it cannot contract, producing a tension T in the wire which caused the strain l/L . If A is the cross-sectional area of the wire, we have

$$Y = \frac{\text{stress}}{\text{strain}} = \frac{\frac{T}{A}}{\frac{l}{L}} = \frac{TL}{Al}$$

$$\text{or } T = \frac{YAl}{L} \quad (i)$$

The mass per unit length of the wire is

$$\begin{aligned} m &= \frac{\text{mass}}{\text{length}} = \frac{\text{volume} \times \text{density}}{\text{length}} \\ &= \frac{(\text{area} \times \text{length}) \times \text{density}}{\text{length}} \\ &= \text{area} \times \text{density} \quad \text{or } m = A\rho \end{aligned} \quad (ii)$$

Speed of transverse wave is given by

$$v = \sqrt{\frac{T}{m}}$$

Using Eqs (i) and (ii) this becomes

$$\begin{aligned} v &= \sqrt{\frac{Yl}{\rho L}} \\ &= \sqrt{\frac{1.3 \times 10^{11}}{9 \times 10^3} \times 34 \times 10^{-5}} \\ &= 70 \text{ m s}^{-1} \end{aligned}$$

Example 17: A metal wire of diameter 1 mm is held on two knife edges separated by a distance of 50 cm. The tension in the wire is 100 N. The wire vibrating with its fundamental frequency and a vibrating tuning fork together produce 5 beats per second. The tension in the wire is then reduced to 81 N. When the two are excited, beats are heard at the same rate. Calculate (a) the frequency of the fork and (b) the density of the material of the wire.

(IIT 1980)

Solution (a) Let N be the frequency of the tuning fork. Then frequency of the wire, when the tension is 100 N will be $(N + 5)$ and when the tension

is 81 N, it is $(N - 5)$; since in each case 5 beats are heard per second. Hence

$$N + 5 = \frac{1}{2L} \sqrt{\frac{T_1}{m}} = \frac{1}{2 \times 0.5} \sqrt{\frac{100}{m}} = \frac{10}{\sqrt{m}} \quad (i)$$

$$\text{and} \quad N - 5 = \frac{1}{2L} \sqrt{\frac{T_2}{m}} = \frac{1}{2 \times 0.5} \sqrt{\frac{81}{m}} = \frac{9}{\sqrt{m}} \quad (ii)$$

Subtracting (ii) from (i) we have

$$10 = \frac{1}{\sqrt{m}} \quad \text{or} \quad m = 0.01 \text{ kg m}^{-1}$$

Using this value of m in (i) or (ii) gives $N = 95 \text{ Hz}$

$$(b) \text{ Now } m = \pi r^2 \rho = \frac{\pi d^2 \rho}{4}$$

where $d = 1 \text{ mm} = 1 \times 10^{-4} \text{ m}$ and ρ = density of the wire. Thus

$$\rho = \frac{4m}{\pi d^2} = \frac{4 \times 0.01}{3.142 \times (10^{-4})^2} = 12731 \text{ kg m}^{-3}$$

Example 18: A string 25 cm long and having a mass of 2.5 g is under tension. A pipe closed at one end is 40 cm long. When the string is set vibrating in its first overtone and the air in the pipe in its fundamental frequency, 8 beats per second are heard. It is observed that decreasing the tension in the string decreases the beat frequency. If the speed of sound in air is 320 m s^{-1} , find the tension in the string. (IIT 1982)

Solution: Frequency of fundamental mode of the closed pipe is

$$n_p = \frac{v}{4L} = \frac{320}{4 \times 0.40} = 200 \text{ Hz}$$

Since the beat frequency is 8, the frequency of the string vibrating in its first overtone is

$$n_s = n_p \pm 8 = 200 \pm 8 = 192 \text{ Hz} \quad \text{or} \quad 208 \text{ Hz}$$

$$\text{where } (n_s)_{\text{first overtone}} = \frac{1}{l} \sqrt{\frac{T}{m}} \quad (i)$$

Here l = length of the wire = 0.25 m

$$m = \text{mass per unit length} = \frac{2.5 \times 10^{-3}}{0.25} = 10^{-2} \text{ kg m}^{-1}$$

and T is the tension in the string

It is given that the beat frequency decreases if the tension in the string is decreased. As the frequency decreases with decrease of tension, it is obvious that $n_s > n_p$. Hence

$$n_s = 208 \text{ Hz and not } 192 \text{ Hz}$$

Substituting the values in Eq. (i) we have

$$208 = \frac{1}{0.25} \sqrt{\frac{T}{10^{-2}}}$$

or

$$T = 27.04 \text{ N}$$

Example 19: A uniform rope of length 12 m and mass 6 kg hangs vertically from a rigid support. A block of mass 2 kg is attached to the free end of the rope. A transverse pulse of wavelength 0.06 m is produced at the lower end of the rope. What is the wavelength of the pulse when it reaches the top of the rope? (IIT 1984)

Solution: If m is the mass per unit length of the rope and T the tension at any point on the rope, then the velocity of the pulse at that point is given by

$$v = \sqrt{\frac{T}{m}}$$

Now, the tension at the lower end of the rope $= 2 \times g$ newtons where g is the acceleration due to gravity. Hence

$$v = \sqrt{\frac{2g}{m}}$$

Let n be the frequency of the pulse and $\lambda = 0.06$ m its wavelength at the lower end, then

$$v = n \lambda$$

$$\text{or } \sqrt{\frac{2g}{m}} = n \times 0.06$$

$$\text{or } n = \frac{1}{0.06} \times \sqrt{\frac{2g}{m}} \quad (i)$$

The tension at the upper end of the rope $= (2 + 6) \times g = 8 \times g$ newton. If λ' is the wavelength and v' the velocity of the pulse when it reaches the top of the rope, we have, since frequency n remains unchanged,

$$v' = n \lambda' = \sqrt{\frac{8g}{m}}$$

$$\text{or } \lambda' = \frac{1}{n} \times \sqrt{\frac{8g}{m}} \quad (ii)$$

Substituting for n from (i) into (ii) we get

$$\lambda' = 0.06 \times \sqrt{\frac{m}{2g}} \times \sqrt{\frac{8g}{m}} = 0.12 \text{ m}$$

Example 20: The vibrations of a string of length 60 cm fixed at both ends are represented by the equation

$$y = 4 \sin \left(\frac{\pi x}{15} \right) \cos (96 \pi t)$$

where x and y are in cm and t in seconds.

- (i) What is the maximum displacement of a point at $x = 5$ cm?
- (ii) Where are the nodes located along the string?
- (iii) What is the velocity of the particle at $x = 7.5$ cm at $t = 0.25$ s?
- (iv) Write down the equations of the component waves whose superposition gives the above vibration. (IIT 1985)

Solution: (i) Displacement is maximum when $\cos(96\pi t) = 1$. At $x = 5$ cm, the maximum displacement is

$$\begin{aligned}(y)_{\max} &= 4 \sin\left(\frac{\pi x}{15}\right) = 4 \sin\left(\frac{\pi \times 5}{15}\right) = 4 \sin \frac{\pi}{3} \\ &= 4 \times \frac{\sqrt{3}}{2} = 3.46 \text{ cm}\end{aligned}$$

(ii) At nodes the displacement is always zero. Hence nodes are located at values of x given by

$$\sin\left(\frac{\pi x}{15}\right) = 0$$

or
$$\frac{\pi x}{15} = p\pi$$

where $p = 0, 1, 2, 3, \dots$ etc. Thus

$x = 15p = 0, 15, 30, 45$ and 60 cm. The end points of the string at $x = 0$ and $x = 60$ are nodes, as expected.

(iii) The velocity of the string at a point x at time t is obtained by differentiating $y = 4 \sin\left(\frac{\pi x}{15}\right) \cos(96\pi t)$ with respect to t .

$$\text{Velocity } \frac{dy}{dt} = -(4 \times 96\pi) \times \sin\left(\frac{\pi x}{15}\right) \times \sin(96\pi t)$$

At $x = 7.5$ cm and $t = 0.25$ s, the velocity is zero because at $t = 0.25$ s, $\sin(96\pi t) = \sin(24\pi) = 0$.

(iv) The standing wave given in the question may be rewritten as

$$y = 2A \sin \alpha \cos \beta$$

where $A = 2$ cm, $\alpha = \frac{\pi x}{15}$ and $\beta = 96\pi t$.

But $2A \sin \alpha \cos \beta = A \sin(\alpha + \beta) + A \sin(\alpha - \beta)$

$$= 2 \sin\left(\frac{\pi x}{15} + 96\pi t\right) + 2 \sin\left(\frac{\pi x}{15} - 96\pi t\right)$$

i.e. $y = y_1 + y_2$

Hence the two constituent waves are

$$y_1 = 2 \sin\left(\frac{\pi x}{15} + 96\pi t\right)$$

and $y_2 = 2 \sin\left(\frac{\pi x}{15} - 96\pi t\right)$

Example 21: An ambulance blowing a siren of frequency 700 Hz is travelling slowly towards a vertical reflecting wall with a speed of 2 m s^{-1} . Calculate the number of beats heard in one second by the driver of the ambulance. Velocity of sound = 350 m s^{-1} .

Solution: The driver will hear two sounds, one coming directly from the siren and the other reflected from the wall or coming from the acoustic image of the car. The reflected sound can be imagined to be coming from the mirror image (see Fig. 9.7).

The observer is approaching the image-source which is also approaching him with the same speed. Hence the frequency of sound heard by him is given by ($u_0 = u_s$)

$$n' = n \times \frac{(v + u_0)}{(v - u_s)} = 700 \times \frac{(350 + 2)}{(350 - 2)} = 708 \text{ Hz}$$

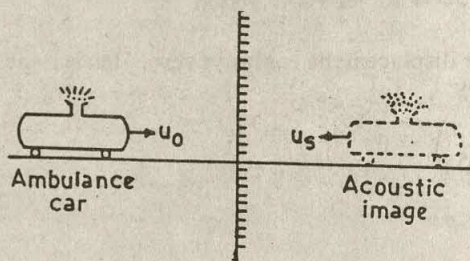


Fig. 9.7 Reflecting wall

Number of beats heard per second = $n' - n = 708 - 700 = 8$

Beat frequency = 8 Hz

Example 22: Two persons A and B , each carrying a source of sound of frequency 600 Hz, are standing a few metres apart in a quiet field. Calculate the beats heard by each when A moves towards B with a velocity of 3 m s^{-1} . Velocity of sound = 340 m s^{-1} .

Solution:

Beats Heard by Person A The person A hears the sound of his own source and that of the source carried by B towards whom he is moving. The apparent frequency of this sound is given by

$$n' = n \left(1 + \frac{u}{v} \right) = 600 \left(1 + \frac{3}{340} \right) = 605.29 \text{ Hz}$$

Beat frequency apparent to $A = 605.29 - 600 = 5.29 \text{ Hz}$, i.e. A hears 529 beats in 100 seconds.

Beats Heard by Person B The person B hears the sound of his own source and that of the source carried by A , which is approaching him with a speed 3 m s^{-1} . The apparent frequency of this sound is given by

$$n'' = \frac{n}{\left(1 - \frac{u}{v} \right)} = \frac{600}{\left(1 - \frac{3}{340} \right)} = 605.34 \text{ Hz}$$

Beat frequency apparent to $B = 605.34 - 600 = 5.34 \text{ Hz}$. i.e. B hears 534 beats in 100 seconds.

Example 23: A sonometer wire under a tension of 64 N vibrating in its fundamental mode is in resonance with a vibrating tuning fork. The vibrating portion of the sonometer wire has a length of 10 cm and a mass of 1 g. The vibrating tuning fork is now moved away from the vibrating wire with a constant speed and an observer standing near the sonometer hears one beat per second. Calculate the speed with which the tuning fork is moved. The speed of sound in air is 300 m s^{-1} . (IIT 1983)

Solution: $T = 64 \text{ N}$
 $L = 10 \text{ cm} = 0.1 \text{ m}$
 $m = \frac{1 \times 10^{-3} \text{ kg}}{0.1 \text{ m}} = 10^{-2} \text{ kg m}^{-1}$

The frequency of the sonometer wire vibrating in the fundamental mode is given by

$$n = \frac{1}{2L} \sqrt{\frac{T}{m}} = \frac{1}{2 \times 0.1} \times \sqrt{\frac{64}{10^{-2}}} = 400 \text{ Hz}$$

As the wire is in resonance with the tuning fork, the frequency of the fork = frequency of the wire, i.e. $n = 400 \text{ Hz}$.

Let u_s be the speed with which the fork is moved away from the observer standing close to the sonometer. The apparent frequency of the sound of the fork is given by

$$n' = n \times \frac{v}{(v + u_s)}$$

where v = speed of sound = 300 m s^{-1}

As 1 beat is heard per second, $n - n' = 1$

or $n' = n - 1 = 400 - 1 = 399 \text{ Hz}$

Now $\frac{n}{n'} = \frac{v + u_s}{v} = 1 + \frac{u_s}{v}$

or $\frac{n - n'}{n'} = \frac{u_s}{v}$

or $u_s = v \times \frac{(n - n')}{n'} = 300 \times \frac{(400 - 399)}{399}$
 $= 0.752 \text{ m s}^{-1}$

Example 24: Two tuning forks with natural frequencies of 340 Hz each move relative to a stationary observer. One fork moves away from the observer, while the other moves towards him at the same speed. The observer hears beats of frequency 3 Hz . Find the speed of the tuning fork. The speed of sound in air = 340 m s^{-1} . (IIT 1986)

Solution: Let v be the speed of sound in air and u_s the speed of each tuning fork. When the source moves towards a stationary observer, its apparent frequency is

$$n' = n \times \frac{v}{(v - u_s)} \quad (\text{i})$$

where $n = 340 \text{ Hz}$ and $v = 340 \text{ m s}^{-1}$

When the source moves away from the stationary observer, its apparent frequency is

$$n'' = n \times \frac{v}{(v + u_s)} \quad (\text{ii})$$

When both the forks are moving relative to a stationary observer, the number of beats heard by him per second = $n' - n'' = 3 \text{ Hz}$ (given).

Since $u_s \ll v$, Eqs (i) and (ii) may be simplified as follows.

$$n' = n \left(1 - \frac{u_s}{v} \right)^{-1} \approx n \left(1 + \frac{u_s}{v} \right)$$

$$n'' = n \left(1 + \frac{u_s}{v} \right)^{-1} \approx n \left(1 - \frac{u_s}{v} \right)$$

where terms of order u_s^2/v^2 have been neglected in the Binomial expansion. Thus,

$$n' - n'' = n \left(1 + \frac{u_s}{v} \right) - n \left(1 - \frac{u_s}{v} \right) = 2n \frac{u_s}{v}$$

$$\text{or} \quad 3 = 2 \times 340 \times \frac{u_s}{340}$$

$$\text{or} \quad u_s = 1.5 \text{ m s}^{-1}$$

Example 25: Two sound waves have intensities of $5 \times 10^{-10} \text{ W m}^{-2}$ and $1 \times 10^{-8} \text{ W m}^{-2}$. Find the intensity level of each sound in decibel. How many decibels is the louder sound above the other?

Solution: $I_1 = 5 \times 10^{-10} \text{ W m}^{-2}$

$$I_2 = 1 \times 10^{-8} \text{ W m}^{-2}$$

Standard intensity $I_0 = 1 \times 10^{-12} \text{ W m}^{-2}$

$$\begin{aligned} \text{Intensity of } I_1 \text{ in decibels (dB)}_1 &= 10 \log \left(\frac{I_1}{I_0} \right) = 10 \log \left(\frac{5 \times 10^{-10}}{1 \times 10^{-12}} \right) \\ &= 10 \log (500) = 27 \text{ dB} \end{aligned}$$

$$\begin{aligned} \text{Intensity level of } I_2 \text{ in decibels (dB)}_2 &= 10 \log \left(\frac{I_2}{I_0} \right) \\ &= 10 \log \left(\frac{1 \times 10^{-8}}{1 \times 10^{-12}} \right) = 10 \log (10^4) = 40 \text{ dB} \end{aligned}$$

$$(\text{dB})_2 - (\text{dB})_1 = 40 - 27 = 13 \text{ dB}$$

9.3 TRUE-FALSE STATEMENTS WITH REASONS

A few statements are given below. We have to decide whether they are true or false giving, in brief, the reason in support of our answers.

- Any function $y(x, t)$ of the form $y(x, t) = f(vt + x)$ represents a travelling wave.
- The velocity, wavelength and frequency do not undergo any change when a wave is reflected from a surface.
- The velocity, wavelength and frequency all undergo a change when a wave travels from one medium into another.
- When an ultrasonic wave travels from air into water, it bends towards the normal to the air-water interface.
- The velocity of sound is generally greater in solids than in gases at NPT.

(IIT 1977)

- The decrease in the speed of sound at high altitudes is due to a fall in pressure.
- The standing wave on a string under tension, fixed at its ends, does not have well-defined nodes.

8. When two waves interfere, the law of conservation of energy is not obeyed at points where destructive interference takes place.
9. The phenomenon of beats is not observed in the case of visible light waves.
10. The apparent frequency is n_1 when a source of sound approaches a stationary observer with a speed u and is n_2 when the observer approaches the same stationary source with the same speed. Then $n_2 > n_1$, if $u < v$, where v is the speed of sound.

ANSWERS

1. *True* $y(x, t) = f(vt + x) = f(z)$ where $z = vt + x$, Differentiating with respect to t we have

$$\frac{\partial y}{\partial t} = \frac{\partial f}{\partial z} \frac{\partial z}{\partial t} = v \frac{\partial f}{\partial z}$$

Differentiating again wrt time t we have

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 f}{\partial z^2}$$

Similarly differentiating twice with respect to x we have

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 f}{\partial z^2}$$

Hence,

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$$

which is the standard equation (in differential form) of a travelling wave.

2. *True* Because the wave is reflected back into the same medium, the velocity remains unchanged. The wavelength cannot change because the frequency cannot change by reflection.
3. *False* The frequency does not change when a wave travels from one medium into another.
4. *False* The ultrasonic wave bends away from the normal because the speed of the wave (being a sound wave) is greater in water than in air.
5. *True* The reason is that solids have a much higher coefficient of elasticity than gases at NTP.
6. *False* A change in pressure has no effect on the speed of sound. The decrease in the speed of sound at high altitudes is due to a fall in temperature.
7. *True* Standing waves are produced due to a superposition of the incident waves and the waves reflected from the fixed ends of the string. Since the ends are never perfectly rigidly fixed, the amplitude of the reflected wave is always less than that of the incident wave. Consequently, the resultant amplitude at nodes is not exactly zero. Thus the nodes are not well-defined.
8. *False* The energy is not destroyed at points where destructive interference takes place. The energy is transferred from regions of destructive interference to regions of constructive interference; the average energy being always equal to the sum of the energies of the individual waves.
9. *True* To observe beats the difference between the two interfering frequencies must be less than about 10 – 16 Hz. Since visible light waves have very high frequencies, beats are not observed due to persistence of vision.
10. *False*

$$n_1 = \frac{n}{1 - \frac{u}{v}} \quad (i)$$

and
$$n_2 = n \left(1 + \frac{u}{v} \right) \quad (ii)$$

The expression for n_1 may be rewritten as

$$n_1 = n \left(1 - \frac{u}{v} \right)^{-1}$$

Expanding Binomially and retaining terms upto order $\frac{u^2}{v^2}$ we have

$$n_1 = n \left(1 + \frac{u}{v} + \frac{u^2}{v^2} \right) \quad (\text{iii})$$

Comparing (ii) and (iii) we find that $n_1 > n_2$.

9.4 MULTIPLE CHOICE QUESTIONS

Choose the correct answer from the given alternatives in each of the following

1. Transverse waves are produced in a long string by attaching its free end to a vibrating tuning fork. Figure 9.8 shows the shape of a part of the string. Which pairs of points are in phase?

- (a) A and D
(b) B and E
(c) C and F
(d) A and G

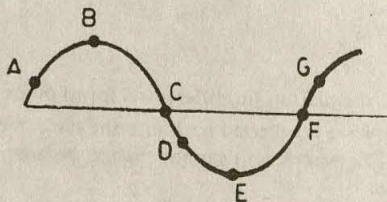


Fig. 9.8

2. When a wave travels in a medium, the particle displacements are given by $y(x, t) = 0.03 \sin \pi (2t - 0.01 x)$ where y and x are in metres and t in seconds. The wavelength of the wave is
- (a) 10 m
(b) 20 m
(c) 100 m
(d) 200 m
3. The velocity of the wave in Q. 2 is
- (a) 100 m s^{-1}
(b) 200 m s^{-1}
(c) 300 m s^{-1}
(d) 400 m s^{-1}
4. What is the phase difference, at a given instant of time, between two particles 25 m apart, when the wave of Q. 2 above travels in a medium?
- (a) $\pi/8$
(b) $\pi/4$
(c) $\pi/2$
(d) π
5. Transverse wave of amplitude 10 cm is generated at one end ($x = 0$) of a long string by a tuning fork of frequency 500 Hz. At a certain instant of time, the displacement of a particle A at $x = 100 \text{ cm}$ is -5 cm and of particle B at $x = 200 \text{ cm}$ is $+5 \text{ cm}$. What is the wavelength of the wave?
- (a) 2 cm
(b) 3 cm
(c) 4 cm
(d) 5 cm
6. How long does the wave of Q. 5 take to travel from A to B?
- (a) 10 s
(b) 1 s
(c) $1/10 \text{ s}$
(d) $1/100 \text{ s}$

7. Transverse waves of the same frequency are generated in two steel wires A and B . The diameter of A is twice that of B and the tension in A is half that in B . The ratio of the velocities of waves in A and B is
- (a) 1 : 2 (b) 1 : $\sqrt{2}$
(c) 1 : $2\sqrt{2}$ (d) 3 : $2\sqrt{2}$
8. A sonometer wire, with a suspended mass of $M = 1$ kg, is in resonance with a given tuning fork. The apparatus is taken to the moon where the acceleration due to gravity is $1/6$ that on earth. To obtain resonance on the moon, the value of M should be
- (a) 1 kg (b) $\sqrt{6}$ kg
(c) 6 kg (d) 36 kg
9. Both light and sound waves
- (a) can be diffracted (b) can be polarized
(c) travel in vacuum (d) are electromagnetic in nature
10. A source of sound vibrates according to the equation $y = 0.05 \cos \pi t$. It sends out waves of velocity 1.5 m s^{-1} . The wavelength of the waves is
- (a) 1.5 m (b) 3.0 m
(c) 4.5 m (d) 6.0 m
11. Two identical waves, each of frequency 10 Hz, are travelling in opposite directions in a medium with a speed of 20 cm s^{-1} . The distance between adjacent nodes is
- (a) 1.0 cm (b) 1.2 cm
(c) 1.5 cm (d) 2.0 cm
12. Figure 9.9 shows the shape of a part of a long string in which transverse waves are produced by attaching one end of the string to a tuning fork of frequency 250 Hz. What is the velocity of the waves?
- (a) 1.0 m s^{-1} (b) 1.5 m s^{-1}
(c) 2.0 m s^{-1} (d) 2.5 m s^{-1}

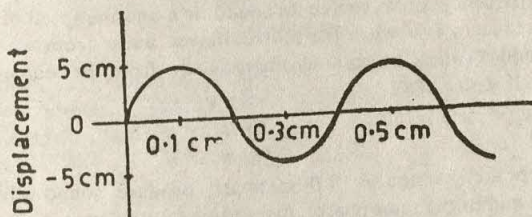


Fig. 9.9

13. Particle displacements (in cm) in a standing wave are given by

$$y(x, t) = 2 \sin(0.1 \pi x) \cos(100 \pi t)$$

The distance between a node and the next anti-node is

- (a) 2.5 cm (b) 5.0 cm
(c) 7.5 cm (d) 10.0 cm
14. If the Young's modulus of the material of a rod is $2 \times 10^{11} \text{ N m}^{-2}$ and its density is 8000 kg m^{-3} , the time taken by a sound wave to traverse 1 m of the rod will be
- (a) 10^{-4} s (b) $2 \times 10^{-4} \text{ s}$
(c) 10^{-2} s (d) $2 \times 10^{-2} \text{ s}$

15. An observer moves towards a stationary source of sound with a velocity one-tenth the velocity of sound. The apparent increase in frequency is
 (a) zero (b) 5%
 (c) 10% (d) 0.1%
16. In which of the following gases will the velocity of sound at a given temperature have the least value?
 (a) N_2 (b) O_2
 (c) SO_2 (d) CO_2
17. Two parts of a sonometer wire, divided by a movable knife-edge, differ in length by 1 cm and produce 1 beat per second when sounded together. If the total length of the wire is 100 cm, the frequencies of the two parts of the wire are
 (a) 51 Hz, 50 Hz (b) 50.5 Hz, 49.5 Hz
 (c) 49 Hz, 48 Hz (d) 49.5 Hz, 48.5 Hz
18. A stone hangs from the free end of a sonometer wire whose vibrating length, when tuned to a tuning fork, is 40 cm. When the stone hangs wholly immersed in water, the resonant length is reduced to 30 cm. The relative density of the stone is
 (a) 16/9 (b) 16/7
 (c) 16/5 (d) 16/3
19. A pipe closed at one end and open at the other will give
 (a) all the harmonics (b) all even harmonics
 (c) all odd harmonics (d) none of the harmonics
20. A cylindrical tube, open at both ends, has a fundamental frequency n . If one of the ends is closed, the fundamental frequency will become
 (a) $n/2$ (b) n
 (c) $2n$ (d) $4n$
21. An organ pipe, open at both ends and another organ pipe, closed at one end, will resonate with each other, if their lengths are in the ratio of
 (a) 1 : 1 (b) 1 : 4
 (c) 2 : 1 (d) 1 : 2
22. The real frequency of a source of sound is n and the speed of sound is v . The apparent frequency is n' when the source moves away from an observer with a velocity u and n'' when the observer moves away from the source with the same velocity u . If $u < v$, then
 (a) $n = 1/2 (n' + n'')$ (b) $n' = n''$
 (c) $n' < n''$ (d) $n' > n''$
23. Two loud speakers A and B , 1.0 m apart, produce sound waves of the same wavelength and in the same phase. A sensitive sound detector, moving along PQ , parallel to line AB , 2.4 m away, detects maximum sound at P (on the perpendicular bisector of AB) and another maximum sound when it first reaches Q directly opposite to B , as shown in Fig. 9.10. What is the wavelength of sound emitted by the speakers?

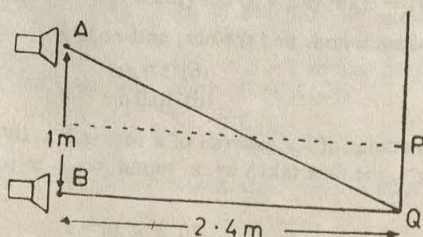


Fig. 9.10

- (a) 0.1 m (b) 0.2 m
(c) 0.3 m (d) 0.4 m
24. Two waves of the same frequency travel in opposite directions in a medium with amplitudes of 3 units and 2 units respectively. The ratio of the amplitude at an anti-node and at a node in the stationary wave is
- (a) 5 : 1 (b) 5 : 3
(c) 3 : 2 (d) infinite

25. A progressive wave in a medium is represented by the equation

$$y = 0.1 \sin \left(10 \pi t - \frac{5}{11} \pi x \right)$$

where y and x are in cm and t in seconds. The maximum speed of a particle of the medium due to the wave is

- (a) 1 cm s⁻¹ (b) 10 cm s⁻¹
(c) π cm s⁻¹ (d) 10π cm s⁻¹
26. A plane progressive wave is represented by the equation

$$y = 0.25 \cos (2 \pi t - \pi x)$$

The equation of a wave with double the amplitude and half the frequency but travelling in the opposite direction will be

- (a) $y = 0.5 \cos (\pi t - \pi x)$ (b) $y = 0.5 \cos (\pi t + \pi x)$
(c) $y = 0.25 \cos (\pi t + 2 \pi x)$ (d) $y = 0.5 \cos (2 \pi t + 2 \pi x)$
27. A tuning fork of frequency 340 Hz is sounded above a cylindrical tube 1 m high. Water is slowly poured into the tube. If the speed of sound is 340 m s⁻¹, at what levels of water in the tube will the sound of the fork be appreciably intensified?
- (a) 25 cm, 75 cm (b) 20 cm, 80 cm
(c) 15 cm, 85 cm (d) 17 cm, 83 cm

28. A sonometer wire, 65 cm long, is in resonance with a tuning fork of frequency N . If the length of the wire is decreased by 1 cm and it is vibrated with the same tuning fork, 8 beats are heard per second. What is the value of N ?

- (a) 256 Hz (b) 384 Hz
(c) 480 Hz (d) 512 Hz
29. Two organ pipes, each closed at one end, give 5 beats per second when emitting their fundamental notes. If their lengths are in the ratio of 50 : 51, their fundamental frequencies (in Hz) are
- (a) 250, 255 (b) 255, 260
(c) 260, 265 (d) 265, 270
30. Two sources A and B are sounding notes of frequency 680 Hz. A listener moves from A to B with a constant velocity u . If the speed of sound is 340 m s⁻¹, what must be the value of u so that he hears 10 beats per second?
- (a) 2.0 m s⁻¹ (b) 2.5 m s⁻¹
(c) 3.0 m s⁻¹ (d) 3.5 m s⁻¹

9.5 NUMERICAL EXERCISES

1. Determine the amplitude, period, frequency, speed and wavelength of the waves given by (all quantities are in MKS units)

- (a) $y(x, t) = 0.01 \sin 2\pi (3t - 2x)$
(b) $y(x, t) = 0.02 \sin (2t + 3x)$
(c) $y(x, t) = 0.03 \cos (\pi t + x)$
(d) $y(x, t) = 0.05 \cos (6\pi t - \pi x)$

2. Write down the equations (using the sine function) of a harmonic travelling along (–) x -axis and having the following characteristics:
 - (a) Amplitude 0.05 m, speed 2 m s^{-1} , frequency 4 Hz
 - (b) Amplitude 0.02 m, frequency 800 Hz, wavelength 0.3 m
 - (c) Amplitude 0.01 m, speed 10 m s^{-1} , wavelength 0.2 m.
3. Two strings A and B , of equal thickness, are made of the same material. The length of A is half that of B while the tension in A is twice that in B . Compare the velocities of the transverse waves on them.
4. Two strings A and B are of the same length and are made of the same material. The cross-sectional area of A is half that of B while the tension in A is twice that in B . Compare the velocities of the transverse waves in them.
5. A string 25 m long has a mass of 0.5 kg. A 10 m long specimen of the string is stretched with force equal to the weight of 5 kg. How long will the transverse wave take to travel the length of the string?
6. The intensity of the sound of a barking dog is 10^{-3} W m^{-2} . If the frequency is 1000 Hz, calculate the amplitude of the sound waves in air at standard conditions.
7. Calculate the amplitude of the sound wave of Ex. 6 if the air temperature were 35°C .

8. Two waves, travelling in a medium, are given by

$$y_1 = 0.02 \sin 4\pi(100t - x)$$

$$\text{and } y_2 = 0.03 \cos 4\pi(100t - x)$$

where the distances are in metres and time in seconds.

- (a) What is the frequency of each wave?
 - (b) What is the phase difference between them?
 - (c) Calculate the amplitude of the resultant motion of a particle at x when the two waves interfere.
 - (d) Will the resultant amplitude depend on where the particle is located in the medium?
 - (e) Will the resultant amplitude change with time?
9. It is desired to tune the string of a sonometer with a tuning fork of frequency 512 Hz. When vibrated together they produce 10 beats per second. (a) By what percentage should the length of the string be altered to achieve tuning, tension remaining the same? (b) By what percentage should the tension in the string be altered to achieve tuning, the length of the string remaining the same?
 10. A string under a tension of 100 N is emitting its fundamental note. While tuning with a tuning fork 5 beats per second are heard. When the tension is increased to 121 N, again 5 beats per second are heard. Determine the frequency of the fork.
 11. Two identical pipes are sounding notes of frequency 500 Hz. The air temperature in both is 25°C . The temperature of the air in one pipe is increased to 30°C . How many beats are heard per second when both the pipes are blown? Velocity of sound at $0^\circ \text{C} = 330 \text{ m s}^{-1}$.
 12. Calculate the percentage change in the frequency of the fundamental note emitted by a closed pipe of length 1.0 m if the temperature of the air is changed from 15°C to 30°C . Velocity of sound at $0^\circ \text{C} = 330 \text{ m s}^{-1}$.
 13. Two tuning forks A and B produce 4 beats per second when vibrated together. If a prong of fork A is filed, the beats reduce to 2 per second. If the frequency of fork B is 480 Hz, what is the frequency of fork A (a) before and (b) after it is filed?
 14. A sonometer wire carries a metal block at the other end. When the block is in air, the resonant length with a tuning fork is found to be 90 cm. When the block is wholly immersed in water, the resonant length with the same tuning fork decreases to 80 cm. Find the relative density of the block.

15. Two tuning forks A and B produce 10 beats per second when sounded together. When each fork is sounded above the water column in a resonance tube apparatus, their respective resonance lengths of the air column are found to be 16 cm and 16.5 cm. Determine the frequencies of forks A and B .
16. A pipe of length 1.5 m closed at one end is filled with a gas and it resonates in its fundamental mode with a tuning fork. Another pipe of the same length but open at both ends is filled with air and it resonates in its fundamental mode with the same tuning fork. Calculate the velocity of sound at 0°C in the gas, given that the velocity of sound in air is 360 m s^{-1} at 30°C where the experiment is performed. (IIT 1974)
17. Two tuning forks A and B produce 8 beats per second when sounded together. A gas column 37.5 cm long in a pipe closed at one end resonates in its fundamental mode with fork A , whereas a column of length 38.5 cm of the same gas in a similar pipe is required for a similar resonance with fork B . Calculate the frequencies of the two forks.
18. A tube of a certain diameter and length 48 cm is open at both ends. Its fundamental frequency of resonance is found to be 320 Hz. The velocity of sound in air is 320 m s^{-1} . Estimate the diameter of the tube. One end of the tube is now closed. Calculate the lowest frequency of resonance for the tube. (IIT 1980)
19. A steel wire of length 1 m, mass 0.1 kg and uniform cross-sectional area 10^{-6} m^2 is rigidly fixed at both ends. The temperature of the wire is lowered by 20°C . If transverse waves are set up by plucking the wire in the middle, calculate the frequency of the fundamental mode of vibration. Coefficient of linear expansion of steel $= 1.21 \times 10^{-5} (^\circ\text{C})^{-1}$, Young's modulus of steel $= 2 \times 10^{11}\text{ N m}^{-2}$. (IIT 1984)
20. The sound of a horn in a car is recorded by a stationary tape recorder as the car is approaching it and also as it is receding from it. The ratio of the two apparent frequencies is found to be 1.2. If the speed of sound is 340 m s^{-1} , determine the speed of the car.
21. At what speed is a car approaching a stationary observer if he hears the music of the car radio with a frequency 10% higher than it actually is? The speed of the sound is 350 m s^{-1} .
22. At what speed must the driver of a car approach a stationary siren so that the frequency of the sound heard by him is 10% higher than it actually is? Speed of sound $= 340\text{ m s}^{-1}$.
23. A railway engine and a car are moving on parallel tracks with speeds of 30 m s^{-1} and 15 m s^{-1} respectively. The engine is continuously sounding a whistle of frequency 1000 Hz. Calculate the frequency of sound heard by the driver of the car when (a) the car and the engine are moving in opposite directions and approaching each other, (b) they are moving in opposite directions but going away from each other, (c) they are moving in the same direction, the car being ahead of the engine and (d) they are moving in the same direction, the car being behind the engine.
24. A source of sound of frequency 256 Hz is moving rapidly towards a wall with a velocity of 5 m s^{-1} . How many beats per second will be heard if sound travels at a speed of 330 m s^{-1} ? (IIT 1981)

ANSWERS

MULTIPLE CHOICE QUESTIONS

- | | | | | |
|---------|---------|---------|---------|---------|
| 1. (d) | 2. (d) | 3. (b) | 4. (b) | 5. (a) |
| 6. (c) | 7. (c) | 8. (c) | 9. (a) | 10. (b) |
| 11. (a) | 12. (a) | 13. (b) | 14. (b) | 15. (c) |
| 16. (c) | 17. (b) | 18. (b) | 19. (c) | 20. (a) |
| 21. (c) | 22. (d) | 23. (b) | 24. (a) | 25. (c) |
| 26. (b) | 27. (a) | 28. (d) | 29. (a) | 30. (b) |

NUMERICAL EXERCISES

- 0.01 m, $1/3$ s, 3 Hz, 1.5 m s^{-1} , 0.5 m
 - 0.02 m, 3.14 s, 0.32 Hz, 0.67 m s^{-1} , 2.09 m
 - 0.03 m, $2\pi/v$ s, $v/2\pi$ Hz, $v \text{ m s}^{-1}$, 2π m
 - 0.05 m, $1/3$ s, 3 Hz, 6 m s^{-1} , 2 m
- $y(x, t) = 0.05 \sin 4\pi(2t - x)$
 - $y(x, t) = 0.02 \sin 2\pi(800t - x/3)$
 - $y(x, t) = 0.01 \sin 10\pi(10t - x)$
- $\sqrt{2} : 1$
- 2 : 1
- 0.2 s
- $3.44 \times 10^{-7} \text{ m}$
- $3.33 \times 10^{-7} \text{ m}$
- 200 Hz, (b) 90° , (c) 0.036 m, (d) No, (e) No
- 1.95%, (b) 4.02% or 3.8%
- 105 Hz
- 4
- 2.63%
- 476 Hz, 478 Hz
- 4.76
- 330 Hz, 320 Hz
- 683.4 m s^{-1}
- 300 Hz, 308 Hz
- 3.33 cm, 163.3 Hz
- 11 Hz
- 30.9 m s^{-1}
- 31.8 m s^{-1}
- 34 m s^{-1}
- 1112.5 Hz, (b) 881.6 Hz, (c) 1046.9 Hz, (d) 360.5 Hz
- zero, 7.7 or 7.9 beats per second depending on the position of the observer.

Thermal Physics

SECTION A : THERMAL EXPANSION OF SOLIDS AND LIQUIDS

10.1A REVIEW OF BASIC CONCEPTS

10.1A.1. Linear Expansion Almost all solids expand on heating. The expansion is practically uniform and the change in length Δl is proportional to the change in temperature Δt and the original length l . Thus the basic equation for the expansion of a solid is

$$\Delta l = l\alpha\Delta t \quad (10.1)$$

where α is the coefficient of linear expansion for the given solid. To find the final length, Δl is added or subtracted from the original length depending on whether the temperature increases or decreases.

The unit for α is per degree celsius ($^{\circ}\text{C}^{-1}$) in the CGS system and per kelvin (K^{-1}) in the SI system. The magnitudes of both these units are equal since the size of the degree is the same on the two temperature scales.

The coefficient of linear expansion is almost constant if Δt is not too large, say not more than 100 K. If a solid of length L_0 at 0°C is heated to a temperature $t^{\circ}\text{C}$, then its length at temperature $t^{\circ}\text{C}$ is given by

$$L_t = L_0 (1 + \alpha t) \quad (10.2)$$

It may be remarked that the elastic properties and thermal expansion of a substance are dependent on the cohesive forces between molecules. Thus, substances which are easily compressed also usually have high expansion coefficients and low melting points. Lead, with relatively weak cohesive forces between atoms in the crystal, is one such example.

10.1A.2 Differential Expansion The difference in the expansion of different materials is used in various practical applications such as in thermostats and inexpensive thermometers. Thin strips of different metals are welded or riveted together along their lengths and clamped at one end. When the temperature changes, the combination bends in an arc, as shown in the schematic diagram of a thermostat (Fig. 10.1), where metal 1 has a greater coefficient of expansion than metal 2. Due to increase in temperature, the electric contact is broken and the appliance is turned off.

Consider two rods XY and $X'Y'$ of different metals rigidly connected at X . If l_1 and l'_1 are their lengths at a temperature t_1 , the difference in lengths is $\Delta l_1 = (l_1 - l'_1)$. If α and α' are the coefficients of linear expansion of the materials of the rods, then the lengths at temperature t_2 will be

$$l_2 = l_1 \{1 + \alpha (t_2 - t_1)\}$$

$$l'_2 = l'_1 \{1 + \alpha' (t_2 - t_1)\}$$

The difference between their lengths will now be

$$\Delta l_2 = l_2 - l'_2 = (l_1 - l'_1) + (t_2 - t_1) (\alpha l_1 - \alpha' l'_1)$$

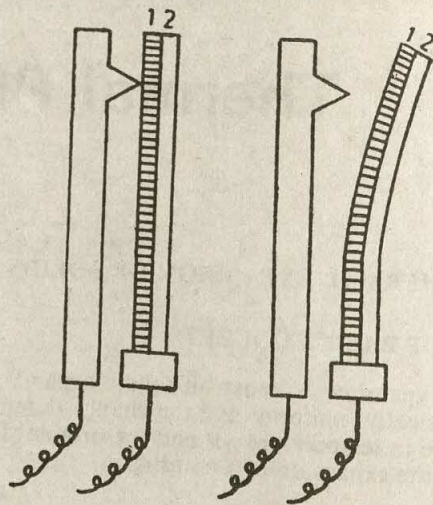


Fig. 10.1

If the materials are suitably selected such that $\alpha l_1 = \alpha' l'_1$ then

$$\Delta l_1 = \Delta l_2$$

This principle is used in compensated pendulums.

10.1A.3 Superficial Expansion If A_0 is the area of a solid at 0°C and A_t its area at $t^\circ\text{C}$ then

$$A_t = A_0 (1 + \beta t) \quad (10.3)$$

where β is known as the coefficient of superficial expansion.

10.1A.4 Volume Expansion The change in the volume of a solid with a change in temperature Δt is given by

$$\Delta V = V \gamma \Delta t \quad (10.4)$$

where γ is the coefficient of volume expansion.

For an isotropic solid,

$$\beta \simeq 2\alpha \quad \text{and} \quad \gamma \simeq 3\alpha$$

10.1A.5 Variation of Density with Temperature If V_0 is the volume of a solid at 0°C and V_t its volume at $t^\circ\text{C}$ then

$$V_t = V_0 (1 + \gamma t) \quad (10.5)$$

Since density $d = \frac{m}{V}$ or $V = \frac{m}{d}$ we have

$$\frac{m}{d_t} = \frac{m}{d_0} (1 + \gamma t)$$

where d_t and d_0 denote the density at $t^\circ\text{C}$ and 0°C respectively.

$$\begin{aligned} \text{Hence} \quad d_0 &= d_t (1 + \gamma t) \\ \text{or} \quad d_t &\simeq d_0 (1 - \gamma t) \end{aligned} \quad (10.6)$$

10.2A EXPANSION OF LIQUIDS

10.2A.1 Volume Expansion Coefficient. In the case of liquids, the cohesive forces are very weak. The liquids are, therefore, generally more compressible than solids and their thermal expansion is greater. The shape of a liquid is determined by the vessel containing it. The only expansivity which is relevant for liquids is, therefore, the volume or cubical expansion. The coefficient of volume expansion of a liquid is defined as

$$\gamma = \frac{V_2 - V_1}{V_1 (t_2 - t_1)} \quad (10.7)$$

where V_1 and V_2 are the volumes of a given mass of a liquid at temperatures t_1 and t_2 respectively.

The expansion observed when a liquid is heated is always less than the true expansion of the liquid itself, as the volume of the containing vessel also increases. The observed expansion is, therefore, called the 'apparent expansion'. If γ_a is the coefficient of the apparent expansion, the coefficient of real expansion γ of the liquid is given by

$$\gamma = \gamma_a + \gamma_g \frac{m_1}{m_2} \quad (10.8)$$

where m_1 and m_2 denote the mass of the same liquid filling the vessel at temperatures t_1 and t_2 respectively.

For $m_1 = m_2$, $\gamma = \gamma_a + \gamma_g$

If d_1 and d_2 are the densities of the liquid at the two temperatures, then

$$d_2 = \frac{d_1}{1 + \gamma (t_2 - t_1)} \quad (10.9)$$

10.2A.2 Correction of the Barometer If d_t is the density of mercury at $t^\circ\text{C}$, then the atmospheric pressure at this temperature is given by

$$p = g d_t h$$

where h , the true height is given by

$$h = h_{\text{obs}} (1 + \lambda t) \quad (10.10)$$

where h_{obs} is the observed height on the scale and λ is the coefficient of linear expansion of the metal scale. The height of a mercury column at 0°C which would exert the same pressure is called the correct height, h_{cor} .

Thus
$$p = g d_0 h_{\text{cor}} \quad (10.11)$$

where d_0 is the density of mercury at 0°C . Therefore,

$$d_t h = d_0 h_{\text{cor}} \quad (10.12)$$

But

$$d_t = \frac{d_0}{(1 + \gamma t)} \quad (10.13)$$

where γ is the coefficient of expansion of mercury. From Eqs (10.10) and (10.13) we have

$$h_{\text{cor}} = h_{\text{obs}} \frac{(1 + \lambda t)}{(1 + \gamma t)} = h_{\text{obs}} (1 + \lambda t) (1 + \gamma t)^{-1}$$

$$\begin{aligned} \text{or } h_{\text{cor}} &\simeq h_{\text{obs}} (1 + \lambda t) (1 - \gamma t) \\ &\simeq h_{\text{obs}} \{1 - (\gamma - \lambda) t\} \end{aligned} \quad (10.14)$$

where terms of order t^2 and higher have been ignored.

Thus the correction to the observed barometric height depends on the difference between the volume expansion coefficient of mercury and the coefficient of linear expansion of the scale.

10.3A COEFFICIENTS OF VOLUME EXPANSION OF SOME SUBSTANCES

Table 10.1

<i>Solids</i>	(K^{-1})	<i>Liquids</i>	(K^{-1})
	$\times 10^{-6}$		$\times 10^{-3}$
Iron or steel	72	Alcohol (ethyl)	750
Aluminium	72	Alcohol (methyl)	1134
Brass	60	Carbon tetrachloride	581
Copper	51	Carbon disulphide	1150
Glass	17-27	Glycerine	485
Pyrex glass	10	Mercury	182
Invar	2.7	Turpentine	900
Fused quartz	1	Water (20 °C)	200
Concrete	36	Water (50 °C)	600
Platinum	27		
Lead	87		

10.4A FORMULAE

1. Linear expansion $l_2 = l_1 \{1 + \alpha (t_2 - t_1)\}$
2. Superficial expansion $A_2 = A_1 \{1 + \beta (t_2 - t_1)\}$
3. Volume expansion $V_2 = V_1 \{1 + \gamma (t_2 - t_1)\}$
4. Density $d_1 = d_2 \{1 + \gamma (t_2 - t_1)\}$
5. Coefficient of real expansion $\gamma = \gamma_a + \gamma_g \frac{m_1}{m_2}$

10.5A NUMERICAL EXAMPLES

Example 1: The metal of a pendulum clock has a linear thermal coefficient of $2 \times 10^{-5} K^{-1}$. If the period is 2 s at 15 °C, calculate the loss or gain in time shown by the clock in 10 hours when its temperature rises to 35 °C.

Solution: Due to increase in temperature from 15 °C to 35 °C the increase in the length of the pendulum is given by

$$\Delta l = l \alpha \Delta t$$

$$\text{or } \frac{\Delta l}{l} = 2 \times 10^{-5} \times 20 = 4 \times 10^{-4} = 0.0004$$

If T is the time period when the length is l and T' the time period when the length is l' , we have

$$\frac{T'}{T} = \sqrt{\frac{l'}{l}} \quad \left(\text{since } T = 2\pi \sqrt{\frac{l}{g}} \right)$$

where $l' = l + \Delta l$

$$\therefore \frac{T'}{T} = \sqrt{\frac{l + \Delta l}{l}} = \sqrt{1 + \frac{\Delta l}{l}} = \sqrt{1.0004} = 1.0002$$

$$\therefore T' = 2 \times 1.0002 \text{ s} = 2.0004 \text{ s}$$

$$\text{Loss of time} = 2.0004 - 2 = 0.0004 \text{ s}$$

This is the loss of time in every two seconds. Hence the loss of time in 10 hours is

$$\frac{10 \times 3600}{2.0004} \times 0.0004 = 7.2 \text{ s}$$

Example 2: A cylinder of diameter exactly 1 cm at 30°C is to be slid into a hole in a steel plate. The hole has a diameter of 0.99970 cm at 30°C . To what temperature must the plate be heated? For steel $\alpha = 1.1 \times 10^{-5} (^\circ\text{C}^{-1})$.

Solution: The hole will expand in the same way as a circle of steel filling it would expand. The diameter of the hole needs to be changed by

$$\Delta l = 1 - 0.99970 = 0.00030 \text{ cm}$$

But $\Delta l = \alpha l \Delta t$

$$\therefore \Delta t = \frac{\Delta l}{\alpha l} = \frac{0.00030}{1.1 \times 10^{-5} \times 0.99970} = 27.3^\circ\text{C}$$

The plate must be raised to a temperature of $30 + 27.3 = 57.3^\circ\text{C}$

Example 3: A steel scale measures the length of a copper rod as 80.00 cm when both are at 20°C , the calibration temperature for the scale. What would the scale read for the length of the rod when both are at 40°C ? α for steel $= 11 \times 10^{-6} (^\circ\text{C}^{-1})$ and α for copper $= 17 \times 10^{-6} (^\circ\text{C}^{-1})$.

Solution: The length of 1 cm division of the steel scale at 40°C is

$$(1 \text{ cm}) \times (1 + 11 \times 10^{-6} \times 20) = 1.00022 \text{ cm}$$

Length of the copper rod at 40°C will be

$$(80) \times (1 + 17 \times 10^{-6} \times 20) = 80.0272 \text{ cm}$$

The number of cm read on the scale will be

$$\frac{80.0272}{1.00022} \text{ cm} = 80.0096 \text{ cm}$$

Example 4: A barometer scale made of steel reads 750 mm on a day when the temperature is 20°C . If the scale is correctly graduated at 0°C , calculate the correct pressure. Linear expansivity for steel is $12 \times 10^{-6} (^\circ\text{C}^{-1})$ and the coefficient of absolute expansion of mercury is $182 \times 10^{-6} (^\circ\text{C}^{-1})$.

Solution: The length of the scale at 20 °C is

$$\begin{aligned} l_{20} &= l_0 (1 + \alpha_s t) \\ &= 750 \times (1 + 12 \times 10^{-6} \times 20) \text{ mm} \end{aligned}$$

The density of mercury at 20 °C is

$$d_{20} = \frac{d_0}{1 + \gamma t} = \frac{d_0}{1 + 182 \times 10^{-6} \times 20}$$

True pressure at 20 °C is given by

$$\begin{aligned} P &= \text{height at 20 °C} \times \text{density at 20 °C} \times g \\ &= l_{20} \times d_{20} \times g \\ &= 750 (1 + 12 \times 10^{-6} \times 20) \frac{d_0 g}{1 + 182 \times 10^{-6} \times 20} \end{aligned}$$

which gives $\frac{P}{d_0 g} = 747.46 \text{ mm}$

Now $P = h_0 d_0 g$

where h_0 is the true barometric height.

$\therefore h_0 = \frac{P}{d_0 g} = 747.46 \text{ mm}$

The above result can also be obtained from Eq. (10.14) :

$$\begin{aligned} h_{\text{cor}} &= h_{\text{obs}} (1 + \lambda t) (1 - \gamma t) \\ &= 750 \times (1 + 12 \times 10^{-6} \times 20) \times (1 - 182 \times 10^{-6} \times 20) \\ &= 747.45 \text{ mm} \end{aligned}$$

Example 5: A mercury barometer has a quartz bulb of volume 0.300 cm³ and a stem of bore 0.0100 cm. How far does the indicating thread of mercury move when the temperature changes from 30 °C to 45 °C ? Ignore the small expansion of the quartz bulb. The volume expansivity of mercury is $182 \times 10^{-6} (\text{°C})^{-1}$.

Solution: The increase in volume due to expansion is given by

$$\begin{aligned} \Delta V &= \gamma V \Delta t \\ &= 182 \times 10^{-6} \times 0.300 \times 15 = 8.19 \times 10^{-4} \text{ cm}^3 \end{aligned}$$

Cross-sectional area of the bore = $\pi (0.01/2)^2 = \pi (0.005)^2 \text{ cm}^2$

$$\begin{aligned} \text{Height through which mercury thread rises} &= \frac{8.19 \times 10^{-4}}{\pi (5 \times 10^{-3})^2} \\ &= 10.4 \text{ cm} \end{aligned}$$

Example 6: A glass flask of volume 250 cm³ is just filled with mercury at 20 °C. How much mercury overflows when the temperature of the system is raised to 100 °C ? The coefficient of volume expansion of glass is $12 \times 10^{-6} (\text{°C})^{-1}$ and that of mercury is $18 \times 10^{-5} (\text{°C})^{-1}$.

Solution: The increase in the volume of the flask is

$$(\Delta V)_f = V \gamma \Delta t = 250 \times 12 \times 10^{-6} \times 80 = 0.24 \text{ cm}^3$$

The increase in the volume of mercury is

$$(\Delta V)_m = 250 \times 18 \times 10^{-5} \times 80 = 3.6 \text{ cm}^3$$

Therefore, the volume of mercury overflowing is

$$3.6 - 0.24 = 3.36 \text{ cm}^3$$

Example 7: A one litre flask contains some mercury. It is found that at different temperatures the volume of air inside the flask remains the same. What is the volume of mercury in the flask? Given the coefficient of linear expansion of glass $= 9 \times 10^{-6} (\text{°C})^{-1}$, the coefficient of volume expansion of mercury is $1.8 \times 10^{-4} (\text{°C})^{-1}$ (IIT 1973)

Solution: Let x be the volume of mercury in the flask. Since the volume of air in the flask remains constant, the volume expansion of the flask must be exactly equal to the expansion of mercury in the flask. Therefore

$$V \gamma_g \Delta t = x \gamma_m \Delta t$$

where γ_g and γ_m are the volume expansivities of glass and mercury respectively.

$$\therefore x = V \frac{\gamma_g}{\gamma_m}$$

Here $\gamma_g = 3 \times 9 \times 10^{-6} = 27 \times 10^{-6} (\text{°C})^{-1}$
 $\gamma_m = 1.8 \times 10^{-4} (\text{°C})^{-1}$, $V = 1000 \text{ cm}^3$

$$\therefore x = \frac{1000 \times 27 \times 10^{-6}}{1.8 \times 10^{-4}} \text{ cm}^3 = 150 \text{ cm}^3$$

Example 8: A sphere of diameter 7 cm and mass 266.5 g floats in a bath of liquid. As the temperature is raised, the sphere begins to sink at a temperature of 35 °C . If the density of the liquid is 1.527 g cm^{-3} at 0 °C , find the coefficient of cubical expansion of the liquid. Neglect the expansion of the sphere. (IIT 1962)

Solution: When the sphere just begins to sink, the upthrust experienced by it must be equal to its weight. Thus,

$$\left(\frac{4}{3} \pi r^3 \right) d_t = 266.5$$

Substituting $r = 3.5 \text{ cm}$ in this equation we get $d_t = 1.484 \text{ g cm}^{-3}$

Also $d_0 = d_t (1 + \gamma t)$
 $= 1.484 (1 + 35 \gamma)$

Now, $d_0 = 1.527 \text{ g cm}^{-3}$. Using this value of d_0 we get
 $\gamma = 8.30 \times 10^{-4} (\text{°C})^{-1}$

Example 9: A metallic bob weighs 50 g in air. If it is immersed in a liquid at a temperature of 25 °C , it weighs 45 g. When the temperature of the liquid is raised to 100 °C , it weighs 45.1 g. Calculate the coefficient of cubical expansion of the liquid, given the coefficient of linear expansion of the metal to be $12 \times 10^{-6} (\text{°C})^{-1}$. (IIT 1973)

Solution: Let γ_m be the coefficient of cubical expansion of the metal.

Then $\gamma_m = 3 \alpha = 3 \times 12 \times 10^{-6} (^\circ\text{C})^{-1} = 36 \times 10^{-6} (^\circ\text{C})^{-1}$

If V_1 and V_2 are the volumes of the metal bob at $t_1 = 25^\circ\text{C}$ and $t_2 = 100^\circ\text{C}$ respectively, then

$$\frac{V_2}{V_1} = 1 + \gamma_m (t_2 - t_1) = 1 + 36 \times 10^{-6} \times 75 = 1.0027$$

Now, Upthrust = loss of weight of the bob in the liquid

$$\therefore V_1 d_1 = 50 - 45 = 5 \text{ g}$$

and $V_2 d_2 = 50 - 45.1 = 4.9 \text{ g}$

Here d_1 and d_2 are the densities of the liquid at 25°C and 100°C respectively. Therefore

$$\frac{V_1 d_1}{V_2 d_2} = \frac{5}{4.9}$$

But $d_1 = d_2 (1 + 75 \gamma)$

$$\therefore 1 + 75 \gamma = \frac{d_1}{d_2} = \frac{5}{4.9} \times 1.0027 = 1.0232$$

$$\therefore \gamma = \frac{0.0232}{75} = 3.088 \times 10^{-4} (^\circ\text{C})^{-1}$$

Example 10: A sinker of weight W_0 has an apparent weight W_1 when placed in a liquid at a temperature t_1 and W_2 when weighed in the same liquid at a temperature t_2 . The coefficient of cubical expansion of the material of the sinker is β . What is the coefficient of volume expansion of the liquid?

Solution: Let V_1 be the volume of the sinker at temperature t_1 and V_2 its volume at temperature t_2 . If d_1 and d_2 are the densities of the liquid at temperatures t_1 and t_2 respectively, then we have

$$W_0 - W_1 = V_1 d_1 g$$

and $W_0 - W_2 = V_2 d_2 g$

because the loss of weight of the sinker must be equal to the weight of the liquid displaced.

$$\therefore \frac{W_0 - W_1}{W_0 - W_2} = \frac{V_1 d_1}{V_2 d_2}$$

But $V_2 = V_1 \{1 + \beta (t_2 - t_1)\}$

and $d_1 = d_2 \{1 + \gamma (t_2 - t_1)\}$

where γ is the coefficient of cubical expansion of the liquid.

Substituting for V_1/V_2 and d_1/d_2 we have

$$\frac{W_0 - W_1}{W_0 - W_2} = \frac{1 + \gamma (t_2 - t_1)}{1 + \beta (t_2 - t_1)}$$

$$(W_0 - W_1) \{1 + \beta (t_2 - t_1)\} = (W_0 - W_2) \{1 + \gamma (t_2 - t_1)\}$$

which gives

$$\gamma = \frac{(W_2 - W_1)}{(W_0 - W_2)(t_2 - t_1)} + \frac{\beta(W_0 - W_1)}{(W_0 - W_2)}$$

10.6A TRUE-FALSE STATEMENTS WITH REASONS

A few statements are given below. We have to decide whether they are true or false giving, in brief, the reasons in support of our answers in each case.

1. A thermometer measures the amount of heat in a body.
2. A temperature change which increases the length of a steel rod by 0.1% will increase its volume by 0.3%.
3. If a copper plate, with a hole in it, is heated, the hole becomes smaller due to thermal expansion.
4. When the distance between two fixed points is measured with a steel tape, the observed reading will be less on a hot day than on a cold day.
5. A temperature change of 1 °C on the celsius scale is equal to a temperature change of 274 K on the kelvin scale.

ANSWERS

1. *False* The working of a thermometer is based on the principle of thermal expansion and hence measures the change in the temperature of a body.
2. *True* The coefficient of volume expansion is nearly three times the coefficient of linear expansion.
3. *False* The hole becomes larger due to expansion as if it were a part of the solid plate. Alternatively, one could imagine the plate to be cut in such a way so as to form a rectangular plate. The expansion would cause an increase in length uniformly. (This is equivalent to an increase in the outer and inner circumference of the circular plate).
4. *True* Because of the expansion of the steel tape on a hot day, the markings on the tape would be farther apart than on a cold day.
5. *False* Since the size of a degree on the celsius and the kelvin is the same, a temperature change of 1 °C corresponds to a temperature change of 1 K.

10.7A MULTIPLE CHOICE QUESTIONS

Choose the correct answer from the given alternatives in each of the following:

1. A bar made of iron for which $\alpha = 11 \times 10^{-6}$ per °C is 10.000 cm at 20 °C. At 19 °C the length is

(a) 11×10^{-6} cm longer	(b) 11×10^{-6} cm shorter
(c) 11×10^{-5} cm shorter	(d) 11×10^{-4} cm shorter
2. A linear accelerator contains a hundred brass discs tightly fitted into a steel tube. The coefficient of expansion of brass is 2×10^{-5} per °C. The system is assembled by cooling the discs in dry ice (-57 °C) to enable them to slide into the close-fitting tube. If the diameter of the disc is 80 mm at 43 °C, its diameter in dry ice will be

(a) 78.40 mm	(b) 79.68 mm
(c) 80.16 mm	(d) 79.84 mm

3. When a copper ball is heated, the largest percentage increase will occur in its
 - (a) diameter
 - (b) area
 - (c) volume
 - (d) density
4. At about 4°C , a certain amount of water has maximum
 - (a) density
 - (b) energy
 - (c) volume
 - (d) specific heat
5. To keep correct time, watches are fitted with a balance wheel made of
 - (a) stainless steel
 - (b) platinum
 - (c) tungsten
 - (d) invar

10.8 A NUMERICAL EXERCISES

1. During the construction of a steel bridge, a steel rivet 4.103 cm in diameter is to be placed in a hole 4.100 cm in diameter. Would it be possible to do the job by first cooling the rivet in dry ice (-78°C)? The temperature of the surroundings is 22°C and α for steel is 11×10^{-6} per $^{\circ}\text{C}$.
2. A steel scale is to be prepared such that the millimeter intervals are to be accurate within 5×10^{-4} mm at a certain temperature. Determine the maximum permissible temperature variation during the ruling of the millimeter marks. For steel the value of α is 13.22×10^{-6} per $^{\circ}\text{C}$. (IIT 1967)
3. A clock with a metallic pendulum is 10 s slow each day at a temperature of 30°C and 5 s fast at a temperature of 15°C . Determine the coefficient of linear expansion of the pendulum metal.
4. The design of a physical apparatus requires that there be a constant difference in length at any temperature between an iron and a copper cylinder laid side by side. What should be the lengths of the cylinders at 0°C , for the difference in length to be 10 cm at all temperatures? $\alpha(\text{Fe}) = 11 \times 10^{-6} (^{\circ}\text{C})^{-1}$, $\alpha(\text{Cu}) = 17 \times 10^{-6} (^{\circ}\text{C})^{-1}$. (IIT 1964)
5. An iron rim is to be fitted round the wheel of a wagon whose circumference at 15°C is 270.562 cm. The rim is the form of a band having a cross-section of 2.50 cm^2 and inner circumference at 15°C of 270.020 cm. (a) To what temperature must the band be heated to fit tightly on the wheel? (b) What will be the tension in the band when it cools to 15°C ? The value of α for iron = $11 \times 10^{-6} (^{\circ}\text{C})^{-1}$.
6. The relative density of aluminium is 2.699 and that of glycerine is 1.248 at 20°C . A block of aluminium whose volume is 100 cm^3 at 20°C is immersed in glycerine at 20°C . (a) What is the apparent weight of the aluminium block? (b) What will be the apparent weight of the block if the system is heated to 80°C ? $\alpha(\text{Al}) = 26 \times 10^{-6} (^{\circ}\text{C})^{-1}$.
7. Aniline is a liquid which does not mix with water and when a small quantity of it is poured into a beaker of water at 20°C , it sinks to the bottom; the densities of the two liquids being 1021 and 998 kg m^{-3} respectively. To what temperature must the beaker and its contents be uniformly heated so that the aniline will form a globule which just floats in the water? The mean coefficients of real expansion of aniline and water over the temperature range are 8.5×10^{-4} and $4.5 \times 10^{-4} \text{ K}^{-1}$ respectively.
8. A glass weight thermometer is filled with mercury at 0°C . When it is heated to 100°C , 7.785 g of mercury overflow and 450 g remain in the thermometer. If glycerine is used in place of mercury, the corresponding figures are 2.173 g and 41 g. Determine (a) the coefficient of linear expansion of glass, (b) the coefficient of volume expansion of glass and (c) the coefficient of volume expansion of glycerine.
9. Using the following data, determine the temperature at which wood will just sink in benzene.

Density of benzene at $0^\circ\text{C} = 900\text{ kg m}^{-3}$

Density of wood at $0^\circ\text{C} = 880\text{ kg m}^{-3}$

Coefficient of volume expansion of benzene $= 1.2 \times 10^{-3}\text{ K}^{-1}$

Coefficient of volume expansion of wood $= 1.5 \times 10^{-4}\text{ K}^{-1}$

10. A certain Fortin's barometer has its pointers, body and scales made of brass. When it is at 0°C , it records a barometric pressure of 760 mm of Hg. What will it read when its temperature is increased to 20°C if the pressure of the atmosphere remains unchanged? The coefficient of real expansion of Hg $= 1.8 \times 10^{-4}\text{ K}^{-1}$ and the coefficient of linear expansion of brass $= 2 \times 10^{-5}\text{ K}^{-1}$.

SECTION B: MEASUREMENT OF HEAT

10.1B REVIEW OF BASIC CONCEPTS

10.1B.1 Heat We know that heat is a form of energy. We may regard it as a form of energy that is *transferred* as a consequence of a difference in temperature between a system and its surroundings; the transfer, of course, taking place in compliance with the law of conservation of energy.

10.1B.2 Units of Heat A popular unit of heat is the *International 15°C calorie*, ordinarily referred to simply as the *calorie*. A calorie is the amount of heat required to raise the temperature of one gram of water from 14.5°C to 15.5°C on the international scale of temperature under the pressure of one normal atmosphere.

The units of heat now used in the SI units is the standard unit of energy, the joule. The relation between the two units is

$$1\text{ cal} = 4.1868\text{ J}$$

10.1B.3 Thermal Capacity The thermal capacity of a body is the quantity of heat required to raise the temperature of the whole of the body through a unit degree. It is measured in calorie per $^\circ\text{C}$ or joule per K.

10.1B.4 Specific Heat Capacity The specific heat capacity (also referred to as specific heat) of a substance is the amount of heat required to raise the temperature of a unit mass of the substance through 1°C . It is measured in $\text{cal g}^{-1} (^\circ\text{C})^{-1}$ or $\text{J kg}^{-1}\text{ K}^{-1}$.

Strictly speaking, the specific heat capacity of a substance depends upon (i) its temperature and (ii) the external conditions under which it is heated, e.g. constant pressure or at constant volume. However, for ordinary purposes, the dependence on temperature is usually neglected. Also the difference in the specific heat capacities of solids and liquids when heated at constant pressure or at constant volume, is small enough to be neglected. For gases, on the other hand, this difference is significant and we have to differentiate between the specific heat of a gas at constant pressure and at constant volume.

10.1B.5 Water Equivalent The 'water equivalent' of a body is the mass of water which would require the same quantity of heat to raise its temperature through a unit degree as the body does. In other words, it is the mass of water which has the same heat capacity as that of the body itself. Hence the units of water equivalent are kg.

If w is the water equivalent of a body of mass m and specific heat s , we have, by definition of w ,

$$w \times s_w = m \times s$$

where s_w is the specific heat capacity of water. Thus

$$w = m \times \left(\frac{s}{s_w} \right)$$

The specific heat capacity of water is $1 \text{ cal g}^{-1} (^\circ\text{C})^{-1}$. Thus, in these units, the water equivalent of a body is simply its mass times its specific heat capacity. In these units, therefore, it is numerically equal to the thermal capacity of the body.

10.1B.6 The Basic Heat Formula The heat Q required to raise the temperature of a mass m of a substance of specific heat capacity s through t degrees is given by

$$Q = m \times s \times t$$

i.e. Heat required = mass $\times$ specific heat $\times$ change in temperature.

This is the basic heat formula.

10.1B.7 Calorimetry Calorimetry is concerned with the measurement of heat, the basic apparatus for this purpose being called the *calorimeter*.

When two bodies at different temperatures are 'mixed', heat 'flows' from the body at a higher temperature to the one at a lower temperature, until a common 'equilibrium' temperature is reached. Assuming this 'heat exchange' to be confined to the two bodies alone (i.e. neglecting any heat loss to the surroundings) we have, from the law of energy conservation:

$$\text{Heat gained by one body} = \text{Heat lost by the other}$$

This is the basic principle of calorimetry.

10.1B.8 Latent Heat We know that energy has to be supplied to a body to make it pass from the solid to the liquid phase. This heat energy is utilised in bringing about the change of state and does not show itself as a rise in temperature on a thermometer. It is, therefore, called latent (hidden) heat.

The *latent heat of fusion* of a substance is the quantity of heat required to change a unit mass of a solid substance from the solid state to the liquid state, the temperature remaining constant.

The *latent heat of vaporisation* of a substance is the quantity of heat required to change a unit mass of the substance from the liquid state to the gaseous state, the temperature remaining constant.

Latent heats are measured in cal g^{-1} or J kg^{-1} . If L is the latent heat for a particular 'phase change' of a substance, the quantity of heat needed to change the 'phase' of a mass m of that substance is given by

$$Q = mL$$

10.1B.9 Principle of Cooling When a body is placed in an enclosure so that it loses heat only by radiation from its surfaces, the cooling of the body takes place in accordance with the general law:

$$-\frac{\delta Q}{\delta t} = Kf(\theta)$$

where δQ = quantity of heat lost by the body in time δt , K is a constant depending on the nature and area of the exposed surface of the body and $f(\theta)$, a function of θ , the excess temperature of the body over the surroundings.

Irrespective of the form of the function f , the above relation signifies that the rate of cooling depends upon the excess temperature in a well defined way and this dependence remains the same under identical conditions. Thus, if equal volumes of water and some other liquid are allowed to cool under identical conditions (i.e. in identical calorimeters kept in identical surroundings) from the same initial temperature θ_1 to the same final temperature θ_2 , their rates of cooling must be the same. This is the underlying principle of the method of cooling used for determining the specific heats of liquids.

10.2B SI UNITS

Table 10.2

Quantity	SI unit
Heat	J (joule)
Thermal capacity	J K ⁻¹
Specific heat capacity	J kg ⁻¹ K ⁻¹
Water equivalent	kg
Latent heat of fusion or vaporisation	J kg ⁻¹

10.3B FORMULAE

1. $1 \text{ cal} = 4.1868 \text{ J} \simeq 4.2 \text{ J}$

2. $H = m \times s$

H = thermal capacity, m = mass, and s = specific heat capacity

3. $w = m \times \frac{s}{s_w}$

w = water equivalent, m = mass,

s = specific heat capacity of the body,

s_w = specific heat capacity of water.

4. $Q = m \times s \times t$

Q = heat required, m = mass, s = specific heat capacity

t = rise or fall in temperature.

5. For heat exchange between two bodies,

Heat gained by one body = Heat lost by the other.

6. $Q = m \times L$

Q = heat required for 'phase change' at constant temperature

m = mass, and L = latent heat of the substance for the phase change being brought about.

7. The general cooling law

$$-\frac{\delta Q}{\delta t} = Kf(\theta)$$

δQ = quantity of heat lost by a body in time δt , K = constant dependent on the nature and area of the exposed surface of the body and θ = excess temperature of the body over the surroundings.

8. Some useful constants are

Specific heat capacity of water = $4.2 \times 10^3 \text{ J kg}^{-1} \text{ K}^{-1}$

Latent heat of fusion of ice at 0°C = $336 \times 10^3 \text{ J kg}^{-1}$

Latent heat of vaporisation of steam at 100°C

$$= 22.5 \times 10^5 \text{ J kg}^{-1}$$

10.4B NUMERICAL EXAMPLES

Example 1: A copper calorimeter of mass 100 g contains a lump of ice at -4°C . When 520 calories of heat are given to the calorimeter and its contents, the temperature rises from -4°C to -2°C . The addition of another 41540 calories of heat brings the temperature of the calorimeter and its contents to $+2^\circ \text{C}$. Determine the specific heat capacity of copper and the mass of ice present in the calorimeter.

Given: Latent heat of fusion of ice = 80 cal g^{-1}

Specific heat capacity of ice = $0.5 \text{ cal g}^{-1} (^\circ \text{C})^{-1}$

Solution: Let s be the specific heat capacity of copper and m the mass of ice present in the calorimeter. We then have

$$100 \times s \times \{-2 - (-4)\} + m \times 0.5 \times \{-2 - (-4)\} = 520$$

$$\text{or } 200s + m = 520 \quad (\text{i})$$

$$\begin{aligned} \text{Also } 100 \times s \times \{2 - (-2)\} + m \times 0.5 \times \{0 - (-2)\} \\ + m \times 80 + m \times 1 \times (2 - 0) = 41540 \end{aligned}$$

$$\text{or } 400s + 83m = 41540 \quad (\text{ii})$$

Solving Eqs (i) and (ii) we get

$$m = 500 \text{ g}$$

$$\text{and } s = 0.1 \text{ cal g}^{-1} (^\circ \text{C})^{-1}$$

Example 2: A thermally insulated vessel contains some water at 0°C . The vessel is connected to a vacuum pump to pump out water vapour. It is observed that this results in some water getting frozen. Explain why this happens and work out the maximum percentage of water that will get solidified in this way. Given

Latent heat of vaporisation of water at 0°C = $21 \times 10^5 \text{ J kg}^{-1}$

Latent heat of freezing of water = $3.36 \times 10^5 \text{ J kg}^{-1}$.

Solution: As the system is 'thermally insulated', the heat required for vaporisation is extracted from water itself. The water in the vessel being at 0°C , this results in part of the water getting frozen.

Let x be the maximum fraction of water frozen in this way. Then $(1-x)$ is the fraction that gets vaporised. Equating the heat given out in the first process to the heat gained in the second, we have

$$x \times 3.36 \times 10^5 = (1 - x) \times 21 \times 10^5$$

which gives

$$x = 0.862$$

Thus a maximum of 86.2 % of water present in the vessel can be frozen in this manner.

Example 3: A copper calorimeter of mass 100 g contains 200 g of a mixture of ice and water. Steam at 100°C under normal pressure is passed into the calorimeter and the temperature of the mixture is allowed to rise to 50°C . If the mass of the calorimeter and its contents is now 330 g, what was the ratio of ice and water in the beginning? Neglect heat losses. Given

Specific heat capacity of copper $= 0.42 \times 10^3 \text{ J kg}^{-1} \text{ K}^{-1}$

Specific heat capacity of water $= 4.2 \times 10^3 \text{ J kg}^{-1} \text{ K}^{-1}$

Latent heat of fusion of ice $= 3.36 \times 10^5 \text{ J kg}^{-1}$

Latent heat of condensation of steam $= 22.5 \times 10^5 \text{ J kg}^{-1}$.

Solution: Heat is lost by steam in getting condensed and gained by water, ice and the calorimeter. Let the calorimeter originally contain x grams of ice and $(200 - x)$ grams of water. We then have

$$\begin{aligned} \text{Heat gained by calorimeter} &= \frac{100}{1000} \times 0.42 \times 10^3 \times (50 - 0) \\ &= 2100 \text{ J} \end{aligned}$$

$$\begin{aligned} \text{Heat gained by ice} &= \frac{x}{1000} \{3.36 \times 10^5 + 4.2 \times 10^3 (50 - 0)\} \\ &= x \{(336 + 210)\} = 546 x \text{ J} \end{aligned}$$

$$\begin{aligned} \text{Heat gained by water} &= \frac{(200 - x)}{1000} \times 4.2 \times 10^3 \times (50 - 0) \\ &= (42000 - 210 x) \text{ J} \end{aligned}$$

Heat lost by steam

$$= \frac{(330 - 200 - 100)}{1000} \times [22.5 \times 10^5 + 4.2 \times 10^3 \times (100 - 50)]$$

$$= 30 (2250 + 210) = 73800 \text{ J}$$

Heat gained = Heat lost

$$546 x + 2100 + 42000 - 210 x = 73800$$

which gives $x = 88.4 \text{ g}$

Mass of ice in the original mixture $= 88.4 \text{ g}$

Mass of water in the original mixture $= 200 - 88.4 = 111.6 \text{ g}$

Ratio of ice present to water present $= 88.4 : 111.6$
 $= 1 : 1.26$

Example 4: An aluminium container of mass 100 g contains 200 g of ice at -20°C . Heat is added to the system at the rate of 100 calories per second. What is the temperature of the system after four minutes?

Draw a rough sketch showing the variation of the temperature of the system as a function of time. Given:

$$\text{Specific heat of ice} = 0.5 \text{ cal g}^{-1} (^{\circ}\text{C})^{-1}$$

$$\text{Specific heat of aluminium} = 0.2 \text{ cal g}^{-1} (^{\circ}\text{C})^{-1}$$

$$\text{Latent heat of fusion of ice} = 80 \text{ cal g}^{-1}. \quad (\text{IIT 1973})$$

Solution: As heat is added to the system, the container and the ice in it will first be brought to 0°C ; next the ice will melt and, if this process is completed before four minutes, the water formed will be heated above 0°C to some temperature $t^{\circ}\text{C}$. Let us calculate the heat required in each step and thence the time needed for that step.

(i) Heat required to bring the container and ice from -20°C to 0°C

$$= 100 \times 0.2 \times \{0 - (-20)\} + 200 \times 0.5 \times \{0 - (-20)\}$$

$$= (400 + 2000) = 2400 \text{ cal}$$

$$\text{Time } t_1 \text{ needed for this} = \frac{2400}{100} = 24 \text{ s}$$

(ii) Heat required to melt the ice at 0°C into water at 0°C

$$= 200 \times 80 = 16000 \text{ cal}$$

$$\text{Time } t_2 \text{ needed for this} = \frac{16000}{100} = 160 \text{ s}$$

$$t_1 + t_2 = 24 + 160 = 184 \text{ s}$$

This is less than 4 minutes ($= 240 \text{ s}$). Hence the heat supplied during the remaining $(240 - 184) = 56 \text{ s}$ will be used up in raising the temperature of the container and 200 g of water in it to some final temperature t .

$$\text{Heat supplied in } 56 \text{ s} = 56 \times 100 = 5600 \text{ cal}$$

$$\text{Hence} \quad 5600 = 100 \times 0.2 \times (t - 0) + 200 \times 1 \times (t - 0)$$

$$\text{or} \quad 220 t = 5600$$

$$\text{or} \quad t = 25.45^{\circ}\text{C}$$

Thus after four minutes, the container and the ice initially in it will have acquired a temperature of 25.45°C .

The variation of temperature of the system as a function of time is as shown in Fig. 10.2.

Example 5: Equal volumes of water and alcohol, when put in similar calorimeters take 100 s and 74 s respectively to cool from 50°C to 40°C . Calculate the specific heat capacity of alcohol given that the thermal capacity of each calorimeter is numerically equal to the volume of either liquid. Take the relative density of alcohol as 0.8 and the specific heat capacity of water as $1 \text{ cal per gram per } ^{\circ}\text{C}$.

Solution: Let $V \text{ cm}^3$ be the volume of either liquid. Then the thermal capacity of each calorimeter is also $V \text{ cal per } ^\circ\text{C}$.

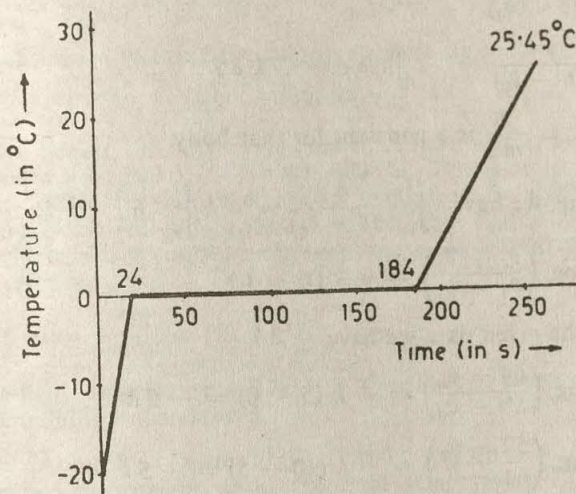


Fig. 10.2

Mass of water = $V \times 1 = V \text{ g}$

Mass of alcohol = $V \times 0.8 = 0.8 V \text{ g}$

The rate of cooling of the 'water calorimeter'

$$= \frac{1}{100} \{V \times (50 - 40) + V \times 1 \times (50 - 40)\}$$

$$= 5 V \text{ cal s}^{-1}$$

The rate of cooling of the 'alcohol calorimeter'

$$= \frac{1}{74} \{V \times (50 - 40) + 0.8 V \times s \times (50 - 40)\}$$

$$= \frac{1}{74} (10 V + 8 V s)$$

Because identical volumes of the liquids are getting cooled under identical conditions, the rate of cooling is the same in both the cases. Hence

$$5 V = \frac{1}{74} (10 V + 8 V s)$$

which gives

$$s = 0.6 \text{ cal g}^{-1} (^\circ\text{C})^{-1}$$

Example 6: A body initially at 80°C cools to 64°C in five minutes and to 52°C in 10 minutes. What will be its temperature after 15 minutes as also the temperature of surroundings?

Solution: Let θ_0 be the temperature of the surroundings. We assume that the cooling of the body takes place in accordance with the law of proportionality:

$$\frac{\delta Q}{\delta t} = K(\theta - \theta_0)$$

Since $\delta Q = m s \delta \theta$, we have

$$\frac{m s \delta \theta}{(\theta - \theta_0)} = -K \delta t$$

or
$$\frac{\delta \theta}{(\theta - \theta_0)} = -\frac{K}{m s} \delta t = -k \delta t$$

where $k = \frac{K}{m s}$ is a constant for that body.

$\therefore$ Integrating, we have
$$\int_{\theta_1}^{\theta_2} \frac{\delta \theta}{(\theta - \theta_0)} = -k \int_{t_1}^{t_2} \delta t$$

or
$$\log_e \left(\frac{\theta_2 - \theta_0}{\theta_1 - \theta_0} \right) = -k (t_2 - t_1)$$

Substituting the given data we have

$$\log_e \left(\frac{64 - \theta_0}{80 - \theta_0} \right) = -k (5 - 0) = -5k$$

and
$$\log_e \left(\frac{52 - \theta_0}{64 - \theta_0} \right) = -k (10 - 5) = -5k$$

Hence
$$\frac{64 - \theta_0}{80 - \theta_0} = \frac{52 - \theta_0}{64 - \theta_0}$$

or
$$(64 - \theta_0)^2 = (80 - \theta_0)(52 - \theta_0)$$

or
$$4096 - 128\theta_0 + \theta_0^2 = 4160 - 132\theta_0 + \theta_0^2$$

which gives $\theta_0 = 16^\circ \text{C}$

Let the temperature of the body be $\alpha^\circ \text{C}$ after 15 minutes. We then have

$$\log_e \left(\frac{\alpha - \theta_0}{52 - \theta_0} \right) = -k (15 - 10) = -5k$$

$\therefore$
$$\frac{\alpha - \theta_0}{52 - \theta_0} = \frac{64 - \theta_0}{80 - \theta_0} = \frac{64 - 16}{80 - 16} = \frac{3}{4}$$

or
$$\frac{\alpha - 16}{52 - 16} = \frac{3}{4}$$

or
$$4\alpha - 64 = 108$$

which gives $\alpha = 43^\circ \text{C}$

Thus the body will have cooled down to 43°C 15 minutes after the start of the cooling process.

Example 7: A mixture of 250 g of water and 200 g of ice is kept in a calorimeter of water equivalent 50 g. If 200 g of steam at 100°C is passed through this mixture, calculate the final temperature and the weight of the contents of the calorimeter. (IIT 1974)

Solution: Let us find out the heat required to raise the temperature of the calorimeter and its contents from 0°C to 100°C . The system is equivalent to (250 + 50) g or 300 g of water at 0°C and 200 g of ice at 0°C .

Heat required to raise the temperature of 300 g of water from

$$0^\circ \text{C to } 100^\circ \text{C} = 300 \times 1 \times (100 - 0) = 30,000 \text{ cal}$$

Heat required to (i) melt 200 g of ice at 0°C into 200 g of water at 0°C and (ii) raise the temperature of 200 g of water from 0°C to 100°C

$$= 200 \times 80 + 200 \times 1 \times (100 - 0) = 36,000 \text{ cal}$$

Total heat needed to bring the calorimeter and its contents to 100°C

$$= 30,000 + 36,000 = 66,000 \text{ cal}$$

If all the steam at 100°C were to condense into water at 100°C , the heat released will be $200 \times 540 = 108,000 \text{ cal}$. This is more than the heat required to raise the calorimeter and its contents from 0°C to 100°C . Hence all the steam will not condense. In fact, only that much mass (say m grams) of steam will condense that can release 66,000 cal of heat on condensation. Hence,

$$m = \frac{66000}{540} = 122.2 \text{ g}$$

Thus the final temperature will be 100°C with $(200 - 122.2) \text{ g} = 77.8 \text{ g}$ of steam remaining uncondensed.

Example 8: About 5 g of water at 30°C and 5 g of ice at -20°C are mixed together in a calorimeter. Find the final temperature of the mixture. Water equivalent of the calorimeter is negligible.

Specific heat of ice = $0.5 \text{ cal g}^{-1} (^\circ\text{C})^{-1}$

Latent heat of fusion of ice = 80 cal g^{-1} (IIT 1977)

Solution: Let us first calculate the heat required to (i) raise the temperature of ice from -20°C to 0°C and (ii) melt the ice at 0°C into water at 0°C .

$$\text{Heat required for step (i)} = 5 \times 0.5 \times \{0 - (-20)\} = 50 \text{ cal}$$

$$\text{Heat required for step (ii)} = 5 \times 80 = 400 \text{ cal}$$

When water at 30°C is allowed to cool to 0°C , the heat given out

$$= 5 \times 1 \times (30 - 0) = 150 \text{ cal}$$

Thus it is clear that all the ice cannot melt and the system will remain at 0°C . Since only $(150 - 50) = 100 \text{ cal}$ are available for melting ice, the mass of ice melted = $100/80 = 1.25 \text{ g}$.

Thus finally the mixture consists of $(5 + 1.25) = 6.25 \text{ g}$ of water at 0°C and $(5 - 1.25) = 3.75 \text{ g}$ of ice at 0°C .

Example 9: Earthen vessels are widely used for cooling water in summer. When water evaporates at the rate of $\sigma \text{ kg s}^{-1}$ from one such vessel (of water equivalent $w \text{ kg}$) containing $M \text{ kg}$ of water, the temperature falls from θ_1 to θ_2 in t seconds ($M \gg \sigma t$). Assuming that a fraction f of the heat needed for evaporation is taken from the vessel and the contained water, obtain a formula for the average rate of fall of temperature. Take L as the average latent heat of vaporisation and s as the specific heat of water.

Solution: The average rate of fall of temperature is $(\theta_1 - \theta_2)/t$. The amount of water evaporating in t seconds = $\sigma t \text{ kg}$. The heat needed for its evaporation over the temperature range θ_1 to θ_2 is $\sigma t L$, where L is the average latent heat of vaporisation in this temperature range.

Since a fraction f of this heat is taken from the vessel and the contained water, the heat lost by the two is $f \sigma t L$.

The mass of water in the vessel is M at the start and $(M - \sigma t)$ after t seconds. Since $M \gg \sigma t$, we may assume that the mass of water will remain constant at M . Therefore, heat lost by the vessel and water in cooling from θ_1 to θ_2 is

$$(M + w) \times s \times (\theta_1 - \theta_2)$$

We thus have

$$f \sigma t L = M s (\theta_1 - \theta_2)$$

$$\text{or} \quad \frac{\theta_1 - \theta_2}{t} = \frac{f \sigma L}{M s}$$

Thus the average rate of cooling is given by $\frac{f \sigma L}{M s}$.

Example 10 A vessel is completely filled with 500 g of water and 1000 g of mercury. When 21200 calories of heat are given to it, water of mass 3.52 g overflows. Calculate the coefficient of volume expansion of mercury. The expansion of the vessel may be neglected.

Coefficient of volume expansion of water = 1.5×10^{-4} per $^{\circ}\text{C}$

Specific heat of mercury = $0.03 \text{ cal g}^{-1} (^{\circ}\text{C})^{-1}$

Density of mercury = 13.6 g cm^{-3} (IIT 1976)

Solution: The volume of water that overflows must equal the total increase in the volumes of mercury and water as a result of their rise in temperature through $t^{\circ}\text{C}$ (say). To calculate t we notice that

$$21200 = 500 \times 1 \times t + 1000 \times 0.03 \times t$$

which gives

$$t = 40^{\circ}\text{C}$$

$$\text{Now initial volume of water} = \frac{500 \text{ g}}{1 \text{ g cm}^{-3}} = 500 \text{ cm}^3$$

$$\text{and initial volume of mercury} = \frac{1000 \text{ g}}{13.6 \text{ g cm}^{-3}} = 73.529 \text{ cm}^3$$

$$\text{Final volume of water} = 500 (1 + 1.5 \times 10^{-4} \times 40) = 503 \text{ cm}^3$$

$$\text{Final volume of mercury} = 73.529 (1 + \gamma \times 40) \text{ cm}^3$$

where γ is the coefficient of volume expansion of mercury

$\therefore$ Increase in volume of water and mercury

$$= 73.529 (1 + 40 \gamma) + 503 - 73.529 - 500$$

$$= (73.529 \times 40 \times \gamma + 3) \text{ cm}^3$$

$$= (2941.16 \gamma + 3) \text{ cm}^3$$

This must equal the volume of water that has overflowed. Now the density of water at the new temperature is given by

$$d_t = d (1 - 1.5 \times 10^{-4} \times 40)$$

$$= 1 (1 - 0.0060) = 0.994 \text{ g cm}^{-3}$$

Mass of water overflowing 3.52 g

$$\begin{aligned}\text{Volume of water overflowing} &= \frac{3.52}{0.994} \\ &= 3.54 \text{ cm}^3\end{aligned}$$

$$\text{Hence } 2941.16 \gamma + 3 = 3.54$$

$$\text{or } \gamma = 1.84 \times 10^{-4} (\text{°C})^{-1}$$

10.5B TRUE-FALSE STATEMENTS WITH REASONS

Given below are some statements. We have to decide whether they are true or false, giving, in brief, reasons to support our answers.

1. It is possible to just melt a given mass of ice completely by using an equal mass of water at 100 °C.
2. The specific heat capacity of a solid is different when the solid is heated at (i) constant volume and (ii) at constant pressure.
3. The water equivalent of a body is numerically equal to its thermal capacity irrespective of the units used.
4. The 'method of cooling' for determining the specific heats of liquids can be used only over the temperature range for which Newton's law of cooling is valid.
5. The heat required to melt 102.35 g of ice at 0 °C into water at 0 °C is roughly 1/7th that needed to convert the same mass of water at 100 °C into steam at 100 °C,

ANSWERS

1. *False* Since the latent heat of fusion of ice is 80 cal g⁻¹, the required temperature is only 80 °C as is given by the formula

$$m \times L = m \times 1 \times (t - 0)$$

With $L = 80 \text{ cal g}^{-1}$, t comes out to be 80 °C.

2. *True* When a solid is heated at constant pressure, it does expand a little and some heat is required for doing the mechanical work associated with this expansion. The difference between the two specific heats (at constant pressure and at constant volume) is small enough to be generally neglected for solids and liquids.
3. *False* The statement is true only for the CGS system in which the specific heat capacity of water is unity. It is not true for the SI system of units.
4. *False* The method of cooling is valid for any law of cooling of the form

$$-\frac{\delta Q}{\delta t} = K f(\theta)$$

irrespective of the form of the function f .

5. *True* The required heats are the same ratio as the latent heat of fusion of ice and the latent heat of vaporisation of water. The ratio is 80 : 540 or roughly 1 : 7.

10.6B MULTIPLE CHOICE QUESTIONS

Choose the correct answer from the given alternatives in each of the following

1. 1080 g of ice at 0 °C is mixed with 1080 g of water at 80 °C. The final temperature of the mixture will be

(a) 0 °C	(b) 40 °C
(c) 60 °C	(d) 80 °C

2. A source of heat supplies heat at a constant rate to a solid cube. The variation of the temperature of the cube with the heat supplied is shown in Fig. 10.3. The slope of the part ST of the graph represents

- the latent heat of the vapour
- the specific heat of the vapour
- the thermal capacity of the vapour
- the reciprocal of the thermal capacity of the vapour.

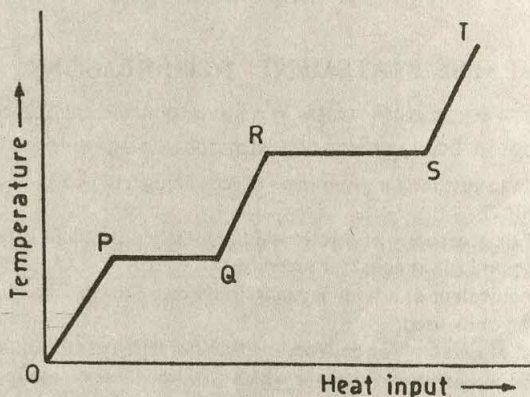


Fig. 10.3

3. In Fig. 10.3 of the above question, it is observed that $RS = 2.5 PQ$. It means that
- the thermal capacity of the liquid is 2.5 times that of the solid
 - the specific heat capacity of the liquid is 2.5 times that of the solid
 - the latent heat of vaporisation of the liquid is 2.5 times the latent heat of fusion of the solid
 - the latent heat of fusion of the solid is 2.5 times the latent heat of vaporisation of the liquid.
4. If there are no heat losses, the heat released by the condensation of x gram of steam at 100°C into water at 100°C can be used to convert y gram of ice at 0°C into water at 100°C . Then the ratio $y : x$ is nearly
- 1 : 1
 - 2 : 1
 - 3 : 1
 - 2.5 : 1
5. The rates of cooling of two different liquids put in exactly similar calorimeters and kept in identical surroundings are the same if
- the masses of the liquids are equal
 - equal masses of the liquids at the same temperature are taken
 - different volumes of the liquids at the same temperature are taken.
 - equal volumes of the liquids at the same temperature are taken.

10.7B NUMERICAL EXERCISES

1. A copper vessel of mass 1 kg is heated to 200°C . What is the maximum mass of a lump of ice that, when put into the vessel, will get completely converted into steam?

Specific heat capacity of copper = $0.1 \text{ cal g}^{-1} (^\circ\text{C})^{-1}$

Latent heat of fusion of ice = 80 cal g^{-1}

Latent heat of vaporisation of water = 540 cal g^{-1} .

2. When 500 g of mercury at 50°C is poured into a vacuum flask containing 90 g of water at 15°C , the temperature of the mixture is 19°C . When 90 g of water at 50°C is poured into 500 g of mercury at 15°C contained in the same flask, the temperature of the mixture is 38°C . Find the specific heat of mercury and the water equivalent of the flask.
3. In an industrial process, 10 kg of water per hour is to be heated from 20°C to 80°C . To this steam at 150°C is passed from a boiler into a copper coil immersed in water. The steam condenses in the coil and is returned to the boiler as water at 90°C . How many kg of steam are required per hour? (IIT 1972)
4. When a quantity of liquid bismuth at its melting point (271°C) is transferred to a calorimeter containing oil, the temperature of the oil rises from 12.5°C to 27.6°C . If the experiment is repeated with solid bismuth at the same temperature, everything else remaining the same, the temperature of the oil rises to 18.1°C . Calculate the latent heat of fusion of bismuth. The specific heat of bismuth is $0.032 \text{ cal g}^{-1} \text{ K}^{-1}$.
5. A copper calorimeter of mass 200 g contains 500 g of water at 10°C . When 15.5 g of 'moist' steam is passed into water, the temperature rises to 30°C . Neglecting heat losses, determine the percentage of water present in moist steam.
 Latent heat of vaporisation of water = 540 cal g^{-1}
 Specific heat of copper = $0.1 \text{ cal g}^{-1} (^{\circ}\text{C})^{-1}$
6. The temperature of equal masses of three different liquids *A*, *B* and *C* are 12°C , 19°C and 28°C respectively. The temperature when *A* and *B* are mixed is 16°C and when *B* and *C* are mixed it is 23°C . What should be the temperature when *A* and *C* are mixed? (IIT 1976)

SECTION C: TRANSMISSION OF HEAT

10.1C REVIEW OF BASIC CONCEPTS

10.1C.1 Modes of Heat Transfer The three main modes of heat transfer are (a) conduction, (b) convection and (c) radiation.

Conduction According to Maxwell, conduction is the flow of heat through an unequally heated body from places of higher temperature to those of lower temperature.

Convection Maxwell defines convection as the flow of heat by the motion of the hot body itself carrying its heat with it.

Radiation Radiation is the mode of heat transfer in which heat travels directly from one place to another without the agency of any intervening medium.

10.1C.2 Steady State When a uniform rod is kept with its one end in contact with a heat source, it attains a steady state after a while. In the steady state, the temperature of any element of the rod is purely a function of its distance from the hot end but is independent of time. In this state, the flow of heat along the rod does not depend on the density or the specific heat of its material but is governed by another characteristic of its material called its thermal conductivity.

10.1C.3 Thermal Conductivity The coefficient of thermal conductivity (k) of a material is defined as the rate of flow of heat through it in a steady

state across each unit area of it perpendicular to the direction of heat flow when there is a unit temperature gradient along the material.

For a bar of cross-sectional area A , length l and having temperatures θ_1 and θ_2 at its two ends, this definition tells us that the amount of heat Q flowing normally across the bar in time t is given by

$$Q = k A \frac{(\theta_1 - \theta_2)}{l} t$$

Here $(\theta_1 - \theta_2)/l$ is the uniform temperature gradient over the whole bar. Denoting the temperature gradient by $-d\theta/dx$, we also write this relation in the form

$$Q = -k A \frac{d\theta}{dx} t$$

This is the basic equation for flow of heat by conduction.

It is clear that the dimensions of k are $M L T^{-3} \theta^{-1}$ and its units are $\text{cal cm}^{-1} \text{s}^{-1} (^\circ\text{C})^{-1}$ or $\text{J s}^{-1} \text{m}^{-1} \text{K}^{-1}$ or $\text{Wm}^{-1} \text{K}^{-1}$ where K denotes kelvin.

10.1C.4 Thermal Resistance The thermal resistance of a body is a measure of its opposition to the flow of heat through it. It is defined as

$$\text{Thermal resistance} = \frac{\text{temperature difference at the two ends}}{\text{rate of flow of heat through it}}$$

It is easily seen that

$$\begin{aligned} \text{Thermal resistance} &= \frac{\text{length or thickness of the material}}{\text{thermal conductivity} \times \text{area}} \\ &= \frac{l}{k A} \end{aligned}$$

This relation is analogous to the corresponding relation for electrical resistance.

10.1C.5 The Case of a Compound Bar The case of a compound bar made up of two materials of thermal conductivities k_1 and k_2 is similar to the case of two resistances in series (see Fig. 10.4). For such a bar, we have, in the steady state

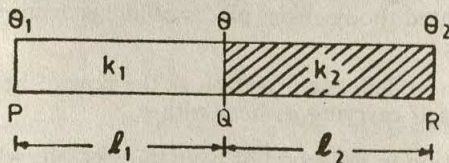


Fig. 10.4

Rate of flow of heat = heat current

$$\begin{aligned} &= \frac{(\theta_1 - \theta_2)}{\left(\frac{l_1}{k_1 A} + \frac{l_2}{k_2 A} \right)} \\ &= \frac{(\theta_1 - \theta_2)}{\Sigma \left(\frac{l}{k A} \right)} \\ &= \frac{\text{temperature difference at the ends}}{\text{total thermal resistance}} \end{aligned}$$

If the rod or bar is made of two materials as shown in Fig. 10.5, the analogy with the law of resistances in parallel holds. The total thermal resistance in this case is given by

$$\Sigma \left(\frac{A k}{l} \right)$$

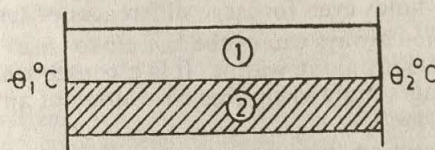


Fig. 10.5

10.1C.6 Increase of Thickness of Ice on a Pond Let the temperatures on the two sides of ice on the pond be θ_1 and θ_2 respectively and let L , ρ and k represent the latent heat, density and thermal conductivity of ice respectively. The time t in which the thickness of ice increases from an initial thickness x_1 to a final thickness x_2 is given by

$$\int_0^t dt = \frac{L \rho}{k (\theta_1 - \theta_2)} \int_{x_1}^{x_2} x dx$$

$$t = \frac{L \rho}{2k (\theta_1 - \theta_2)} (x_2^2 - x_1^2)$$

10.1C.7 Experimental Results on Conductivity

1. The conductivity of metals is higher than that of other substances.
2. Conductivity of liquids (except mercury) is small.
3. Conductivity of gases is even smaller than that of liquids; hydrogen and helium being the best conductors among gases.
4. The best conductors of heat are also the best conductors of electricity. This fact is also expressed mathematically as the Wiedemann-Franz-Lorentz law:

$$\frac{k}{\sigma T} = \frac{3}{J} \left(\frac{R}{\text{Ne}} \right) = \text{a constant}$$

Here k and σ stand for the thermal and electrical conductivities and T is the absolute temperature of the material.

10.1C.8 Two Types of Convection There are two types of convection: (i) free or natural convection and (ii) forced convection. Forced convection takes place when a steady stream of a fluid is blown as a draught, past a hot body.

For forced convection, Newton discovered a law of cooling according to which, in a steady stream of air, the rate of cooling of a body is directly proportional to the excess of its temperature over that of its surroundings'. Thus

$$-\frac{dQ}{dt} = -ms \frac{d\theta}{dt} \propto (\theta - \theta_0)$$

where θ_0 is the temperature of the surroundings. Hence

$$-\frac{d\theta}{dt} = k(\theta - \theta_0)$$

where k is a constant dependent on the mass and the specific heat of the body being cooled.

We must remember that Newton's law of cooling refers to the cooling mainly by convection. It is not a law of radiation and, in fact, if radiation losses are small, it holds even for large differences of temperature.

Natural convection always causes the hot air to rise upwards carrying the heat of the hot body along with it. It is a consequence of gravity and, in general, continues only when *free circulation* of air is permitted by the surrounding conditions.

For natural convection, we have an approximate Langmuir-Lorentz law according to which

$$-\frac{d\theta}{dt} \propto (\theta - \theta_0)^{5/4}$$

Thus the rate of loss of heat by convection is proportional to (i) the excess temperature $(\theta - \theta_0)$ for forced convection and $(\theta - \theta_0)^{5/4}$ for natural or free convection.

10.1C.9 The Lapse Rate The *lapse rate* in the atmosphere is the rate of fall of temperature per unit increase in altitude. It is given approximately by

$$\frac{dT}{dz} = -\frac{Mg(\gamma - 1)}{R\gamma}$$

where M = molecular weight of air, R = gas constant, γ = the ratio of the two specific heats of air and g = acceleration due to gravity. For dry air, the lapse rate is about 10°C per kilometre. When the air is saturated with water vapour, the lapse rate is about 1°C per 150 m.

10.1C.10 Properties of Thermal Radiation Thermal radiations are electromagnetic waves (just like visible light) except that they are not visible. They travel in free space at the speed of light. They obey all the laws of light such as rectilinear propagation, reflection, refraction, interference, diffraction and polarisation.

10.1C.11 Emissive Power The emissive power of a surface is defined as the rate of emission of thermal radiation per unit area of the surface at a given temperature compared with that of a black body under similar conditions.

10.1C.12 Absorptive Power The absorptive power of a surface is the rate of absorption of thermal radiation per unit area of the surface compared with that of a black body under similar conditions.

It may be noted that the absorptive power of a perfectly black body is unity as it absorbs all the radiation incident on it.

10.1C.13 Prevost's Theory of Heat Exchanges According to this theory, all bodies emit thermal radiation at all temperatures above absolute

zero, the amount of radiation emitted increasing with increase in temperature. Also there is a continual *exchange* of radiation between a body and its surroundings and the rise or fall in its temperature is purely a consequence of this exchange.

10.1C.14 Kirchhoff's Law The ratio of the emissive and absorptive powers for radiation of a given wavelength is the same for all substances at the same temperature and is equal to emissive power of a perfectly black body at the same temperature.

In simple qualitative terms, Kirchhoff's law implies that 'good emitters are good absorbers and poor emitters are poor absorbers'.

10.1C.15 Stefan Boltzmann Law The rate of loss of heat by radiation per unit area of the surface of a perfectly black body is directly proportional to the fourth power of its absolute temperature.

$$E = \sigma T^4$$

where σ , the Stefan's constant, is a universal constant being equal to $5.672 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}$ in SI units.

We often extend this law by saying that if a black body at a temperature T is surrounded by a constant temperature enclosure at temperature T_0 , the net rate of loss of heat per unit area of the former is given by

$$E_R = \sigma (T^4 - T_0^4)$$

For a body of emissivity e we have

$$E_R' = e \sigma (T^4 - T_0^4)$$

For small differences of temperature ($T \approx T_0$), we have

$$E_R \propto (T - T_0)$$

This is similar to what Newton's law of cooling states. However, we must not think of Newton's law as a special case of the Stefan Boltzmann law. This is because whereas the former is a law of cooling under *forced convection* conditions (with radiation playing only a minor role), the latter is a law of pure radiation and is basically a law of *emission* and not one of cooling.

10.1C.16 Wien's Displacement Law According to this law, for a black body the wavelength (λ_m) for which the emissive power (E_R) has its maximum value is inversely proportional to its absolute temperature (T), i.e.

$$\lambda_m T = \text{a constant}$$

The value of the constant is found to be $2.93 \times 10^{-3} \text{ m K}$.

10.2C SI UNITS

Table 10.3

Physical Quantity	SI Unit
Thermal conductivity	$\text{W s}^{-1} \text{ K}^{-1}$
Thermal resistance	$\text{J}^{-1} \text{ s K}$
Rate of heat loss	J s^{-1}
Rate of fall of temperature	K s^{-1}
Lapse rate	K m^{-1}
Emissive power	W m^{-2}
Stefan's constant	$\text{W m}^{-2} \text{ K}^{-4}$
Wien's displacement law constant	m K

10.3C FORMULAE

$$1. \quad Q = k A \frac{(\theta_1 - \theta_2)}{l} t$$

Q = amount of heat flowing normally across a bar in time t

A = cross-sectional area

l = length of bar

θ_1, θ_2 = temperatures at the two ends of the bar

k = thermal conductivity of the material of the bar

$$2. \text{ Thermal resistance} = \frac{l}{k A}$$

3. Thermal resistance of two bars

$$(i) \text{ in series} = \Sigma \left(\frac{l}{k A} \right)$$

$$(ii) \text{ in parallel} = \frac{1}{\Sigma \left(\frac{k A}{l} \right)}$$

4. Heat current for a compound bar = $\frac{\text{temperature difference at two ends}}{\text{total thermal resistance}}$

$$5. \quad t = \frac{L \rho}{2k (\theta_1 - \theta_2)} (x_2^2 - x_1^2)$$

L = latent heat of ice, ρ = density of ice, k = thermal conductivity of ice, t = time taken in increasing the thickness of the ice block from an initial thickness x_1 to a final thickness x_2 and θ_1, θ_2 = temperatures at the two ends of the block of ice.

$$6. \quad \frac{k}{\sigma T} = \text{a constant}$$

k = thermal conductivity σ = electrical conductivity

T = absolute temperature.

7. For forced convection (Newton's law of cooling)

$$-\frac{d\theta}{dt} = K(\theta - \theta_0)$$

$\frac{d\theta}{dt}$ = rate of fall of temperature of a body at a temperature θ

θ_0 = temperature of the surroundings

8. For natural convection

$$-\frac{d\theta}{dt} \propto (\theta - \theta_0)^{5/4}$$

This is known as the five-fourths power law or Langmuir-Lorentz law.

$$9. \quad \frac{dT}{dz} = \frac{Mg}{R} \frac{(\gamma - 1)}{\gamma}$$

$\frac{dT}{dz}$ = lapse rate (rate of fall of temperature per unit increase in altitude)

M = molecular weight of air, R = gas constant γ = ratio of the two specific heats of air

10. $E = \sigma T^4$

E = rate of loss of heat per unit area of the surface of a black body

σ = Stefan's constant T = absolute temperature

11. $E_R = \sigma (T^4 - T_0^4)$

E_R = net rate of loss of heat per unit area of the surface of a black body at absolute temperature T

T_0 = absolute temperature of the surroundings.

For small differences of temperature ($T \approx T_0$) we have

$$E_R \propto (T - T_0)$$

12. $\lambda_m T = \text{constant} = 2.93 \times 10^{-3} \text{ mK}$

λ_m = wavelength for which the emissive power (E_e) is maximum

T = absolute temperature of the black body.

10.4C NUMERICAL EXAMPLES

Example 1: A slab of stone of area 3600 sq. cm and thickness 10 cm is exposed on the lower surface to steam at 100°C . A block of ice at 0°C rests on the upper surface of the slab. In one hour 4800 g of ice is melted. Calculate the thermal conductivity of the stone. The latent heat of fusion of ice = 80 cal g^{-1} . (IIT 1972)

Solution: The quantity of heat flowing through the stone in one hour

$$= 4800 \times 80 = 384 \times 10^3 \text{ cal}$$

Now we know that

$$Q = k A \frac{\Delta T}{\Delta x} t$$

where k is the thermal conductivity of the stone, A its surface area and Δx its thickness and ΔT is the temperature difference across the two faces of the slab and t is the time for which heat flows.

Substituting the values we get

$$\begin{aligned} k &= \frac{Q}{t} \frac{\Delta x}{\Delta T} \frac{1}{A} \\ &= \frac{384 \times 10^3}{3600} \times \frac{10}{100} \times \frac{1}{3600} \\ &= 2.96 \times 10^{-3} \text{ cal cm}^{-1} \text{ s}^{-1} (^\circ\text{C})^{-1} \end{aligned}$$

Example 2: A bar of copper of length 75 cm and a bar of steel of length 125 cm are joined together end to end. Both are of circular cross-section with diameter 2 cm. The free ends of the copper and steel bars are maintained at 100°C and 0°C respectively. The surfaces of the bars are thermally insulated. What is the temperature of the copper-steel junction? What is the heat transmitted per unit time across the junction? Thermal conductivity of copper is $9.2 \times 10^{-2} \text{ k cal m}^{-1} (^\circ\text{C})^{-1} \text{ s}$ and that of steel is $1.1 \times 10^{-2} \text{ k cal m}^{-1} \text{ s}^{-1} (^\circ\text{C})^{-1}$. (IIT 1977)

Solution: The temperature T of the junction Q of the bars PQ and QR can be determined by remembering that, in the steady state, the heat flow-

ing per second through PQ must equal the heat flowing per second through QR (Fig. 10.6)

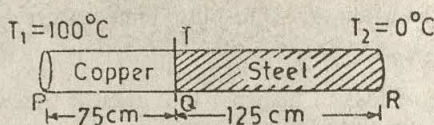


Fig. 10.6

Since the areas of cross-section of the two bars are the same, we have

$$\frac{k_1 A (T_1 - T)}{75} = \frac{k_2 A (T - T_2)}{125}$$

$$T \left(\frac{k_2}{125} + \frac{k_1}{75} \right) = \frac{k_1 T_1}{75} + \frac{k_2 T_2}{125}$$

Hence

$$\begin{aligned} T &= \frac{k_1 T_1 \times 125 + k_2 T_2 \times 75}{k_2 \times 75 + k_1 \times 125} \\ &= \frac{9.2 \times 10^{-2} \times 100 \times 125 + 1.1 \times 10^{-2} \times 0 \times 75}{1.1 \times 10^{-2} \times 75 + 9.2 \times 10^{-2} \times 125} \\ &= \frac{1150}{0.825 + 11.50} \\ &= 93.3^\circ\text{C} \end{aligned}$$

Thus the temperature of the junction is 93.3°C . The heat flowing through the junction per second is now given by

$$\begin{aligned} Q &= \frac{k_1 A (100 - 93.3)}{0.75} \\ &= \frac{9.2 \times 10^{-2} \times \pi \times (10^{-2})^2 \times 6.7}{0.75} \\ &= 258.2 \times 10^{-6} \text{ k cal s}^{-1} \\ &= 0.2582 \text{ cal s}^{-1} \end{aligned}$$

Example 3: An ice box is built of wood 1.75 cm thick, lined inside with cork 2 cm thick. If the temperature of the inner surface of the cork is 0°C and that of the outer surface of the wood is 12°C , what is the temperature of the interface? The thermal conductivity of wood is 0.0006 and that of cork is 0.00012 CGS units.

Solution: Let α be the temperature of the interface in the steady state. Since the same amount of heat must flow through wood and cork, we have

$$\frac{k_1 A (12 - \alpha)}{l_1} = \frac{k_2 A (\alpha - 0)}{l_2}$$

where l_1 and l_2 are the thicknesses of wood and cork respectively. Their respective conductivities are k_1 and k_2 (Fig. 10.7).

Substituting the values we get

$$\frac{6 \times 10^{-4}}{1.75} \times (12 - \alpha) = \frac{12 \times 10^{-5}}{2} \times \alpha$$

or $12 - x = 1.75 x \times 10^{-1}$

or $x = \frac{12}{1.175} = 10.2^\circ\text{C}$

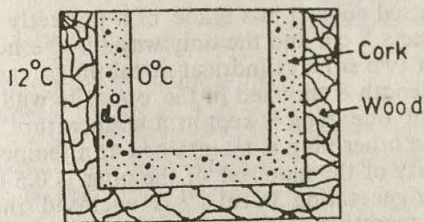


Fig. 10.7

Example 4: A uniform copper bar, 50 cm long, is lagged and has its ends exposed to ice and steam respectively. If there is a layer of water 0.1 mm thick at each end, calculate the temperature gradient in the bar. Thermal conductivities of copper and water are 1.04 and 0.0014 CGS units respectively. Also calculate the temperatures at the two ends of the bar.

Solution: Let θ_1 and θ_2 be the temperatures at the two ends of the bar. Since the same amount of heat must be flowing per second throughout the bar in the steady state, we have (Fig. 10.8)

$$\frac{k_1 A (100 - \theta_1)}{0.01} = \frac{k_2 A (\theta_1 - \theta_2)}{50} = \frac{k_1 A (\theta_2 - 0)}{0.01}$$

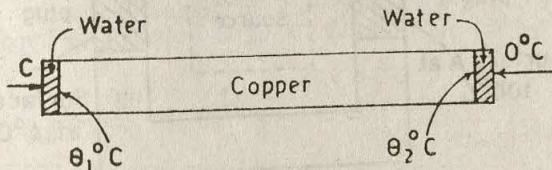


Fig. 10.8

Here $k_1 = 0.0014 \text{ cal cm}^{-1} \text{ s}^{-1} (^\circ\text{C})^{-1}$ = thermal conductivity of water
 $k_2 = 1.04 \text{ cal cm}^{-1} \text{ s}^{-1} (^\circ\text{C})^{-1}$ = thermal conductivity of copper

Hence we get

$$100 - \theta_1 = \theta_2$$

and
$$\frac{k_1 A (100 - \theta_1)}{0.01} = \frac{k_2 A (\theta_1 - \theta_2)}{50}$$

or
$$5000 \times \frac{k_1}{k_2} \times (100 - \theta_1) = (2\theta_1 - 100)$$

or
$$5000 \times \frac{0.0014}{1.4} \times (100 - \theta_1) = (2\theta_1 - 100)$$

which gives $\theta_1 = 88.54^\circ\text{C}$

and $\theta_2 = 100 - \theta_1 = 11.46^\circ\text{C}$

The temperature gradient along the bar is

$$\frac{\theta_1 - \theta_2}{l} = \frac{88.54 - 11.46}{50} = 1.542 \text{ } ^\circ\text{C per cm}$$

Example 5: A closed cubical box made of a perfectly insulating material has walls of thickness 8 cm and the only way for the heat to enter or leave the box is through two solid cylindrical metal plugs, each of cross-sectional area 12 cm^2 and length 8 cm fixed in the opposite walls of the box. The outer surface A of one plug is kept at a temperature of $100 \text{ } ^\circ\text{C}$ while the outer surface of the other plug is maintained at a temperature of $4 \text{ } ^\circ\text{C}$. The thermal conductivity of the material of the plug is $0.5 \text{ cal cm}^{-1} \text{ s}^{-1} (^\circ\text{C})^{-1}$. A source of energy generating 36 cal s^{-1} is enclosed inside the box. Find the equilibrium temperature of the inner surface of the box assuming that it is the same at all points on the inner surface. (IIT 1972)

Solution: At equilibrium, the total energy generated by the source per second must equal the heat leaving per second through the two metal plugs (Fig. 10.9). Let $T \text{ } ^\circ\text{C}$ be the equilibrium temperature. Then heat leaving the box per second through surface A

$$= \frac{k (T - 100) \times 12}{8} \text{ cal s}^{-1}$$

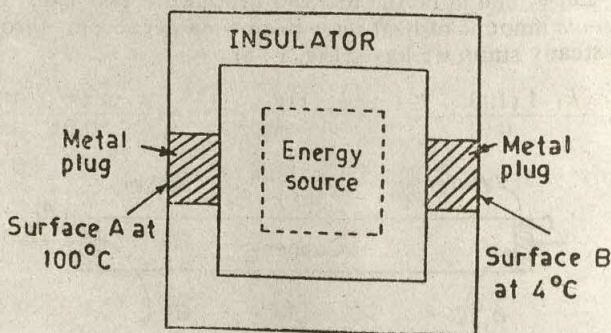


Fig. 10.9

and heat leaving the box per second through the surface B

$$= \frac{k (T - 4) \times 12}{8} \text{ cal s}^{-1}$$

Hence
$$\frac{12k}{8} (T - 100 + T - 4) = 36$$

or
$$2T - 104 = \frac{36 \times 8}{12k} = \frac{36 \times 8}{12 \times 0.5} = 48$$

or
$$T = 76 \text{ } ^\circ\text{C}$$

Example 6: Find the time in which a layer of ice 3 cm thick on the surface of a pond will increase in thickness by 1 mm when the temperature of the surroundings is $-20 \text{ } ^\circ\text{C}$. The thermal conductivity of ice is 0.005 CGS units , its latent heat is 80 cal g^{-1} and its density is 0.91 g cm^{-3} at $0 \text{ } ^\circ\text{C}$.

Solution: Let a layer of ice of surface area A increase in thickness from y to $y + dy$ in time dt . The amount of heat flowing across a thickness y of

ice from the water at the bottom (at θ_1 °C) to the surroundings (at θ_2 °C) in time dt is given by (Fig. 10.10)

$$\Delta Q = \frac{k A \theta dt}{y}$$

where $\theta = (\theta_1 - \theta_2) = 20$ °C in this case.

The mass of ice formed when the layer of ice increases in thickness by dy is $A dy \rho$, where ρ is the density of ice. Hence the heat released to the surroundings also equals $(A dy \rho) \times L$ where L is the latent heat of ice. Thus we have

$$\frac{k A \theta}{y} dt = A \rho L dy$$

$$\therefore y dy = \frac{k \theta}{\rho L} dt$$

Hence if the thickness increases from y_1 to y_2 in time t , we have

$$\int_{y_1}^{y_2} y dy = \frac{k \theta}{\rho L} \int_0^t dt$$

$$t = \frac{\rho L}{k \theta} \frac{(y_2^2 - y_1^2)}{2}$$

Substituting the numerical values of $L = 80$ cal g⁻¹, $\rho = 0.91$ g cm⁻³, $k = 0.005$ cal cm⁻¹ s⁻¹ (°C)⁻¹, $\theta = 20$ °C, $y_2 = 3.1$ cm and $y_1 = 3.0$ cm we get

$$t = \frac{0.91 \times 80}{0.005 \times 20} \times \frac{(3.1)^2 - (3.0)^2}{2} = 222.04 \text{ s} = 3 \text{ min } 42 \text{ s}$$

Example 7: A body cools from 60 °C to 50 °C in 10 minutes. Find its temperature at the end of next 10 minutes if the room temperature is 25 °C. Assume Newton's law of cooling.

Solution: According to Newton's law of cooling, we have

$$\frac{d\theta}{dt} = -K(\theta - \theta_0)$$

where θ is the temperature of the body and θ_0 that of the surroundings. Hence

$$\frac{d\theta}{\theta - \theta_0} = -K dt$$

$$\text{Integrating we have } \int_{\theta_1}^{\theta_2} \frac{d\theta}{\theta - \theta_0} = -K \int_0^t dt$$

$$\text{or } \log_e \left(\frac{\theta_2 - \theta_0}{\theta_1 - \theta_0} \right) = -Kt$$

$$\text{or } \log_e \left(\frac{60 - 25}{50 - 25} \right) = -K \times 10 \times 60$$

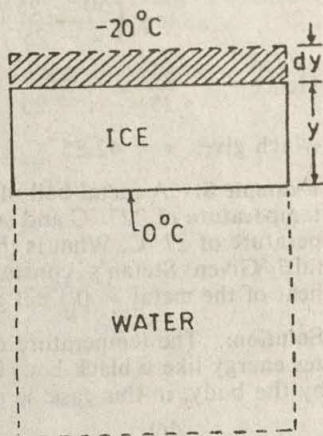


Fig. 10.10

If α is the temperature at the end of the next 10 minutes, we have

$$\log_e \left(\frac{50 - 25}{\alpha - 25} \right) = -k \times 10 \times 60$$

Hence
$$\frac{35}{25} = \frac{25}{\alpha - 25}$$

which gives $\alpha = 42.85^\circ\text{C}$

Example 8: A metal ball of 14 cm diameter and mass 10 kg is heated to a temperature of 227°C and suspended in a box whose walls are at a temperature of 27°C . What is the maximum rate at which its temperature will fall? Given Stefan's constant $= 5.7 \times 10^{-5} \text{ erg cm}^{-2} \text{ s}^{-1} (^\circ\text{C})^{-4}$, specific heat of the metal $= 0.1 \text{ cal g}^{-1} (^\circ\text{C})^{-1}$.

Solution: The temperature of the body will fall most rapidly when it radiates energy like a black body (emmissivity $= 1$). The net rate of loss of heat by the body, in this case, is given by Stefan's law:

$$\begin{aligned} \frac{\Delta Q}{\Delta t} &= \sigma A (T^4 - T_0^4) \\ &= \frac{5.7 \times 10^{-5}}{4.2 \times 10^7} \times \{4\pi(7)^2\} \times \{(500)^4 - (300)^4\} \\ &= 45.46 \text{ cal s}^{-1} \end{aligned}$$

If the temperature falls by $\Delta\theta$ in one second, we have

$$\begin{aligned} \frac{\Delta Q}{\Delta t} &= ms \frac{\Delta\theta}{\Delta t} \\ \text{Hence } \frac{\Delta\theta}{\Delta t} &= \frac{1}{ms} \frac{\Delta Q}{\Delta t} \\ &= \frac{1}{10 \times 10^3 \times 0.1} \times 45.46 \\ &= 0.04546^\circ\text{C s}^{-1} \end{aligned}$$

Example 9: A room heater is made up of 5 polished thin-walled tubes of copper, each 1 m long and 2.5 cm in radius and hot water at 70°C circulates constantly through the tubes. Calculate the amount of heat radiated in one hour in the room where the average temperature is 15°C . The emissivity of copper $= 4 \times 10^{-6} \text{ cal s}^{-1} \text{ cm}^{-2} \text{ K}^{-1}$.

Solution: The emissivity e gives the heat radiated per unit area per second per degree difference of temperature between the body and the surroundings. Hence the total heat radiated is given by

$$Q = e A t \Delta\theta$$

where A is the area of the radiating surface emitting Q units of heat in time t , $\Delta\theta$ being the difference of temperature between the body and the surroundings.

Here $A = 5 \times (2\pi \times 2.5 \times 100) \text{ cm}^2 = 7855 \text{ cm}^2$

$$t = 60 \times 60 = 3600 \text{ s}$$

$$\Delta\theta = (70 + 273) - (15 + 273) = 55 \text{ K}$$

$$e = 4 \times 10^{-6} \text{ cal s}^{-1} \text{ cm}^{-2} \text{ K}^{-1}$$

$$\therefore Q = 4 \times 10^{-6} \times 7855 \times 3600 \times 55 = 6221.2 \text{ cal}$$

Example 10: The radiant energy received from the sun on the earth's surface is 8.2 J cm^{-2} per minute. Assuming the radiation to be black body in character, calculate the surface temperature of the sun. Stefan's constant $= 5.67 \times 10^{-5} \text{ erg cm}^{-2} \text{ s}^{-1} \text{ K}^{-4}$ and angular diameter of the sun $= 0^\circ 32'$.

Solution: Let R be the radius of the sun and r its distance from the earth (Fig. 10.11). Then the angular diameter of the sun is

$$\theta = \frac{2R}{r} = 32' = \frac{32}{60} \times \frac{\pi}{180} \text{ radian}$$

$$R = \frac{r}{2} \times \frac{32}{60} \times \frac{\pi}{180} = 0.004665 r$$

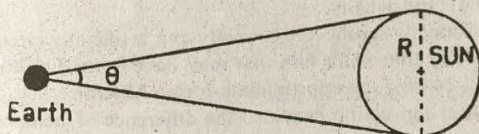


Fig. 10.11

Now the total surface area of the sun $= 4\pi R^2$. Assuming the sun to be a black body, the total energy radiated by it in one second is

$$\sigma T^4 \times 4\pi R^2$$

where T is the surface temperature of the sun. This energy is uniformly spread over an area $4\pi r^2$ at points as far distant from the sun as the earth. Hence the radiant energy received per unit area per second on the earth is

$$\begin{aligned} \frac{\sigma T^4 \times 4\pi R^2}{4\pi r^2} &= \sigma T^4 \left(\frac{R}{r}\right)^2 \\ &= \sigma T^4 (0.004665)^2 \\ &= 0.000022 \sigma T^4 \end{aligned}$$

$$\begin{aligned} \text{Hence } 0.000022 \sigma T^4 &= \frac{8.2}{60} \text{ J cm}^{-2} \text{ s}^{-1} \\ &= \frac{8.2}{60} \times 10^7 \text{ erg cm}^{-2} \text{ s}^{-1} \end{aligned}$$

Substituting the value of $\sigma = 5.67 \times 10^{-5} \text{ erg cm}^{-2} \text{ s}^{-1}$ in the above equation we get

$$T = 5770 \text{ K}$$

10.5C TRUE-FALSE STATEMENTS WITH REASONS

We give below a few statements. We have to decide whether they are true or false, giving in brief the reasons in support of our answers in each case.

1. It is sometimes possible for a block of wood and a block of metal to feel equally cold or hot when touched.
2. Two layers of cloth of equal thickness provide as warm a covering as a single layer of double the thickness.
3. The thermal conductivity of air being less than that of felt, we prefer felt to air for thermal insulation.

4. On a winter night, we feel warmer when clouds cover the sky than when the sky is clear.
5. Two thermometers are constructed in such a way that one has a spherical bulb and the other an elongated cylindrical bulb. The bulbs are made of the same material and their walls have equal thickness. The spherical bulb will respond more quickly to temperature changes.
6. A sphere, a cube and a thin circular plate of the same mass are made of the same material. If all of them are heated to the same high temperature, the rate of cooling is maximum for the plate and minimum for the sphere.
7. Before the steady state is reached, the rate of heat flow is proportional to the thermal diffusivity (or thermometric conductivity $= k/c\rho$); the ratio between the thermal conductivity k and thermal capacity per unit volume ($= c\rho$), where c is the specific heat and ρ the density.
8. Boilers are occasionally scrubbed by rapidly and artificially circulating water inside them to remove any thin water film that may have formed on their inside.
9. Newton's law of cooling, an approximate form of Stefan's law is valid both for forced and natural convection provided the difference of temperature between the body and its surroundings is smaller than about 30°C .
10. During clear nights, the temperature rises steadily upward near the ground level, i.e. the lapse rate there is negative.

ANSWERS

1. *True* An object appears hot (or cold) to us, if more (or less) heat flows from it into our body than the heat flowing out of our body to it. If, therefore, an object is at our body temperature, there will be no exchange of heat between it and our body and it will appear equally hot or cold no matter whether it is made of wood or metal.
2. *False* The two layers of cloth provide us with greater warmth as they trap a layer of air between them and air is a poor conductor of heat.
3. *False* The thermal conductivity of air is less than that of felt. However, we prefer felt which traps pockets of air between its fibres. These air pockets cannot carry heat by convection either as they are 'contained' by the fibres.
4. *True* The cloud cover does not let the heat radiated by the ground at night to escape easily into the space. This heat is 'trapped' between the ground and the cloud cover making us feel warmer. On a clear night, no such 'trapping' of heat takes place.
5. *False* Because of its greater surface area the cylindrical bulb will respond more quickly to temperature changes.
6. *True* Since the mass and material are the same, the volumes must also be the same. For the same volume, the surface area of the plate is the greatest and of the sphere the least. The rate of loss of heat by radiation being proportional to the surface area, the plate cools the fastest and the sphere the slowest.
7. *True* Before the steady state is reached, heat is not only 'transferred across the cross-section' but is also used up in heating the section of the bar through which it is flowing. The latter heat will be less if the thermal capacity of the bar per unit volume ($= c\rho$) is small. Thus the rate of heat flow will be proportional to k and inversely proportional to $c\rho$, i.e. the rate of heat flow will be proportional to $(k/c\rho)$.
8. *True* A water film, if formed, will act as a very poor conductor of heat and will not easily let the heat of the furnace pass into the boiler. An oil film, if present, is even worse than water film and the formation of such films inside the boiler must be avoided.

9. *False* Firstly, Newton's law of cooling, though similar in mathematical terms to an approximate form of Stefan's law, is not an approximate form of the latter. It is a law of cooling by convection, whereas the Stefan's law tells us about the heat given out by a body through radiation. Secondly Newton's law is valid only for forced convection and not for natural convection, for which the five-fourths law holds. For forced convection, Newton's law is valid for quite large temperature differences.
10. *True* On clear nights, the ground cools more quickly than the air and, therefore, we have conduction from air to ground. The layers of air near the ground are now cooler than the ones above and we can regard the temperature as rising upwards, near the ground level. The lapse rate, i.e. the rate of fall of temperature with altitude, is therefore negative.

10.6C MULTIPLE CHOICE QUESTIONS

Select the correct answer from the given alternatives in each of the following.

- A solid sphere and a hollow sphere of the same material and size are heated to the same temperature and allowed to cool in the same surroundings. If the temperature difference between the body and that of the surroundings is T , then
 - both spheres will cool at the same rate for all values of T
 - both spheres will cool at the same rate only for small values of T
 - the hollow sphere will cool at a faster rate for all values of T
 - the solid sphere will cool at a faster rate for all values of T .
- The top of a lake gets frozen at a place where the surrounding air is at a temperature of -20°C . Then
 - the temperature of the layer of water in contact with the lower surface of the ice block will be at 0°C and that at the bottom of the lake will be 4°C
 - the temperature of water below the lower surface of ice will be 4°C right up to the bottom of the lake
 - the temperature of the water below the lower surface of ice will be 0°C right up to the bottom of the lake
 - the temperature of the layer of water immediately in contact with the lower surface of ice will be about -20°C and that of water at the bottom will be 0°C .
- A block of metal is heated well above the room temperature and then left to cool in the room. Which of the graphs shown in Fig. 10.12 correctly represents the rate of cooling?

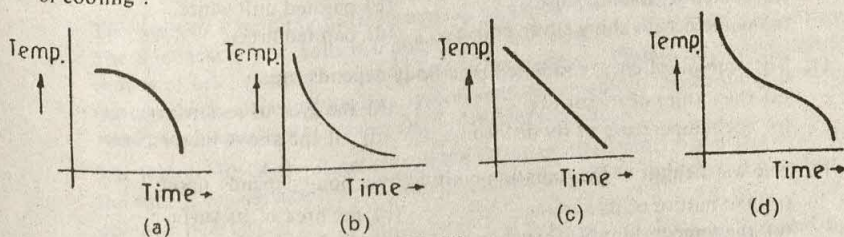


Fig. 10.12

- A piece of blue glass heated to a high temperature and a piece of red glass at room temperature are taken inside a dimly-lit room. Then
 - the blue piece will look blue and the red piece will look red as usual.
 - the red piece will look brighter red and the blue piece will look ordinary blue.
 - the blue piece will look brighter red compared to the red piece.
 - both the pieces will look equally red.

5. A 'fire screen' produces a sensation of cooling as it
- allows only the visible light to pass through and cuts out the longer wavelength infrared radiations from the fire.
 - cuts out both the visible as well as the infrared radiations from the fire.
 - allows the long wavelength infrared radiations to pass through but cuts out the shorter wavelength visible radiations from the fire.
 - allows both the infrared and the visible radiations to pass through but cuts out the very short wavelength ultraviolet radiations from the fire.
6. Two rods of the same length and areas of cross-section A_1 and A_2 have their ends at the same temperature. If k_1 and k_2 are their thermal conductivities, c_1 and c_2 their specific heats and ρ_1 , ρ_2 their densities, then the condition that the rate of flow of heat is the same in both rods is
- $\frac{A_1}{A_2} = - \frac{k_1}{k_2}$
 - $\frac{A_1}{A_2} = \frac{k_1 c_1 \rho_1}{k_2 c_2 \rho_2}$
 - $\frac{A_1}{A_2} = \frac{\frac{k_2}{c_2 \rho_2}}{\frac{k_1}{c_1 \rho_1}}$
 - $\frac{A_1}{A_2} = \frac{k_1}{k_2}$
7. Which one of the following statements is not true about thermal radiations?
- All bodies emit thermal radiations at all temperatures
 - Thermal radiations are electromagnetic waves
 - Thermal radiations are not reflected from a mirror
 - Thermal radiations travel in free space with a velocity of $3 \times 10^8 \text{ m s}^{-1}$.
8. Which of the following devices will you use to detect thermal radiations?
- Liquid thermometer
 - Six's maximum and minimum thermometer
 - Constant volume air thermometer
 - Thermopile
9. The absorptive power of a perfectly black body is
- zero
 - 0.5
 - 1
 - infinite
10. A Leslie's cube filled with hot water is slowly rotated about the vertical axis. A thermopile is placed in front of it. The galvanometer of the thermopile will show the greatest deflection when the face of the cube in front of it is
- coated with lampblack
 - painted dull white
 - painted with shiny silver polish
 - painted green
11. The amount of energy radiated by a body depends upon
- the nature of its surface
 - the area of its surface
 - the temperature of its surface
 - all the above factors.
12. The wavelength of the radiation emitted by a body depends upon
- the nature of its surface
 - the area of its surface
 - the temperature of its surface
 - all the above factors.
13. The SI unit of Stefan's constant is
- $\text{N m}^{-2} \text{ K}^{-4}$
 - $\text{J m}^{-2} \text{ K}^{-4}$
 - $\text{J m}^{-1} \text{ s}^{-1} \text{ K}^{-4}$
 - $\text{W m}^{-2} \text{ K}^{-4}$
14. Which one of the following devices is used for measuring very high temperatures?
- Mercury thermometer
 - Gas thermometer
 - Platinum-resistance thermometer
 - Pyrometer

15. When a coil of copper is kept at a certain distance above a flame, the candle keeps burning, but when the coil is placed over the flame, it is extinguished. This happens because
- the coil reduces the amount of oxygen necessary for burning
 - the coil prevents the setting up of convection currents in air
 - the coil reduces the radiation losses
 - the coil conducts away heat very quickly and reduces the temperature of the flame to a value below the ignition temperature.

10.7C NUMERICAL EXERCISES

- Water is being boiled in a flat-bottomed kettle placed on a stove. The area of the bottom is 300 cm^2 and the thickness is 2 mm . If the amount of steam produced is 1 g per minute, calculate the difference of temperature between the inner and outer surfaces of the bottom of the kettle. The thermal conductivity of the kettle material is $0.5 \text{ cal cm}^{-1} \text{ s}^{-1} (^\circ\text{C})^{-1}$ and the latent heat of steam is 540 cal g^{-1} . (IIT 1972)
- A bar of length 30 cm and of uniform cross-section of 5 cm^2 consists of two halves AB of copper and BC of iron, welded together at B . The end A is maintained at 200°C and the end C at 0°C and the sides are thermally insulated. Find the rate of flow of heat along the bar when the steady state has been reached. Thermal conductivities are: copper 0.9 and iron 0.12 in the CGS units.
- Two flat sheets of thicknesses d_1 and d_2 and thermal conductivities k_1 and k_2 are joined together. If the open face of the first sheet is maintained at a temperature θ_1 and the open face of the second sheet at a temperature θ_2 , calculate the (a) temperature of the interface of the two plates and (b) the conductivity of the composite plate.
- Show that the increase in the thickness of ice on a pond in a time t from an initial thickness x_0 to a final thickness x is given by

$$x^2 - x_0^2 = \frac{2k\theta}{L\rho} t$$

when $-\theta^\circ\text{C}$ is the air temperature, L the latent heat, ρ , the density and k the thermal conductivity of ice.

Hence calculate the time in which the thickness of the ice layer on a lake will get doubled from an initial thickness of 10 cm when the surrounding air is at -20°C . Given thermal conductivity of ice $= 0.004 \text{ cal cm}^{-1} \text{ s}^{-1} (^\circ\text{C})^{-1}$, latent heat of fusion of ice $= 80 \text{ cal g}^{-1}$ and the density of ice $= 0.92 \text{ g cm}^{-3}$.

- The walls of a building have an area A , of which the windows occupy an area a . The thickness of the walls is d and that of the windows is d' . The thermal conductivities of bricks and glass are k_b and k_g respectively. Obtain an expression for the wattage of an electrical heating system that will maintain the building interior at a steady temperature $\theta^\circ\text{C}$ above that of the surroundings.
- The luminosity of the Rigel star in the Orion constellation is 17000 times that of the sun. If the surface temperature of the sun is 6000 K , calculate the surface temperature of Rigel.

(Hint : The luminosity of a star depends on the total heat energy radiated by it.)

- Compare the rates of emission of heat by two black bodies maintained at temperatures 727°C and 227°C .
- Each square cm of the sun's surface radiates energy at the rate of $1.5 \times 10^3 \text{ cal s}^{-1} \text{ cm}^{-2}$. Calculate the temperature of the surface of the sun assuming that Stefan's law is applicable. The Stefan's constant is $5.7 \times 10^{-8} \text{ erg cm}^{-2} \text{ s}^{-1} \text{ K}^{-4}$.

ANSWERS

Section A

MULTIPLE CHOICE QUESTIONS

1. (b) 2. (d) 3. (c) 4. (a) 5. (d)

NUMERICAL EXERCISES

1. Yes, 0.0015 cm to spare
2. 37.83°C
3. $2.315 \times 10^{-5} (^{\circ}\text{C})^{-1}$
4. 28.33 cm for Fe, 18.33 cm for Cu
5. (a) 197°C , (b) $9.5 \times 10^4 \text{ N}$
6. (a) 1.422 N, (b) 1.452 N
7. 79°C
8. (a) $3.33 \times 10^{-6} (^{\circ}\text{C})^{-1}$, (b) $1.0 \times 10^{-5} (^{\circ}\text{C})^{-1}$, (c) $5.4 \times 10^{-4} (^{\circ}\text{C})^{-1}$
9. 21.7°C
10. 762.4 mm of Hg

Section B

MULTIPLE CHOICE QUESTIONS

1. (a) 2. (d) 3. (c) 4. (c) 5. (d)

NUMERICAL EXERCISES

1. 27.7 g
2. $0.031 \text{ cal g}^{-1} (^{\circ}\text{C})^{-1}$, 31.3 g
3. 1 kg
4. 14.04 cal g^{-1}
5. 11.3 %
6. 20.26°C

Section C

MULTIPLE CHOICE QUESTIONS

1. (c) 2. (a) 3. (b) 4. (c) 5. (a)
 6. (d) 7. (c) 8. (d) 9. (c) 10. (a)
 11. (d) 12. (c) 13. (d) 14. (d) 15. (d)

NUMERICAL EXERCISES

1. 0.012°C
2. 7.058 cal s^{-1}
3. $\frac{k_1 \theta_1}{d_1} + \frac{k_2 \theta_2}{d_2}, \frac{2 k_1 k_2}{k_1 + k_2}$
4. 38 hours 20 minutes
5. $\frac{\theta}{d d'} \{k_b d' A + (k_g d - k_b d') a\}$
6. 68520 K
7. 16 : 1
8. 5492 K

Kinetic Theory of Gases

11.1 REVIEW OF BASIC CONCEPTS

11.1.1 The Gaseous State Every substance (solid, liquid and gas) consists of a very large number of very tiny particles called molecules. In a gas, the molecules move very rapidly because the forces between them are very weak. They move everywhere, filling every container they are put into.

11.1.2 Postulates of Kinetic Theory The kinetic theory of gases is based on the following assumptions regarding the model of an ideal gas.

1. A gas consists of tiny particles called molecules.
2. The molecules of a gas are in random motion and obey Newton's laws of motion.
3. The total number of molecules is large.
4. The volume of the molecules is a negligibly small fraction of the volume occupied by the gas.
5. No appreciable force acts on the molecules during collision.
6. Collisions are elastic and of negligible duration.

11.1.3 Kinetic Theory and Gas Pressure The molecules of a gas move in all directions with various speeds. When a gas is contained in a vessel, the molecules strike against the walls of the container. These molecules collectively exert a sizeable force on the walls. The force experienced per unit area is the pressure of the gas. Thus, *the pressure of a gas is the result of continuous bombardment of the gas molecules against the walls of the container*. According to the kinetic theory, the pressure P exerted by an ideal gas is given by

$$P = \frac{1}{3} \rho \bar{c}^2 \quad (11.1)$$

where ρ is the density of the gas and $\bar{c}^2$ is the mean square speed of the gas molecules which is defined as

$$\bar{c}^2 = \frac{1}{n} (c_1^2 + c_2^2 + \dots + c_n^2) \quad (11.2)$$

where n is the number of molecules having speeds $c_1, c_2, \dots, c_n$.

11.1.4 Equation of State of an Ideal Gas The relationship between pressure P , volume V and absolute temperature T of a gas is called its equation of state. The equation of state of an ideal gas is

$$PV = nRT \quad (11.3)$$

where n is the number of moles of the enclosed gas and R is the *molar gas constant* which is the same for all gases and its value is

$$R = 8.315 \text{ J K}^{-1} \text{ mol}^{-1}$$

11.1.5 Ideal Gas Laws

1. $P \propto T$, when n and V are constant (Gay-Lussac's law)
2. $V \propto T$, when n and P are constant (Charles' law)
3. $P \propto \frac{1}{V}$, when n and T are constant (Boyle's law)

11.1.6 Kinetic Interpretation of Temperature The total average kinetic energy of all the molecules of a gas is proportional to its absolute temperature (T). Thus, the temperature of a gas is a measure of the average kinetic energy U of the molecules of the gas.

$$U = \frac{3}{2} R T \quad (11.4)$$

According to this interpretation of temperature, the average kinetic energy U is zero at $T = 0$, i.e. the motion of molecules ceases altogether at absolute zero.

11.1.7 Degrees of Freedom and Equipartition of Energy *The total number of co-ordinates or independent quantities required to completely specify the configuration of molecules of a system are called its degrees of freedom.*

1. A *monoatomic* gas, such as helium, argon, etc. consists of a single atom which has only three translational degrees of freedom along x , y and z axes. Thus a monoatomic gas has only *three* degrees of freedom.

2. A *diatomic* gas, such as hydrogen, oxygen, nitrogen, etc. consists of two atoms and has three translational + two rotational = *five* degrees of freedom.

3. A *triatomic (or higher polyatomic)* gas, has three translational + three rotational = *six* degrees of freedom.

Law of Equipartition of Energy *In any dynamical system, at a uniform absolute temperature T , the total energy is distributed equally among all the degrees of freedom and the average energy per degree of freedom per molecule equals $\frac{1}{2} k T$ where k is a constant called the Boltzmann's constant and its value is*

$$k = 1.38 \times 10^{-23} \text{ J K}^{-1} \text{ per molecule}$$

Thus the average energy per molecule for a monoatomic gas = $3 \times \frac{1}{2} k T = \frac{3}{2} k T$, for a diatomic gas = $\frac{5}{2} k T$ and for a polyatomic gas = $\frac{6}{2} k T = 3 k T$.

11.1.8 Specific Heat of a Gas The molar specific heat of a gas at constant volume (C_v) is the amount of heat energy required to raise the temperature of 1 mole of the gas through 1 K when its volume is kept constant.

The molar specific heat of a gas at constant pressure (C_p) is the amount of heat energy required to raise the temperature of 1 mole of the gas through 1 K when its pressure is kept constant.

Both C_v and C_p are expressed in $\text{J K}^{-1} \text{mol}^{-1}$. They are related as

$$C_p - C_v = R \quad (11.5)$$

11.1.9 Isothermal and Adiabatic Processes If a system is perfectly conducting to the surroundings and the temperature remains constant throughout the process, the process is called *isothermal*. During an isothermal process, the temperature of the system remains constant, but it can absorb heat from or give heat to the surroundings. In an isothermal process, the pressure and volume changes of an ideal gas obey the relation

$$P V = \text{constant} \quad (11.6)$$

In an *adiabatic* process, the system is completely insulated from the surroundings. It can neither give heat to nor take heat from the surroundings. In an adiabatic process, the pressure and volume changes of an ideal gas obey the relation

$$P V^\gamma = \text{constant} \quad (11.7)$$

where $\gamma = \frac{C_p}{C_v}$

11.2 SI UNITS

Table 11.1

Physical Quantity	SI Unit	Symbol
Heat energy	joule	J
Temperature	kelvin	K
Amount of gas	mole	mol
Pressure	newton/(metre) ²	N m^{-2}
Volume	(metre) ³	m^3
Root mean square velocity	metre/second	m s^{-1}
Molar specific heat	joule per kelvin per mole	$\text{J K}^{-1} \text{mol}^{-1}$
Molar gas constant	joule per kelvin per mole	$\text{J K}^{-1} \text{mol}^{-1}$
Boltzmann's constant	joule per kelvin per molecule	$\text{J K}^{-1} (\text{molecule})^{-1}$

11.3 FORMULAE

1. Pressure P exerted by an enclosed gas

$$P = \frac{1}{3} \rho \bar{c^2} = \frac{1}{3} \frac{N m}{V} \bar{c^2}$$

ρ = density, $\bar{c^2}$ = mean square molecular velocity, V = volume, N = total number of molecules of enclosed gas, m = mass of each molecule.

2. Root-mean square velocity

$$c_{\text{rms}} = \sqrt{\bar{c^2}} = \sqrt{\frac{3P}{\rho}}$$

3. Average translational kinetic energy of a molecule of the gas

$$\bar{E} = \frac{1}{2} m \bar{c^2} = \frac{1}{2} m c_{\text{rms}}^2$$

Total energy of all the N molecules of a gas

$$U = N \bar{E} = \frac{1}{2} m N \bar{c^2} = \frac{3}{2} R T$$

U is called the internal energy of the gas.

$$4. \quad P = \frac{2}{3} \frac{U}{V}$$

5. Equation of state of an ideal gas consisting of n moles at absolute temperature T

$$P V = n R T$$

6. Average kinetic energy of a molecule per degree of freedom
 $= \frac{1}{2} k T$

7. Molecular kinetic energy per gram mole of a gas $= \frac{3}{2} R T$.

8. Molecular kinetic energy per unit volume of a gas $= \frac{3}{2} P$.

$$9. \quad C_p - C_v = R$$

10. Pressure-volume relationship for an isothermal process

$$P V = \text{constant}$$

11. Pressure-volume relationship for an adiabatic process

$$P V^\gamma = \text{constant}; \quad \gamma = \frac{C_p}{C_v}$$

$$\text{or} \quad T V^{(\gamma-1)} = \text{constant}$$

$$\text{or} \quad T^\gamma P^{(1-\gamma)} = \text{constant}$$

12. Work done by a gas during expansion from volume V_1 to volume V_2

$$W = \int_{V_1}^{V_2} P dV$$

For an isothermal process

$$\begin{aligned} W &= n R T \log_e \left(\frac{V_2}{V_1} \right) = 2.303 n R T \log_{10} \left(\frac{V_2}{V_1} \right) \\ &= n R T \log_e \left(\frac{P_1}{P_2} \right) = 2.303 n R T \log_{10} \left(\frac{P_1}{P_2} \right) \end{aligned}$$

For an adiabatic process

$$W = \frac{1}{(\gamma - 1)} (P_1 V_1 - P_2 V_2) = \frac{n R}{(\gamma - 1)} (T_1 - T_2)$$

By convention, the work done by the gas is taken to be positive and the work done on the gas is taken to be negative.

11.4 NUMERICAL EXAMPLES

Example 1: Assuming hydrogen to be an ideal gas, calculate the root-mean square speed of the gas molecules at 0.0°C and a pressure of 1 atm. The density of hydrogen at $0.0^\circ\text{C} = 9.0 \times 10^{-2} \text{ kg m}^{-3}$.

Solution: $P = 1 \text{ atm} = 0.76 \text{ m of Hg}$
 $= 0.76 \times 13600 \times 9.8 = 1.01 \times 10^5 \text{ N m}^{-2}$
 $\rho = 9.0 \times 10^{-2} \text{ kg m}^{-3}$

The root-mean square speed is given by

$$C_{\text{rms}} = \sqrt{\frac{3P}{\rho}} = \sqrt{\frac{3 \times 1.01 \times 10^5}{9.0 \times 10^{-2}}}$$

$$= 1840 \text{ m s}^{-1}$$

Example 2: At what temperature would the root-mean square speed of a gas molecule have twice its value at 100°C ?

Solution: We know that

$$\bar{c}^2 = 3 \frac{R}{N} \frac{T}{m} = 3 \frac{k}{m} T$$

Thus $c_1^2 = \frac{3kT_1}{m}$

and $c_2^2 = \frac{3kT_2}{m}$

$$\therefore \frac{c_1^2}{c_2^2} = \frac{T_1}{T_2}$$

Here $c_2 = 2c_1$, $T_1 = 273 + 100 = 373 \text{ K}$

$$T_2 = T_1 \times \frac{c_2^2}{c_1^2} = 373 \times 4$$

$$= 1492 \text{ K} = 1219^\circ\text{C}$$

Example 3: Calculate the average kinetic energy of a gas molecule at a temperature of 300 K . The Boltzmann constant $k = 1.38 \times 10^{-23} \text{ J K}^{-1}$.

Solution: $T = 300 \text{ K}$

$$k = 1.38 \times 10^{-23} \text{ J K}^{-1}$$

The average kinetic energy of a gas molecule is given by

$$E = \frac{3}{2} k T$$

$$= \frac{3}{2} \times 1.38 \times 10^{-23} \times 300$$

$$= 6.21 \times 10^{-21} \text{ J}$$

Example 4: Calculate the root-mean square speed of oxygen molecules at 1092 K . Density of oxygen at STP $= 1.424 \text{ kg m}^{-3}$.

Solution: We first calculate the root-mean square speed of oxygen at STP.

$$P_0 = 0.76 \text{ m of Hg} = 1.01 \times 10^5 \text{ N m}^{-2}$$

$$\rho_0 = 1.424 \text{ kg m}^{-3}$$

The root-mean square speed at 0°C is given by

$$\begin{aligned} c_0 &= \sqrt{\frac{3 P_0}{\rho_0}} \\ &= \sqrt{\frac{3 \times 1.01 \times 10^5}{1.424}} \\ &= 4.61 \times 10^2 \text{ m s}^{-1} \end{aligned}$$

Now c_{rms} is also given by

$$c_{\text{rms}} = \sqrt{\frac{3 k T}{m}}$$

$$\therefore \frac{c_{\text{rms}}}{c_0} = \sqrt{\frac{T}{T_0}}$$

Here

$$T_0 = 273 \text{ K} \quad \text{and} \quad T = 1092 \text{ K}$$

$$\begin{aligned} c_{\text{rms}} &= c_0 \sqrt{\frac{T}{T_0}} = 4.61 \times 10^2 \times \sqrt{\frac{1092}{273}} \\ &= 9.22 \times 10^2 \text{ m s}^{-1} \end{aligned}$$

Example 5: Argon is a monoatomic gas. Assuming that the gas is an ideal one, calculate C_p and C_v for argon. Given $R = 8.32 \text{ J K}^{-1} \text{ mol}^{-1}$.

Solution: The kinetic energy of 1 mole of a gas is given by

$$U = \frac{3}{2} R T$$

The heat energy is completely used up in increasing the kinetic energy of the molecules of the gas if it is not allowed to expand. Thus, if the volume is kept constant, the gain in kinetic energy for 1 K rise in temperature is the same as the amount of heat energy gained by the gas for 1 K rise in temperature. Thus, from the definition of C_v , we have

$$C_v = \frac{3}{2} R (T + 1) - \frac{3}{2} R T = \frac{3}{2} R$$

Now $R = 8.32 \text{ J K}^{-1} \text{ mol}^{-1}$,

$$\therefore C_v = \frac{3}{2} \times 8.32 = 12.48 \text{ J K}^{-1} \text{ mol}^{-1}$$

Since $C_p = C_v + R$, we have

$$\begin{aligned} C_p &= \frac{3}{2} R + R = \frac{5}{2} R \\ &= \frac{5}{2} \times 8.32 = 20.80 \text{ J K}^{-1} \text{ mol}^{-1} \end{aligned}$$

Example 6: One mole of an ideal gas is heated from 0°C to 100°C at a constant pressure of 1 atmosphere. Calculate the work done in the process.

Solution: One mole of a gas occupies a volume of 22400 cm^3 at 0°C and at 1 atmosphere pressure. Thus the initial volume of the gas is $V_1 = 22400 \text{ cm}^3 = 0.0224 \text{ m}^3$. The final volume V_2 can be calculated from the expression

$$\frac{P_1 V_1}{T_1} = \frac{P_2 V_2}{T_2}$$

Here $T_1 = 0^\circ\text{C} = 273\text{ K}$, $T_2 = 100^\circ\text{C} = 373\text{ K}$

$$P_1 = P_2 = 1 \text{ atmosphere} = 1.013 \times 10^5 \text{ N m}^{-2}$$

$$V_2 = V_1 \left(\frac{T_2}{T_1} \right) = 0.0224 \times \frac{373}{273}$$

$$= 0.0306 \text{ m}^3$$

$$\text{Work done } W = \int_{V_1}^{V_2} P dV = P \int_{V_1}^{V_2} dV$$

$$= P (V_2 - V_1)$$

$$= 1.013 \times 10^5 \times (0.0306 - 0.0224)$$

$$= 830.7 \text{ J}$$

The positive sign indicates that the work is done by the gas.

Example 7: Three moles of an ideal gas are compressed to half the initial volume at a constant temperature of 300 K. Compute the work done in the process.

Solution: Since the temperature remains constant, the process is isothermal. The work done in the process is given by

$$W = 2.303 n RT \log \left(\frac{V_2}{V_1} \right)$$

Here $n = 3$, $R = 8.315 \text{ J K}^{-1} \text{ mol}^{-1}$,

$$T = 300 \text{ K} \quad \text{and} \quad \frac{V_2}{V_1} = 1/2$$

$$W = 2.303 \times 3 \times 8.315 \times 300 \times \log (1/2)$$

$$= -5188 \text{ J}$$

The negative sign indicates that work is done on the gas.

Example 8: One mole of air at 27°C is compressed adiabatically to half its volume. Calculate (a) the final temperature and (b) the work done in the process. Given γ for air = 1.4.

Solution: (a) In an adiabatic process the relation between pressure and volume is

$$PV^\gamma = \text{constant} \quad (i)$$

The relation between volume and temperature can be obtained by using the gas equation: ($\because n = 1$)

$$PV = RT \quad \text{or} \quad P = \frac{RT}{V} \quad (ii)$$

Using Eqs (i) and (ii) we have

$$TV^{(\gamma-1)} = \text{constant}$$

Thus $T_1 V_1^{(\gamma-1)} = T_2 V_2^{(\gamma-1)}$

or $T_2 = T_1 \left(\frac{V_1}{V_2} \right)^{(\gamma-1)}$

Here $T_1 = 27^\circ\text{C} = 300\text{ K}$, $\frac{V_1}{V_2} = 2$ and $\gamma = 1.4$

$$T_2 = 300 \times (2)^{(1.4-1)}$$

$$= 300 (2)^{0.4}$$

$$= 395.9\text{ K} = 122.9^\circ\text{C}$$

(b) The work done is given by

$$W = \frac{R}{(\gamma - 1)} (T_1 - T_2)$$

$$= \frac{8.315}{(1.4 - 1)} \times (300 - 395.9)$$

$$= -1994\text{ J}$$

The negative sign indicates that the work is done on the gas.

Example 9: An electric bulb of volume 250 cm^3 was sealed off during manufacture at a pressure of 10^{-3} mm of Hg at 27°C . Find the number of air molecules in the bulb. (IIT 1974)

Solution: Let N be the number of air molecules in the bulb. It is given that

$$P = 10^{-3}\text{ mm of Hg} = 10^{-4}\text{ cm of Hg}$$

$$V = 250\text{ cm}^3$$

$$T = 273 + 27 = 300\text{ K}$$

Now $PV = NkT$, where k is a constant. (i)

We know that at STP, one mole of a gas occupies a volume $V_0 = 22400\text{ cm}^3$ and contains $N_0 = 6.02 \times 10^{23}$ molecules (N_0 is the Avogadro's number) and the pressure $P_0 = 76\text{ cm of Hg}$ and $T_0 = 273\text{ K}$. Also

$$P_0 V_0 = N_0 k T_0 \quad \text{(ii)}$$

Dividing Eqs (i) and (ii) we get

$$N = N_0 \times \frac{T_0}{T} \times \frac{P}{P_0} \times \frac{V}{V_0}$$

$$= (6.02 \times 10^{23}) \times \left(\frac{273}{300}\right) \times \left(\frac{10^{-4}}{76}\right) \times \left(\frac{250}{22400}\right)$$

$$= 8.045 \times 10^{15}\text{ molecules}$$

Example 10: A cyclic process $ABCA$ shown in the $V - T$ diagram (see Fig. 11.1) is performed with a constant mass of an ideal gas. Show the

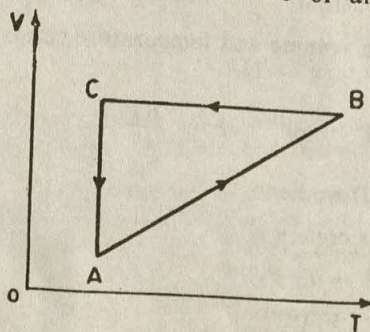


Fig. 11.1

same process on a $P - V$ diagram. (In the figure, CA is parallel to the V -axis and BC is parallel to the T -axis.) (IIT 1981)

Solution: To draw the corresponding P - V diagram, we use the ideal gas equation

$$PV = RT$$

Since the temperature T remains constant along the path CA , P will be inversely proportional to V along this path. Hence, as P increases, V must decrease in a non-linear fashion. This is represented by the curve $C'A'$ in Fig. 11.2.

Along the path BC , the volume V is constant. Hence the graph of P against V is a straight line perpendicular to the V -axis. On a P - V diagram, the corresponding path is $B'C'$ as shown in Fig. 11.2.

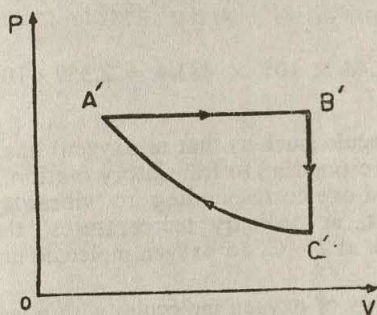


Fig. 11.2

For the path AB , V is directly proportional to T : pressure remaining constant. The corresponding path $A'B'$ is, therefore a straight line parallel to the V -axis. Thus the cyclic process on a P - V diagram is represented by the curve $A'B'C'A'$ as shown in Fig. 11.2.

Example 11: One gram mole of oxygen at 27°C and one atmospheric pressure is enclosed in a vessel. (i) Assuming the molecules to be moving with v_{rms} , find the number of collisions per second which the molecules make with one square-metre area of the vessel wall; (ii) the vessel is next thermally insulated and moved with a constant speed v_0 . It is then suddenly stopped. The process results in a rise of temperature of the gas by 1°C . Calculate the speed v_0 . (Boltzmann constant $= 1.38 \times 10^{-23} \text{ J K}^{-1}$ per molecule, Avogadro's number $= 6.02 \times 10^{23}$ molecules per mole, 1 atmosphere $= 1.01 \times 10^5 \text{ N m}^{-2}$) (IIT 1983)

Solution: (i) Let n be the number of molecules of oxygen per unit volume at pressure P and absolute temperature T . Then

$$n = \frac{P}{kT}$$

where $P = 1 \text{ atmosphere} = 1.01 \times 10^5 \text{ N m}^{-2}$, $T = 273 + 27 = 300 \text{ K}$ and k (Boltzmann constant) $= 1.38 \times 10^{-23} \text{ J K}^{-1}$ per molecule.

$$n = \frac{1.01 \times 10^5}{(1.38 \times 10^{-23}) \times 300} = 2.44 \times 10^{25} \text{ molecules per m}^3$$

Now

$$v_{\text{rms}} = \sqrt{\frac{3P}{\rho}} = \sqrt{\frac{3RT}{M}} = \sqrt{\frac{3NkT}{M}}$$

where N (Avagadro's number) = 6.02×10^{23} molecules per mole,

M = molecular weight of oxygen = $32 \text{ g} = 32 \times 10^{-3} \text{ kg}$.

$$\therefore v_{\text{rms}} \times \sqrt{\frac{3 \times (6.02 \times 10^{23}) \times (1.38 \times 10^{-23}) \times 300}{32 \times 10^{-3}}} \\ = 483.4 \text{ m s}^{-1}$$

The number of collisions which the molecules make per second with one square metre area of the vessel wall

$$= \frac{(\text{number of molecules per m}^3) \times (\text{r.m.s. velocity in m s}^{-1})}{(\text{number of degrees of freedom})}$$

$$= \frac{n \times v_{\text{rms}}}{5}$$

$$= \frac{1}{5} \times 2.44 \times 10^{25} \times 483.4 = 2.359 \times 10^{27}$$

Note: A diatomic molecule (such as that of oxygen) has six degrees of freedom in all—three corresponding to translatory motion, two corresponding to rotatory motion and one corresponding to vibratory motion. Experiments have shown that, at ordinary temperatures, the vibratory motion does not occur. Hence, at 27°C , an oxygen molecule has just five degrees of freedom.

(ii) The kinetic energy of oxygen molecules with a velocity $v_0 = \frac{1}{2} M v_0^2$, where M = molecular weight of oxygen.

Now heat energy = $C_v dT = C_v \times 1 = C_v$

But $C_p - C_v = R$ or $\frac{C_p}{C_v} - 1 = \frac{R}{C_v}$ or $(\gamma - 1) = \frac{R}{C_v}$

or $C_v = \frac{R}{(\gamma - 1)}$. Therefore,

$$\frac{1}{2} M v_0^2 = \frac{R}{(\gamma - 1)}$$

or $v_0 = \sqrt{\frac{2R}{M(\gamma - 1)}} = \sqrt{\frac{2 \times 8.315}{(32 \times 10^{-3}) \times (1.4 - 1)}} \text{ (for air, } \gamma = 1.4)$
 $= 35.6 \text{ m s}^{-1}$

Example 12: A motor car tyre has a pressure of 2 atms at 27°C . If the tyre suddenly bursts, find the resulting temperature.

Solution: The process is adiabatic. The final temperature is obtained from the relation

$$\frac{P_1^{(\gamma-1)}}{T_1^\gamma} = \frac{P_2^{(\gamma-1)}}{T_2^\gamma}$$

where Initial pressure $P_1 = 2 \text{ atms}$

Initial temperature $T_1 = 300 \text{ K}$

Final pressure $P_2 = 1 \text{ atm}$.

γ for air = 1.4

$$\text{Now} \quad \left(\frac{P_1}{P_2}\right)^{(\gamma-1)} = \left(\frac{T_2}{T_1}\right)^\gamma$$

$$\text{or} \quad (1/2)^{0.4} = \left(\frac{T_2}{300}\right)^{1.4}$$

$$\text{or} \quad 0.4 \log (1/2) = 1.4 (\log T_2 - \log 300)$$

$$\text{or} \quad -0.1204 = 1.4 \log T_2 - 2.4680$$

$$\text{or} \quad \log T_2 = 2.3911$$

$$T_2 = 246.1 \text{ K} = -26.9^\circ \text{C}$$

Example 13: An enclosure of volume four litres contains a mixture of 8 g of oxygen, 14 g of nitrogen and 22 g of carbon dioxide. If the temperature of the mixture is 27°C , find the pressure of the mixture of gases. Given $R = 8.315 \text{ J K}^{-1} \text{ mol}^{-1}$.

Solution: Temperature $T = 300 \text{ K}$

$$\text{Volume } V = 4 \text{ litre} = 4 \times 10^{-3} \text{ m}^3$$

The pressure exerted by a gas is given by

$$P = \frac{nRT}{V} = \frac{\text{mass}}{\text{molecular weight}} \times \frac{RT}{V}$$

$$\text{Pressure exerted by oxygen } P_1 = \frac{8}{32} \frac{RT}{V} = \frac{1}{4} \frac{RT}{V}$$

$$\text{Pressure exerted by nitrogen } P_2 = \frac{14}{28} \frac{RT}{V} = \frac{1}{2} \frac{RT}{V}$$

$$\text{Pressure exerted by carbon dioxide } P_3 = \frac{22}{44} \frac{RT}{V} = \frac{1}{2} \frac{RT}{V}$$

From Dalton's law of partial pressures, the total pressure exerted by the mixture is given by

$$\begin{aligned} P &= P_1 + P_2 + P_3 \\ &= \frac{1}{4} \frac{RT}{V} + \frac{1}{2} \frac{RT}{V} + \frac{1}{2} \frac{RT}{V} \\ &= \frac{5}{4} \frac{RT}{V} = \frac{5}{4} \times \frac{8.315 \times 300}{4 \times 10^{-3}} \\ &= 7.79 \times 10^5 \text{ N m}^{-2} \end{aligned}$$

Example 14: One mole of oxygen is heated from 0°C , at constant pressure till its volume increases by 10%. (a) Find the quantity of heat required. The specific heat of oxygen, under these conditions, is $0.22 \text{ cal g}^{-1} \text{ K}^{-1}$. (b) If the same quantity of heat is supplied to the gas at constant volume, what will be the final temperature?

Solution: (a) For an enclosed gas at constant pressure,

$$V \propto T \quad (\text{Charles' law})$$

Hence, to increase the volume by 10%, the absolute temperature must be increased by 10%. The initial temperature of oxygen is $T = 273 \text{ K}$. The final temperature must be $1.1 \times T = 300.3 \text{ K}$.

$$\text{Rise in temperature} = 27.3 \text{ K}$$

Quantity of heat supplied is given by

$$\begin{aligned} Q &= (\text{mass of 1 mole}) \times (\text{specific heat}) \times (\text{rise in temp}) \\ &= 32 \text{ g} \times 0.22 \text{ cal g}^{-1} \text{ K}^{-1} \times 27.3 \text{ K} \\ &= 192.2 \text{ cal} = 807.24 \text{ J} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad C_p &= \text{molar specific heat at constant pressure} \\ &= 0.22 \times 32 \text{ cal K}^{-1} \text{ mol}^{-1} \\ &= 0.22 \times 32 \times 4.2 = 29.568 \text{ J K}^{-1} \text{ mol}^{-1} \end{aligned}$$

$$\begin{aligned} \text{But} \quad C_v &= C_p - R \\ &= 29.568 - 8.315 \\ &= 21.253 \text{ J K}^{-1} \text{ mol}^{-1} \end{aligned}$$

When 807.24 J of heat energy is supplied to 1 mole of oxygen at 0°C , at constant volume, the rise in temperature is given by

$$\Delta T = \frac{Q}{C_v} = \frac{807.24}{21.253} = 38 \text{ K}$$

$$\therefore \text{Final temperature} = 38^\circ\text{C}$$

Example 15: Calculate the work done when one mole of a perfect gas is compressed adiabatically. The initial pressure and volume of the gas are 10^5 N m^{-2} and 6 litres respectively. The final volume of the gas is 2 litres. Molar specific heat of the gas at constant volume is $3R/2$. (IIT 1982)

Solution: Initial pressure, $P_1 = 10^5 \text{ N m}^{-2}$

Initial volume, $V_1 = 6 \text{ litres} = 6 \times 10^{-3} \text{ m}^3$

Final volume, $V_2 = 2 \text{ litres} = 2 \times 10^{-3} \text{ m}^3$

Let P_2 be the final pressure. Now

$$\begin{aligned} C_v &= \frac{3}{2} R, \text{ therefore, } \gamma = \frac{C_p}{C_v} = \frac{C_v + R}{C_v} = \frac{\frac{3}{2} R + R}{\frac{3}{2} R} \\ &= \frac{5}{3} \end{aligned}$$

For an adiabatic change, $P_1 V_1^\gamma = P_2 V_2^\gamma$

$$P_2 = P_1 \left(\frac{V_1}{V_2} \right)^\gamma = 10^5 \times \left(\frac{6}{2} \right)^{5/3} = 6.24 \times 10^5 \text{ N m}^{-2}$$

The work done in the adiabatic process is

$$\begin{aligned} W &= \frac{1}{(\gamma - 1)} (P_1 V_1 - P_2 V_2) \\ &= \frac{1}{\left(\frac{5}{3} - 1 \right)} (10^5 \times 6 \times 10^{-3} - 6.24 \times 10^5 \times 2 \times 10^{-3}) \\ &= -972 \text{ J} \end{aligned}$$

The negative sign shows that the work is done on the gas.

11.5 TRUE-FALSE STATEMENTS WITH REASONS

A few statements are given below. We have to decide whether they are true or false, giving in brief, reasons in support of our answers

1. The pressure exerted by an enclosed ideal gas depends on the shape of the container.
2. The root-mean square speeds of the molecules of different ideal gases, maintained at the same temperature, are the same. (IIT 1981)
3. The average kinetic energy of the molecules in one mole of all ideal gases, at the same temperature, is the same.
4. The average kinetic energy of 1 g of all ideal gases, at the same temperature, is the same.
5. The volume V versus temperature T graphs for a certain amount of a perfect gas at two pressures P_1 and P_2 are as shown in Fig. 11.3. It follows from the graphs that $P_1 > P_2$. (IIT 1982)

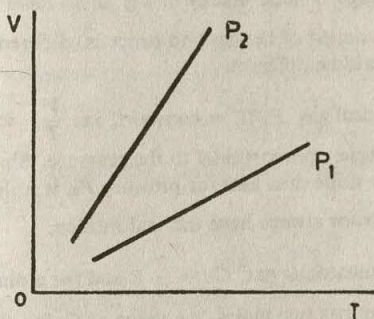


Fig. 11.3

6. Electrons in a conductor have no motion in the absence of a potential difference across it. (IIT 1982)
7. One mole of a monoatomic ideal gas is mixed with one mole of a diatomic ideal gas. The molar specific heat of the mixture at constant volume is $2R$, where R is the molar gas constant.
8. The curves A and B in Fig. 11.4 show the P - V graphs for an isothermal process and an adiabatic process for an ideal gas. The isothermal process is represented by the curve A . (IIT 1985)

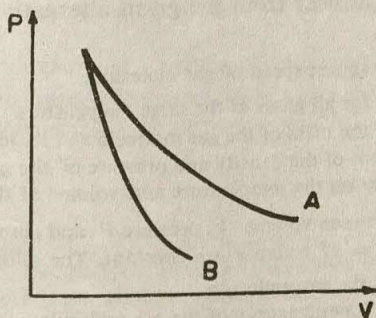


Fig. 11.4

ANSWERS

1. *False* The pressure of an ideal gas depends on the number of moles, temperature and volume of the enclosure; not its shape.
2. *False* The root-mean square speed of the molecules of a gas is given by

$$c_{rms} = \sqrt{\frac{3kT}{m}}$$

and hence varies inversely as the square-root of the mass of the molecule.

3. *True* The average kinetic energy of one mole of any ideal gas is given by

$$\bar{E} = \frac{3}{2} R T$$

where R is the molar gas constant which is the same for all gases. Thus, E depends only on the temperature of the gas.

4. *False* The average kinetic energy of 1 g of an ideal gas = $\frac{3}{2} \frac{RT}{M}$ where M is the molecular weight of the gas and hence is different for different gases, as the value of M will be different.
5. *True* For an ideal gas $PV/T = \text{constant}$, i.e. $\frac{V}{T} \propto \frac{1}{P}$ or the slope of the V - T curve is inversely proportional to the pressure. Since the graph for pressure P_1 has a smaller slope than that for pressure P_2 , it follows that $P_1 > P_2$.
6. *False* The electrons always have thermal motion.
7. *True* For a monoatomic gas, $C_v = \frac{3}{2} R$ and for a diatomic gas, $C_v = \frac{5}{2} R$. Since the mixture has two moles, the value of C_v for the mixture

$$= \frac{1}{2} \left(\frac{3}{2} R + \frac{5}{2} R \right) = 2 R.$$

8. *True* For an isothermal process, $PV = \text{constant}$ and for an adiabatic process, $PV^\gamma = \text{constant}$, where γ , the ratio of the two specific heats of the gas, has a value greater than unity. Hence the slope of the P - V graph is less for an isothermal process than for an adiabatic process. The curve A , whose slope is less than that of curve B , therefore, represents an isothermal process.

11.6 MULTIPLE CHOICE QUESTIONS

Choose the correct answer from the given alternatives in each of the following.

1. The root-mean square speed of gas molecules
 - (a) is the same for all gases at the same temperature
 - (b) depends on the mass of the gas molecule and its temperature
 - (c) is independent of the density and pressure of the gas
 - (d) depends only on the temperature and volume of the gas.
2. The relation between volume V , pressure P and absolute temperature T of an ideal gas is $PV = xT$ where x is a constant. The value of x depends upon
 - (a) the mass of the gas molecule
 - (b) the average kinetic energy of the gas molecules
 - (c) P , V and T
 - (d) the number of gas molecules in volume V .

3. The equation of state for n moles of an ideal gas is $P V = n R T$, where R is a constant. The SI unit for R is
 - (a) J K^{-1} per molecule
 - (b) $\text{J kg}^{-1} \text{K}^{-1}$
 - (c) $\text{J K}^{-1} \text{mol}^{-1}$
 - (d) $\text{J K}^{-1} \text{g}^{-1}$
4. Four molecules of a gas have speeds 1, 2, 3 and 4 km s^{-1} . The value of the root-mean square speed of the gas molecules is
 - (a) $\frac{1}{2}\sqrt{15} \text{ km s}^{-1}$
 - (b) $\frac{1}{2}\sqrt{10} \text{ km s}^{-1}$
 - (c) 2.5 km s^{-1}
 - (d) $\sqrt{15/2} \text{ km s}^{-1}$
5. The average kinetic energy of a molecule of a gas at absolute temperature T is proportional to
 - (a) $1/T$
 - (b) $\sqrt{T}$
 - (c) T
 - (d) T^2
6. The root-mean square speed of the molecules of a gas at absolute temperature T is proportional to
 - (a) $1/T$
 - (b) $\sqrt{T}$
 - (c) T
 - (d) T^2
7. The following four gases are at the same temperature. In which gas do the molecules have the maximum root-mean square speed?
 - (a) Hydrogen
 - (b) Oxygen
 - (c) Nitrogen
 - (d) Carbon dioxide.
8. If k is the Boltzmann constant, the average kinetic energy of a gas molecule at absolute temperature T is
 - (a) $k T/2$
 - (b) $3 k T/4$
 - (c) $k T$
 - (d) $3 k T/2$
9. E_0 and E_h respectively represent the average kinetic energy of a molecule of oxygen and hydrogen. If the two gases are at the same temperature, which of the following statements is true?
 - (a) $E_0 > E_h$
 - (b) $E_0 = E_h$
 - (c) $E_0 < E_h$
 - (d) Nothing can be said about the magnitude of E_0 and E_h as the information given is insufficient.
10. Choose the correct statement from the following:
 - (a) The average kinetic energy of a molecule of any gas is the same at the same temperature
 - (b) The average kinetic energy of a molecule of a gas is independent of its temperature
 - (c) The average kinetic energy of 1 g of any gas is the same at the same temperature
 - (d) The average kinetic energy of 1 g of a gas is independent of its temperature.
11. The root-mean square speed of the molecules of an enclosed gas is v . What will be the root-mean square speed if the pressure is doubled, the temperature remaining the same?
 - (a) $v/2$
 - (b) v
 - (c) $2 v$
 - (d) $4 v$
12. The mass of an oxygen molecule is about 16 times that of a hydrogen molecule. At room temperature the rms speed of oxygen molecules is v . The rms speed of the hydrogen molecule at the same temperature will be
 - (a) $v/16$
 - (b) $v/4$
 - (c) $4 v$
 - (d) $16 v$

13. The average kinetic energy of hydrogen molecules at 300 K is E . At the same temperature, the average kinetic energy of oxygen molecules will be
 (a) $E/16$ (b) $E/4$
 (c) E (d) $4E$
14. The average kinetic energy of a gas molecule at 27°C is 6.21×10^{-21} J. The average kinetic energy at 227°C will be
 (a) 9.35×10^{-21} J (b) 10.35×10^{-21} J
 (c) 11.35×10^{-21} J (d) 12.35×10^{-21} J
15. By what percentage should the pressure of a given mass of a gas be increased so as to decrease its volume by 10% at a constant temperature?
 (a) 8.1% (b) 9.1%
 (c) 10.1% (d) 11.1%
16. A gas at a temperature 250 K is contained in a closed vessel. If the gas is heated through 1°C , the percentage increase in its pressure is
 (a) 0.4% (b) 0.6%
 (c) 0.8% (d) 1.0%
17. A vessel containing 0.1 m^3 of air at 76 cm of Hg pressure is connected to an evacuated vessel of capacity 0.09 m^3 . The resultant air pressure will be
 (a) 20 cm of Hg (b) 30 cm of Hg
 (c) 40 cm of Hg (d) 50 cm of Hg.
18. A vessel is filled with a gas at a pressure of 10 atm and a temperature of 27°C . One-half of the mass of the gas is removed from the vessel and the temperature of the remaining gas is increased to 87°C . At this temperature the pressure of the gas in the vessel will be
 (a) 5 atm (b) 6 atm
 (c) 7 atm (d) 8 atm
19. Figure 11.5 shows a horizontal tube sealed at both ends and containing a pellet of mercury which occupies negligible volume. The region marked I contains a mass m of a gas and the region marked II contains a mass $2m$ of the same gas. If both regions have the same temperature, what fraction of the volume of the tube will be occupied by the gas of mass $2m$ when the pellet is in equilibrium?
 (a) $1/4$ (b) $1/2$
 (c) $2/3$ (d) $3/4$

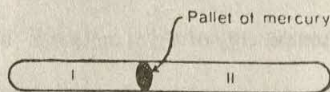


Fig. 11.5

20. Figure 11.6 shows the P - V curves for a certain mass of an ideal gas at two constant temperatures T_1 and T_2 . Which one of the following inferences is correct?
 (a) $T_1 = T_2$
 (b) $T_1 > T_2$

7. At what temperature, pressure remaining constant, will the rms speed of gas molecules be half its value at 0°C ?
8. Helium gas occupies a volume of 0.04 m^3 at a pressure of $2 \times 10^5\text{ N m}^{-2}$ and temperature 300 K . Calculate (i) the mass of helium, (ii) the rms speed of its molecules and (iii) the rms speed when the gas is heated at constant pressure to a temperature of 432 K .
9. A certain mass of a diatomic ideal gas ($\gamma = 1.4$) at a pressure of 75 cm of Hg , is compressed to half its original volume. What is the pressure of the compressed gas, if the compression is carried out (i) isothermally and (ii) adiabatically?
10. A certain mass of a monoatomic ideal gas ($\gamma = 5/3$) at a temperature of 300 K is compressed to half its original volume. What is the temperature of the compressed gas, if the compression is carried out (i) isothermally and (ii) adiabatically?
11. A gas expands isothermally from a volume of 100 cm^3 to 250 cm^3 under a constant pressure of 2 atmospheres . Calculate the work done by the gas.
12. Two moles of an ideal gas expand to double the initial volume at a constant temperature of 400 K . Calculate the work done in the process.
13. Two moles of air at 300 K expand adiabatically to twice the original volume. Calculate the work done by the gas. Given $R = 8.315\text{ J K}^{-1}\text{ mol}^{-1}$ and $\gamma = 1.4$.
14. A certain mass of a diatomic gas ($\gamma = 1.4$) occupies a volume of 100 cm^3 at a pressure of 75 cm of Hg . It is compressed adiabatically until the pressure increases to 80 cm of Hg . Calculate the work done in the process.
15. One mole of air at 300 K is compressed adiabatically until its temperature rises to 400 K . If the specific heat of air at constant volume is $20.75\text{ J K}^{-1}\text{ mol}^{-1}$, what is the work done?

ANSWERS

MULTIPLE CHOICE QUESTIONS

- | | | | | |
|---------|---------|---------|---------|---------|
| 1. (b) | 2. (d) | 3. (c) | 4. (d) | 5. (c) |
| 6. (b) | 7. (a) | 8. (d) | 9. (b) | 10. (a) |
| 11. (b) | 12. (c) | 13. (c) | 14. (b) | 15. (d) |
| 16. (a) | 17. (c) | 18. (b) | 19. (c) | 20. (c) |
| 21. (b) | 22. (b) | | | |

NUMERICAL EXERCISES

1. $4.61 \times 10^2\text{ m s}^{-1}$
2. $1.93 \times 10^3\text{ m s}^{-1}$
3. 100 K
4. 2768 K
5. $4.84 \times 10^2\text{ m s}^{-1}$
6. $6.06 \times 10^{-20}\text{ J}$
7. 68.25 K
8. (i) $1.28 \times 10^{-3}\text{ kg}$ (ii) $1.37 \times 10^3\text{ m s}^{-1}$ (iii) $1.64 \times 10^3\text{ m s}^{-1}$
9. (i) 150 cm of Hg (ii) 197.9 cm of Hg
10. (i) 300 K (ii) 395.8 K
11. 30 J
12. 554.6 J
13. 3020 J
14. 29.2 J
15. -2075 J

Electrostatics

12.1 REVIEW OF BASIC CONCEPTS

12.1.1 Coulomb's Law Force F between two point charges q_1 and q_2 , distance r apart is directly proportional to the product of their magnitudes and inversely proportional to the square of the distance between them, i.e.

$$F = k \frac{q_1 q_2}{r^2} \quad (12.1)$$

where k is a positive constant. The force F acts along the straight line joining the two charges and is directed away from each charge (repulsion) for like charges. If the charges have opposite signs, the force is directed towards each charge (attraction).

In SI units, r is in metres, F in newton and the charges are in a unit called coulomb (C). In SI units the constant k is given by

$$k = \frac{1}{4 \pi \epsilon_0} = 9 \times 10^9 \text{ N m}^2 \text{ C}^{-2}$$

where $\epsilon_0 = 8.85 \times 10^{-12} \text{ C}^2 \text{ N}^{-1} \text{ m}^{-2}$ is called the permittivity of free space. Thus, Coulomb's law for vacuum (or air) becomes

$$F = \frac{1}{4 \pi \epsilon_0} \frac{q_1 q_2}{r^2} \quad (12.2)$$

In a material medium, ϵ_0 is replaced by $\epsilon = K \epsilon_0$, where ϵ is called the permittivity of the medium and K is known as the dielectric constant of the medium.

Coulomb's law also applies to uniform spherical shells or spheres of charge. In that case, the distance r is measured from the centres of the spheres.

12.1.2 Electric Field and Electric Intensity An electric field exists at any point in space where a test charge, if placed at that point, would experience an electric force. A test charge is a fictitious charge of such a small magnitude that it does not disturb the electric field due to the neighbouring charges. The direction of the electric field at a point is the same as the direction of the force experienced by a positive test charge placed at that point.

The electric intensity $\vec{E}$ at a given point in an electric field is the force experienced per unit positive charge when a test charge is placed at that point. The SI unit of E is N C^{-1} or V m^{-1} . If a charge q is placed at a

point where the electric field due to other charges is $\vec{E}$, the charge will experience a force $\vec{F}$ given by

$$\vec{F} = q \vec{E} \quad (12.3)$$

12.1.3 Electric Intensity Due to a Point Charge If a test charge q is brought near a point charge q at a distance r from it, then it will experience a force

$$F = \frac{1}{4 \pi \epsilon_0} \frac{q' q}{r^2} = q' \left(\frac{1}{4 \pi \epsilon_0} \frac{q}{r^2} \right)$$

But the force experienced by a test charge q in an electric field E is given by

$$F = q' E$$

$$\therefore E = \frac{1}{4 \pi \epsilon_0} \frac{q}{r^2} \quad (12.4)$$

This is the electric field intensity at a distance r from a point charge q . For a positive charge, $\vec{E}$ is directed radially outwards from q , while for a negative charge ($-q$), $\vec{E}$ is directed radially inwards. Equation 12.4 also applies to points outside a finite spherical charge q .

To find the electric field $\vec{E}$ at a point due to several charges, the vector sum of the intensities due to the individual charges is taken. This is known as the *principle of superposition*, which may be used to calculate the coulomb force experienced by a charge due to a number of other charges.

12.1.4 Electric Potential The potential difference between two points A and B is the work done against electrical force in transporting a unit positive charge from A to B and is represented by $(V_B - V_A)$ or simply V . If W is the work done in carrying a test charge q from A to B , then

$$W = q (V_B - V_A) = q V$$

$$\text{or} \quad V = \frac{W}{q}$$

The SI unit of potential difference is the volt (V). Thus

$$1 \text{ volt} = 1 \text{ joule/coulomb}$$

$$\text{or} \quad 1 \text{ V} = 1 \text{ J C}^{-1}$$

Like work, potential is a scalar quantity and may be positive or negative.

The electric potential (or absolute potential) at any point in an electric field is the work done against electric force in carrying a unit positive charge from infinity to that point. The potential at a point P at a distance r from a charge q in free space is

$$V = k \frac{q}{r}$$

which, in SI units, can be written as

$$V = \frac{1}{4 \pi \epsilon_0} \frac{q}{r} \quad (12.5)$$

Because potential is a scalar quantity, the absolute potential at a point due to several point charges is given by (using the principle of superposition)

$$V = \frac{1}{4 \pi \epsilon_0} \sum_i \frac{q_i}{r_i} \quad (12.6)$$

where r_i are the distance of the charges q_i from that point respectively. The terms in the summation are positive or negative depending on whether the charges are positive or negative.

12.1.5 Electric Potential Energy Every charge possesses a potential energy because work is done to carry the charge from out of the field to the point in question. The work done qV on a charge q to bring it from infinity to that point against the electrical force is stored as potential energy in the charge. The potential energy can be positive or negative depending on whether the work is done against the electric force or by the electric force during the transport of charge. For a positive charge, there is an increase in potential for positive potential energy. But for negative potential energy, there will be a drop in potential if charge q is positive.

12.1.6 Relation Between E and V Because E is force per unit charge and V is work per unit charge, E and V are related in the same way as work and force. If $\Delta V (= V_B - V_A)$ is the increase in potential over a short displacement $\vec{\Delta s}$, then (see Fig. 12.1)

$$V = - \vec{E} \cdot \vec{\Delta s}$$

where the negative sign indicates that the work is done against the field. If θ is the angle between E and Δs , we have

$$V = - E (\Delta s) \cos \alpha = - E \Delta x$$

where $\Delta x = (\Delta s) \cos \theta$ is the component of $\vec{\Delta s}$ along $\vec{E}$.

$$\text{Therefore, } E = - \frac{\Delta V}{\Delta x} \quad (12.7)$$

Thus electric intensity E is the negative gradient of potential. This means that the potential decreases along the direction of $\vec{E}$. The SI unit of E is therefore, volt per metre ($V m^{-1}$).

If two points separated by a distance d in a uniform electric field have a potential difference V , then

$$|V| = E d$$

$$\text{or } E = \frac{|V|}{d} \quad (12.8)$$

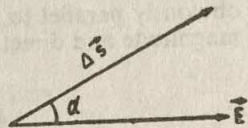


Fig. 12.1

The electric field between two large, parallel and oppositely charged plates is uniform. Hence, the potential difference between them, when they are separated by distance d , is $E d$.

12.1.7 Electron Volt The work done in transporting a charge of $+e$ (charge on a proton) through a potential of 1 volt is called one electron volt, written as eV. This is the unit of energy commonly used in atomic physics.

Therefore, $1 \text{ eV} = (1.6 \times 10^{-19} \text{ C})(1 \text{ V}) = 1.6 \times 10^{-19} \text{ J}$

$1 \text{ k eV} = 1000 \text{ eV}$, $1 \text{ M eV} = 10^6 \text{ eV}$,

$1 \text{ B eV} = 1 \text{ G eV} = 10^9 \text{ eV}$

Work or energy (in eV) = $\frac{\text{Work (in joules)}}{e}$

12.1.8 Lines of Force As an aid on visualizing electric fields, Faraday and others drew lines of force that everywhere have the direction of the electric field. Fig. 12.2 shows the lines of force that represent the electric field surrounding a point charge. A line of force in an electric field is an imaginary line, the tangent to which at any point on it represents the direction of the electric intensity at that point. The lines of force are directed away from a positive charge, since like charges repel each other and the imaginary test charge is a positive one. The lines extend all the way to infinity in this case, since there is a small force on a test charge no matter how far away it is. The lines of force originate at a positive charge and terminate at a negative charge. The number of lines crossing a unit area is related to the electric intensity. For a uniform electric field lines of force are obviously parallel to each other and equally spaced because only then the magnitude and direction of the field will be same in the region.

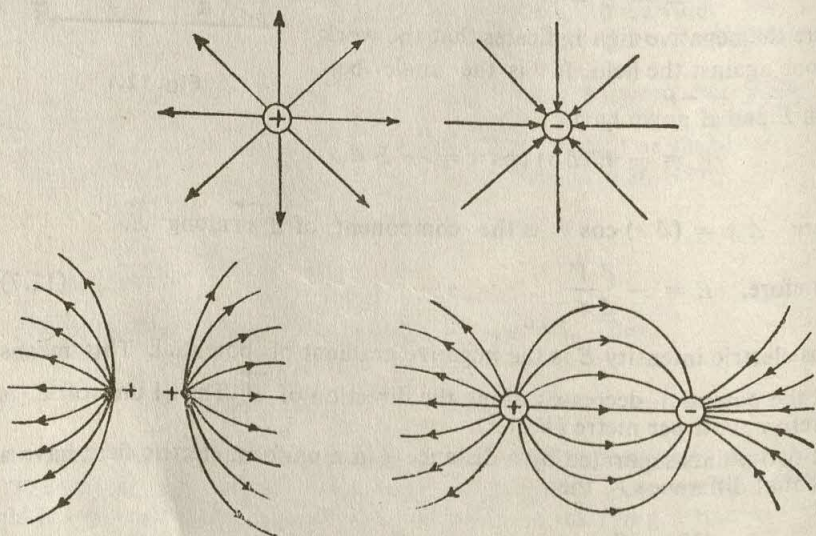


Fig. 12.2

12.1.9 Motion of a Charged Particle in a Uniform Electric Field A charged particle will experience a constant force in a uniform electric field and hence a constant acceleration. Let us consider a beam of electrons, moving with a velocity V_x between two parallel plates as shown in Fig. 12.3. The force on an electron of charge e between the plates is eE .

Using Eq. (12.8)

$$F = e E = \frac{eV}{d}$$

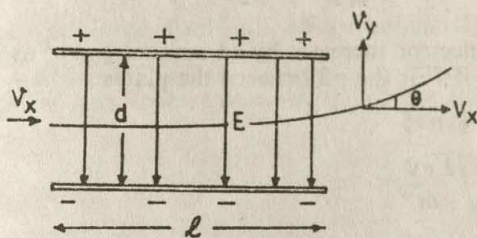


Fig. 12.3

and is directed towards the +ve plate. Since E is vertical, there is no horizontal force on the electrons and the velocity v_x remains unaffected. The situation is similar to the motion of a projectile projected horizontally under the action of gravity. The vertical acceleration due to gravity does not affect the horizontal motion.

The vertical displacement y is given by

$$y = \frac{1}{2} a t^2 \quad (s = u t + \frac{1}{2} a t^2, u = 0)$$

where a is the vertical acceleration given by

$$a = \frac{F}{m} = \frac{e E}{m}, \quad m \text{ being the electron mass}$$

$$\therefore y = \frac{1}{2} \frac{e E}{m} t^2 \quad (i)$$

The horizontal displacement,

$$x = v_x t \quad (ii)$$

Substitution of t from Eq. (ii) into Eq. (i) gives

$$y = \frac{1}{2} \left(\frac{e E}{m} \right) \frac{x^2}{v_x^2} \quad (12.9)$$

The path of the electron is therefore a parabola in the plane defined by the electric intensity and the initial velocity.

The value of y after travelling the distance l between the ends of the plates is

$$y = \frac{e E}{2 m} \frac{l^2}{v_x^2} \quad (12.10)$$

The beam then moves in a straight line as shown in Fig. 12.3. The time spent by the electrons between the plates is l/v_x . The vertical component of velocity v_y attained during this time is given by

$$\begin{aligned} v_y &= \text{acceleration} \times \text{time} \\ &= \frac{e E}{m} \frac{l}{v_x} \end{aligned} \quad (12.11)$$

The angle θ at which beam emerges from the field is given by

$$\tan \theta = \frac{v_y}{v_x} = \frac{e E l}{m v_x^2} = \frac{y}{l/2} = \frac{2y}{l} \quad (12.12)$$

The energy of the electron increases by an amount $\frac{1}{2} m v_y^2$ as it comes out of the plates. Thus if V is the pd between the plates

$$eV = \frac{1}{2} m v_y^2$$

$$\text{or } v_y = \sqrt{\frac{2 eV}{m}} \quad (12.13)$$

The above equations hold if the speed is much less than the velocity of light. The kinetic energy of the particle does not remain constant throughout the path. It increases as its path progresses in the field.

12.1.10 Electric Dipole A pair of point charges of equal magnitude and opposite signs separated by a short distance is called an electric dipole. In

Fig. 12.4 the dipole is in a uniform electric field $\vec{E}$, whose direction makes an angle θ with line joining the two charges, called the dipole axis. The net force on the dipole is zero, but a couple acts on it, whose moment is given by

$$\tau = (q E) l \sin \theta \quad (12.14)$$

where $l \sin \theta$ is the perpendicular distance between the lines of action of the two forces.

The product $q l$ of the charge q and the separation l is called the electric dipole moment, p .

The torque exerted by the couple is then

$$\tau = p E \sin \theta \quad (12.15)$$

In the vector form

$$\vec{\tau} = \vec{p} \times \vec{E} \quad (12.16)$$

$\vec{p}$ is the dipole moment along the axis of the dipole. The torque tends to rotate the dipole to a position in which $\vec{p}$ is parallel to $\vec{E}$. In a uniform electric field, the dipole is in equilibrium in this position, as in the bottom of Fig. 12.4.

12.1.11 Capacitance

(a) *Definition* If a conductor has a charge Q on it and V is its surface potential then

$$Q = C V$$

where C is a constant for the conductor and depends only on its size and shape but is independent of the nature of the material of which the conductor is made. This constant is called the capacity of the conductor.

SI unit of capacity is farad.

$$1 \text{ farad} = \frac{1 \text{ coulomb}}{1 \text{ volt}}$$

Since 1 farad is very large, 1 mF, μF , pF are commonly used.

(b) *Energy of a Charged Conductor* When a conductor is being charged, let the charge at any instant be q and the potential of the conductor V . Then

$$V = \frac{q}{C}$$

Work done in giving a further charge dq to the conductor is

$$dW = V dq = \frac{q dq}{C}$$

Total work done in charging the conductor to a final charge Q from an initial value of zero is

$$W = \int_0^Q dW = \frac{1}{C} \int_0^Q q dq = \frac{1}{2} \frac{Q^2}{C}$$

If V_0 is the final potential,

$$C = \frac{Q}{V_0}$$

Energy stored in a charged conductor is

$$W = \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} C V_0^2 = \frac{1}{2} Q V_0$$

(c) *Capacitor or Condenser* A capacitor or condenser consists of two conductors separated by an insulator or dielectric. The presence of a second conductor, which is usually connected to the earth, enhances the capacity of the system to store charge.

The dielectric between the conductors reduces the electric intensity between them because of the polarisation of the dielectric. This decreases the potential on the charged conductor and hence increases the capacity of the condenser.

The capacity of a condenser in the presence of a dielectric of a specific inductive capacity K (also known as dielectric constant) is K times the capacity when there is air (or vacuum) between the plates. Thus

$$C_e = K C_0 \quad (12.19)$$

where C_e and C_0 are the capacitance for a condenser with and without dielectric respectively.

The energy of a charged condenser is the same as that of a charged conductor, i.e.

$$W = \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} Q V = \frac{1}{2} C V^2$$

where Q is the charge on either plate and V , the potential difference between them. For the same charge, the presence of a dielectric results in a fall of potential between the two conductors, increases capacity and hence

reduces the stored energy. This loss in energy is due to the work done in polarising the dielectric medium.

(d) *Spherical Condenser* A spherical condenser consists of a solid charged sphere *A* surrounded by a concentric hollow sphere *B* insulated from outside, as shown in Fig. 12.5.

The capacity of such a condenser is given by

$$C = 4 \pi \epsilon_0 K \left(\frac{a b}{b - a} \right) \quad (12.20)$$

where *K* is the dielectric constant of the material between the two spheres, and *a* and *b* are the outer and inner radii of the two spheres. For vacuum, *K* = 1.

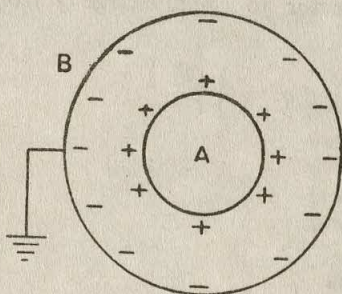


Fig. 12.5

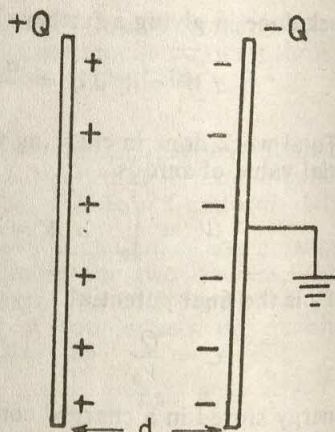


Fig. 12.6

(e) *Parallel Plate Condenser* The capacitance of a parallel plate capacitor with two large plates (Fig. 12.6) is given by

$$C = K \epsilon_0 \frac{A}{d} \quad (12.21)$$

where *A* is the area of one side of one plate which is opposite the other plate and *d* is the distance between them. This result holds for a capacitor of arbitrary shape.

(f) *Cylindrical Condenser* A cylindrical condenser consists of two co-axial cylinders with an inner solid cylinder and an outer hollow cylinder, as shown in Fig. 12.7.

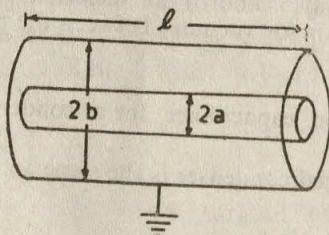


Fig. 12.7

The capacitance of this capacitor is given by

$$C = \frac{(2 \pi \epsilon_0) K l}{\log_e (b/a)} \quad (12.22)$$

(g) *Capacitors in Parallel and Series* In parallel arrangement of capacitors, as shown in Fig. 12.8 (a), the potential difference across individual capacitors is the same and the total charge is shared by them in the ratio of their capacities.

$$Q = Q_1 + Q_2 + Q_3 + \dots$$

and
$$V = \frac{Q_1}{C_1} = \frac{Q_2}{C_2} = \frac{Q_3}{C_3} = \dots$$

$\therefore C = C_1 + C_2 + C_3 + \dots$ (12.23)

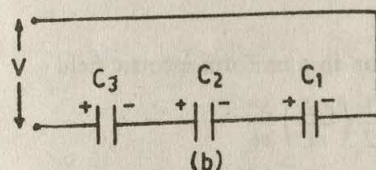
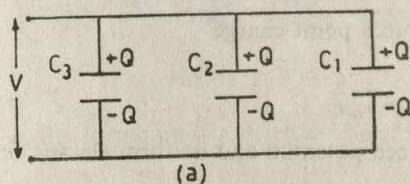


Fig. 12.8

In series arrangement of capacitors, as shown in Fig. 12.8 (b), the charge on each capacitor is the same and the total potential difference is shared by them in the inverse ratio of their capacities.

$$Q = C_1 V_1 = C_2 V_2 = C_3 V_3 = \dots$$

and
$$V = V_1 + V_2 + V_3 + \dots$$

Therefore, the effective capacity of the combination is given by

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \dots$$
 (12.24)

12.2 SI UNITS

Table 12.1

Physical Quantity	SI Units	Symbol
Electric charge	coulomb	C
Permittivity of free space	$8.85 \times 10^{-12} \text{ C}^2 \text{ N}^{-1} \text{ m}^{-2}$ or F m^{-1}	ϵ_0
Dielectric constant $K = \frac{\epsilon}{\epsilon_0}$	dimensionless	K
Electric intensity	newton per coulomb or volt per meter	N C^{-1} V m^{-1}
Electric potential	volt (joule per coulomb)	V or J C^{-1}
Electric potential energy	joule, electron volt	J, eV
Electric dipole moment	coulomb meter	C m
Capacity	farad	F

12.3 FORMULAE

1. Coulomb's Law:

$$F = \frac{1}{4 \pi \epsilon_0} \frac{q_1 q_2}{r^2}, \quad (1/4 \pi \epsilon_0) = 9 \times 10^9 \text{ N m}^2 \text{ C}^{-2}$$

2. Electric intensity due to a point charge

$$E = \frac{1}{4 \pi \epsilon_0} \frac{q}{r^2}$$

3. Potential due to a point charge

$$V = \frac{1}{4 \pi \epsilon_0} \frac{q}{r}$$

4. Relation between potential and uniform electric field

$$E = \frac{V}{d}$$

5. Path of an electron in a uniform electric field

$$y = \frac{1}{2} \left(\frac{e E}{m} \right) \frac{x^2}{v_x^2}$$

$$\tan \theta = \frac{v_y}{v_x} = \frac{e E l}{m v_x^2}$$

$$v_y = \sqrt{\frac{2 e V}{m}}$$

6. Moment of an electric dipole
- $p = q l$

7. Torque due to an electric dipole
- $\vec{\tau} = \vec{p} \times \vec{E}$

8. Capacitance
- $C = \frac{Q}{V}$

9. Energy stored in a charged conductor or capacitor

$$W = \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} C V^2 = \frac{1}{2} Q V$$

10. Capacity of spherical condenser

$$C = 4 \pi \epsilon_0 K \left(\frac{a b}{b - a} \right)$$

11. Capacity of a cylindrical condenser

$$C = \frac{2 \pi \epsilon_0 K l}{\log_e (b/a)}$$

12. Capacity of a series combination of capacitors

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \dots$$

13. Capacity of a parallel combination of capacitors

$$C = C_1 + C_2 + C_3 + \dots$$

12.4 NUMERICAL EXAMPLES

Example 1: Two negative charges of unit magnitude and a positive charge q are placed along a straight line. At what position and for what value of q will the system be in equilibrium? Check whether it is a stable or neutral equilibrium. (IIT 1973)

Solution: Let the charge q be at distances r_1 and r_2 from the two charges P and Q respectively, as shown in Fig. 12.9.

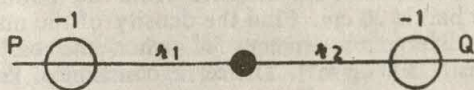


Fig. 12.9

For equilibrium of q , the forces on it exerted by P and Q must be co-linear, equal and opposite.

Force on q by P

$$F_{qP} = \frac{q}{4\pi\epsilon_0 r_1^2} \quad \text{towards } P$$

Force on q by Q

$$F_{qQ} = \frac{q}{4\pi\epsilon_0 r_2^2} \quad \text{towards } Q$$

$$\therefore |F_{qP}| = |F_{qQ}| \quad \text{or} \quad \frac{q}{r_1^2} = \frac{q}{r_2^2}$$

$$\therefore r_1 = r_2 = r \text{ (say). Hence charge } q \text{ should be equidistant from } P \text{ and } Q$$

For the system to be in equilibrium, the charges P and Q must also be in equilibrium. Now

$$F_{Pq} = \text{force on } P \text{ by } q = \frac{q}{4\pi\epsilon_0 r^2} \quad (\text{towards } q)$$

$$F_{PQ} = \text{force on } Q \text{ by } P = \frac{1}{4\pi\epsilon_0 (2r)^2}$$

(away from P and away from q)

Since F_{Pq} and F_{PQ} are oppositely directed along the same line, we have, for equilibrium,

$$\frac{q}{4\pi\epsilon_0 r^2} = \frac{1}{4\pi\epsilon_0 (4r^2)}$$

$$\text{or} \quad q = 1/4$$

Similarly for the equilibrium of Q , we would get $q = 1/4$. Thus $q = 1/4$ in magnitude of either charge P or Q .

Stability A slight displacement of q towards P increases the magnitude of F_{qP} and decreases the magnitude of F_{qQ} . Consequently, the displacement of q is increased. Thus the three charges are no longer in equilibrium. Hence the original equilibrium is unstable for displacement along the axis on which the charges are located.

For a displacement of q along a direction normal to the line PQ , the resultant of the two forces of attraction F_{qP} and F_{qQ} will bring the charge q back to its original position. Thus the equilibrium is stable for displacement in the vertical direction.

Example 2: Two balls of the same radius and weight are suspended on threads so that their surfaces are in contact. A charge of $Q_0 = 4 \times 10^{-7} \text{ C}$ is given to the balls which makes them repel each other to an angle of 60° . Find the mass of the balls if the distance from the point of suspension to the centre of a ball is 20 cm. Find the density of the material of the balls if the angle of divergence becomes 54° when the balls are immersed in kerosene of density 800 kg m^{-3} . Dielectric constant of kerosene = 2.

Solution: Let 2α be the angle between the two threads as shown in Fig. 12.10.

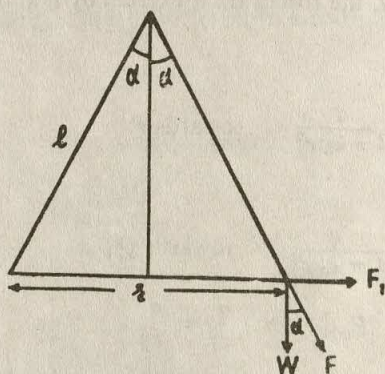


Fig. 12.10

Each ball is acted upon by two forces: the weight W of the ball and Coulomb's force of repulsion F_1 . The resultant of these two forces is F . But

$$F_1 = W \tan \alpha = \frac{q^2}{4 \pi \epsilon_0 r^2}$$

and $\frac{r}{2} = l \sin \alpha$

Hence $F_1 = W \tan \alpha = \frac{1}{4 \pi \epsilon_0} \frac{q^2}{4 l^2 \sin^2 \alpha}$

or $W = \left(\frac{1}{4 \pi \epsilon_0} \right) \frac{q^2}{4 l^2 \sin^2 \alpha \tan \alpha} \quad (1)$

Each ball carries a charge $q = Q_0/2$. Thus we have

$$W = \left(\frac{1}{4 \pi \epsilon_0} \right) \frac{Q_0^2}{16 l^2 \sin^2 \alpha \tan \alpha}$$

Now $Q_0 = 4 \times 10^{-7} \text{ C}$, $l = 20 \text{ cm} = 0.20 \text{ m}$, $\alpha = 30^\circ$

and $\frac{1}{4 \pi \epsilon_0} = 9 \times 10^9 \text{ N m}^2 \text{ C}^{-2}$

$$\begin{aligned}\text{Therefore } W &= \frac{9 \times 10^9 \times (4 \times 10^{-7})^2}{16 \times (0.20)^2 \times (0.5)^2 \times 0.577} \\ &= 0.0156 \text{ N}\end{aligned}$$

Hence $mg = 0.0156 \text{ N}$, where $g = 9.8 \text{ m s}^{-2}$ is the acceleration due to gravity. Thus we have

$$m = \frac{0.0156}{9.8} = 1.592 \times 10^{-3} \text{ kg} = 1.592 \text{ g}$$

When the balls are immersed in kerosene, each ball is acted upon by the upward thrust W_1 (Archimedes' principle). For a ball immersed in kerosene, we have

$$W - W_1 = \left(\frac{1}{4 \pi \epsilon_0 K} \right) \frac{q^2}{4 l^2 \sin^2 \alpha_2 \tan \alpha_2} \quad (2)$$

where α_2 is the angle between the threads now. Also

$$W - W_1 = (\rho_1 - \rho_2) V g \quad (3)$$

where ρ_1 is the density of the ball and ρ_2 is the density of kerosene. V is the volume of the ball. From Eqs (1), (2) and (3) we have

$$\frac{W - W_1}{W} = \frac{\sin^2 \alpha_1 \tan \alpha_1}{K \sin^2 \alpha_1 \tan \alpha_2} = \frac{\rho_1 - \rho_2}{\rho_1} = 1 - \frac{\rho_2}{\rho_1}$$

$$\therefore \rho_1 = \frac{\rho_2 K \sin^2 \alpha_2 \tan \alpha_2}{K \sin^2 \alpha_2 \tan \alpha_2 - \sin^2 \alpha_1 \tan \alpha_1}$$

$$\alpha_2 = 27^\circ, \sin^2 \alpha_2 = (0.4540)^2, \tan \alpha_2 = 0.5095$$

$$\alpha_1 = 30^\circ, \sin^2 \alpha_1 = 0.2500, \tan \alpha_1 = 0.5774$$

$$K = 2, \rho_2 = 800 \text{ kg m}^{-3}$$

$$\begin{aligned}\therefore \rho_1 &= \frac{800 \times 2 \times (0.4540)^2 \times 0.5095}{2 \times (0.4540)^2 \times 0.5095 - 0.2500 \times 0.5774} \\ &= 2559 \text{ kg m}^{-3}\end{aligned}$$

Example 3: Two identical balls, each of mass 0.20 g , carry identical charges and are suspended by two threads of equal length. At equilibrium, they position themselves as shown in Fig. 12.11. Find the charge on either ball. Given

$$g = 9.8 \text{ m s}^{-2}.$$

Solution: Consider the ball on the right. There are three forces acting on it. They are

(i) Tension T in the thread, (ii) the force due to gravity

$$\begin{aligned}mg &= 0.20 \times 10^{-3} \times 9.8 \\ &= 1.96 \times 10^{-3} \text{ N}\end{aligned}$$

and (iii) the force F due to Coulomb's repulsion.

For equilibrium, the sum of the forces along x and y directions must be zero, i.e.

$$\sum F_x = 0, \quad \sum F_y = 0$$

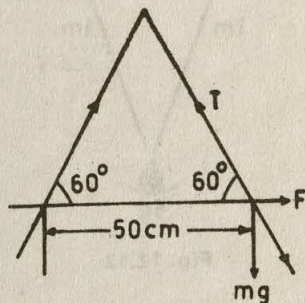


Fig. 12.11

Thus for the ball on the right, writing the x and y components of forces, we have

$$T \cos 60^\circ - F = 0 \quad (i)$$

$$T \sin 60^\circ - mg = 0 \quad (ii)$$

From Eq. (ii) we have

$$T = \frac{mg}{\sin 60^\circ} = \frac{1.96 \times 10^{-3}}{0.866} = 2.26 \times 10^{-3} \text{ N}$$

Substituting in Eq. (i) we get

$$F = T \cos 60^\circ = 2.26 \times 10^{-3} \times 0.50 = 1.13 \times 10^{-3} \text{ N}$$

But this is the Coulomb's force which is (since the two charges are equal)

$$\frac{q_1 q_2}{4 \pi \epsilon_0 r^2} = \frac{q^2}{4 \pi \epsilon_0 r^2}$$

$$F = \frac{q^2}{4 \pi \epsilon_0 r^2}$$

or

$$q^2 = (4 \pi \epsilon_0) F r^2$$

Now $r = 50 \text{ cm} = 0.50 \text{ m}, \quad \frac{1}{4 \pi \epsilon_0} = 9 \times 10^9 \text{ N m}^2 \text{ C}^{-2}.$

Therefore, $q^2 = \frac{1.13 \times 10^{-3} \times (0.50)^2}{9 \times 10^9}$

$$q = 1.77 \times 10^{-7} \text{ C} = 0.177 \mu\text{C}$$

Example 4: Two similar helium-filled spherical balloons tied to a 5 g weight with strings and each carrying a charge q float in equilibrium as shown in

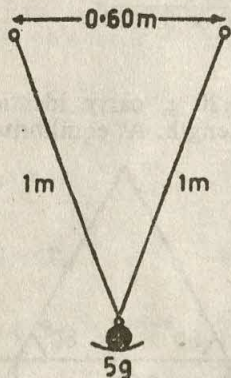


Fig. 12.12

Fig. 12.12. Find (a) the magnitude of q , assuming that the charge on each balloon acts as if it were concentrated at the centre and (b) the volume of each balloon. (Neglect the weight of the unfilled balloons and assume that the density of air = 1.29 kg m^{-3} and the density of helium in the balloons = 0.2 kg m^{-3} .) (IIT 1967)

Solution: Let T be the tension in the string. Resolve T into rectangular components at X . The horizontal components $T \sin \theta$ balance each other. For equilibrium,

$$Mg = 2 T \cos \theta$$

In triangle PRX ,

$$\sin \theta = \frac{0.3}{1.0} = 0.3$$

$$\cos \theta = \sqrt{1 - \sin^2 \theta} = 0.954$$

$$T = \frac{Mg}{2 \cos \theta} = \frac{5 \times 10^{-3} \times 9.8}{2 \times 0.954} = 2.568 \times 10^{-2} \text{ N}$$

If v is the volume of each balloon and ρ_1 , the density of helium, the mass of helium in each balloon is given by

$$m = v \rho_1 = 0.2 v$$

Let W be the upward thrust on each balloon. Then the weight of the air displaced is

$$W = V \rho g = 1.29 v g$$

Each balloon is in equilibrium under the action of the following forces.

- (i) its weight
- (ii) Coulomb force of repulsion F
- (iii) tension T in the string
- (iv) upthrust W

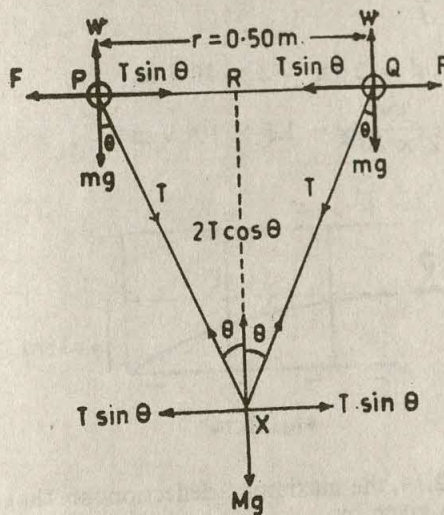


Fig. 12.13

Resolve T along horizontal and vertical directions at P . For the equilibrium of the balloons, we have

$$F = T \sin \theta \quad \text{and} \quad W = mg + T \cos \theta$$

But
$$F = \frac{q^2}{4 \pi \epsilon_0 (0.60)^2} = T \sin \theta$$

$$\begin{aligned} q^2 &= \frac{T \sin \theta \times 0.36}{(1/4 \pi \epsilon_0)} \\ &= \frac{2.568 \times 10^{-2} \times 0.3 \times 0.36}{9 \times 10^9} \\ &= 3.082 \times 10^{-13} \end{aligned}$$

or
$$q = 5.55 \times 10^{-7} \text{ C} = 0.555 \mu \text{ C}$$

Now
$$W = mg + T \cos \theta = 1.29 v g. \quad \text{But } mg = 0.2 v g$$

and
$$T \cos \theta = Mg/2$$

$$\therefore 1.29 v g = 0.2 v g + \frac{Mg}{2}$$

$$\text{or } 1.09 v = Mg/2 = 2.5 \times 10^{-3}$$

$$\text{or } v = 2.294 \times 10^{-3} \text{ m/s}$$

Example 5: A particle having a charge of $1.6 \times 10^{-19} \text{ C}$ enters midway between the plates of a parallel plate condenser. The initial velocity of the particle is parallel to the plates. A potential difference of 300 V is applied to the capacitor plates. If the length of the plates is 10 cm and their separation is 2 cm, calculate the greatest initial velocity for which the particle will not be able to come out of the plates. The mass of the particle is $12 \times 10^{-24} \text{ kg}$. (IIT 1976)

Solution: $E = \frac{V}{d}$

where $V = 300 \text{ V}$, $d = 2 \text{ cm} = 2 \times 10^{-2} \text{ m}$.

Therefore $E = \frac{300}{2 \times 10^{-2}} = 1.5 \times 10^4 \text{ V m}^{-1}$

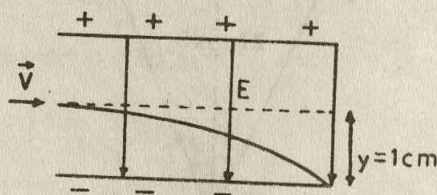


Fig. 12.14

Referring to Fig. 12.14, the maximum deflection so that the particle just fails to come out is given by

$$y = 1 \text{ cm or } 10^{-2} \text{ m}$$

But $y = \frac{q E x^2}{2 m V_x^2}$

$$V_x^2 = \frac{q E x^2}{2 m y}$$

Here $q = 1.6 \times 10^{-19} \text{ C}$, $E = 1.5 \times 10^4 \text{ V m}^{-1}$,
 $x = 10 \text{ cm} = 0.1 \text{ m}$

Therefore

$$V_x^2 = \frac{1.6 \times 10^{-19} \times 1.5 \times 10^4 \times (0.1)^2}{2 \times 12 \times 10^{-24} \times 10^{-2}}$$

$$= 1.0 \times 10^8$$

$$V_x = 1.0 \times 10^4 \text{ m s}^{-1}$$

Example 6: A simple pendulum consists of a small sphere of mass m suspended by a thread of length l . The sphere carries a positive charge q . The pendulum is placed in a uniform electric field of strength E directed verti-

cally upwards. With what period will the pendulum oscillate if the electrostatic force acting on the sphere is less than the gravitational force? Assume the oscillations to be small. (IIT 1977)

Solution: Let x be the small displacement given to the pendulum such that the angle θ is small (Fig. 12.15).

The forces acting at A are

- (i) tension T along the thread
- (ii) weight mg acting vertically downwards
- (iii) electrical force qE vertically upwards.

The resultant force vertically downwards is $(mg - qE)$. Therefore

Net acceleration

$$g' = g - \frac{qE}{m}$$

Time period

$$\begin{aligned} T &= 2\pi \sqrt{\frac{l}{g'}} \\ &= 2\pi \sqrt{\frac{l}{\left(g - \frac{qE}{m}\right)}} \end{aligned}$$

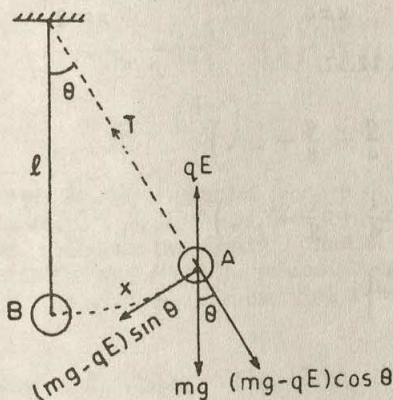


Fig. 12.15

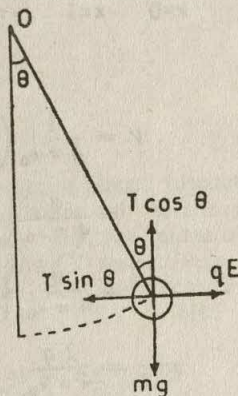


Fig. 12.16

Example 7: A pendulum bob of mass 80 milligrams and carrying a charge of 2×10^{-8} C is at rest in a uniform horizontal electric field of 20,000 V m^{-1} . Find the tension in the thread of the pendulum and the angle it makes with the vertical. (IIT 1979)

Solution: The forces acting on the bob are shown in Fig. 12.16. Since the bob is in equilibrium, we have

$$mg = T \cos \theta, \quad qE = T \sin \theta$$

$$\tan \theta = \frac{qE}{mg}$$

Here $q = 2 \times 10^{-8}$ C, $E = 2 \times 10^4$ V m $^{-1}$ and $m = 80 \times 10^{-6}$ kg. Substituting these values, we get

$$\tan \theta = \frac{2 \times 10^{-8} \times 2 \times 10^4}{80 \times 10^{-6} \times 9.8} = 0.51$$

$$\theta = 27^\circ 2'$$

Substituting the value of θ in $mg = T \cos \theta$, we get

$$\begin{aligned} T &= \frac{mg}{\cos \theta} \\ &= \frac{80 \times 10^{-6} \times 9.8}{0.8908} \\ &= 8.8 \times 10^{-4} \text{ N} \end{aligned}$$

Example 8: An infinite number of charges each equal to q are placed along the x -axis at $x = 1, x = 2, x = 8, \dots$ and so on. Find the potential and the electric field at the point $x = 0$ due to this set of charges. What will be the potential and electric field if, in the above set up, the consecutive charges have opposite sign? (IIT 1974)

Solution: The potential at $x = 0$ due to this set of charges is given by

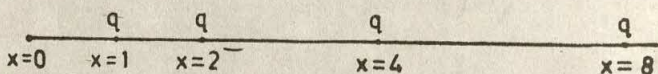


Fig. 12.17

$$\begin{aligned} V &= \frac{1}{4 \pi \epsilon_0} \left(\frac{q}{1} + \frac{q}{2} + \frac{q}{4} + \frac{q}{8} + \dots \right) \\ &= \frac{q}{4 \pi \epsilon_0} \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \right) \\ &= \frac{q}{4 \pi \epsilon_0} \left\{ \frac{1 - (1/2)^x}{1 - 1/2} \right\} \\ &= \frac{2q}{4 \pi \epsilon_0} \end{aligned}$$

Since the point charges are along the same straight line, the intensities at $x = 0$ are also along the x -axis.

$$\begin{aligned} E &= \frac{1}{4 \pi \epsilon_0} \left(\frac{q}{1^2} + \frac{q}{2^2} + \frac{q}{4^2} + \frac{q}{8^2} + \dots \right) \\ &= \frac{q}{4 \pi \epsilon_0} \left(1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \dots \right) \\ &= \frac{q}{4 \pi \epsilon_0} \left\{ \frac{1}{1 - (1/4)} \right\} \\ &= \frac{1}{4 \pi \epsilon_0} \left(\frac{4q}{3} \right) = \frac{q}{3 \pi \epsilon_0} \end{aligned}$$

If the consecutive charges have opposite sign, the potential at $x = 0$ is

$$\begin{aligned}
 V &= \frac{1}{4\pi\epsilon_0} \left(\frac{q}{1} - \frac{q}{2} + \frac{q}{4} - \frac{q}{8} + \frac{q}{16} - \frac{q}{32} + \dots \right) \\
 &= \frac{q}{4\pi\epsilon_0} \left\{ \left(1 + \frac{1}{4} + \frac{1}{16} + \dots \right) - \left(\frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \dots \right) \right\} \\
 &= \frac{q}{4\pi\epsilon_0} \left\{ \frac{1}{1 - (1/4)} - \left(\frac{1}{2} \right) \frac{1}{1 - (1/4)} \right\} \\
 &= \frac{q}{4\pi\epsilon_0} \left(\frac{4}{3} - \frac{2}{3} \right) = \frac{q}{6\pi\epsilon_0} \\
 \therefore V &= \frac{1}{4\pi\epsilon_0} \left(\frac{2q}{3} \right)
 \end{aligned}$$

The electric intensity of $x = 0$ is given by

$$\begin{aligned}
 E &= \frac{1}{4\pi\epsilon_0} \left\{ \frac{q}{(1)^2} - \frac{q}{(2)^2} + \frac{q}{(4)^2} - \frac{q}{(8)^2} + \frac{q}{(16)^2} - \frac{q}{(32)^2} + \dots \right\} \\
 &= \frac{q}{4\pi\epsilon_0} \left\{ \left(1 + \frac{1}{16} + \frac{1}{256} + \dots \right) - \left(\frac{1}{4} + \frac{1}{64} + \frac{1}{1024} + \dots \right) \right\} \\
 &= \frac{q}{4\pi\epsilon_0} \left\{ \frac{1}{1 - (1/16)} - \left(\frac{1}{4} \right) \frac{1}{1 - (1/16)} \right\} \\
 &= \frac{q}{4\pi\epsilon_0} \left\{ \frac{16}{15} - \left(\frac{1}{4} \right) \left(\frac{16}{15} \right) \right\} \\
 &= \frac{1}{4\pi\epsilon_0} \left(\frac{4q}{5} \right)
 \end{aligned}$$

Example 9: Four charges $+q$, $+q$, $-q$ and $-q$ are placed respectively at the corners A , B , C and D of a square of side a , arranged in the given order. Calculate the electric potential and intensity at O , the centre of the square. If E and F are the midpoints of sides BC and CD respectively, what will be the work done in carrying a charge e from O to E and from O to F ? (IIT 1977)

Solution: Referring to Fig. 12.18, the potential at O is given by

$$\begin{aligned}
 V_0 &= \frac{1}{4\pi\epsilon_0} \left(\frac{q}{OA} + \frac{q}{OB} - \frac{q}{OC} - \frac{q}{OD} \right) \\
 &= \frac{q}{4\pi\epsilon_0} \left(\frac{1}{r} + \frac{1}{r} - \frac{1}{r} - \frac{1}{r} \right) = 0
 \end{aligned}$$

Since each charge has the same magnitude and is at the same distance, the electric intensities E_A , E_B , E_C and E_D due to charges at A , B , C and D respectively, have the same magnitudes. Therefore

$$\begin{aligned}
 E_A &= E_B = E_C = E_D = \frac{q}{4\pi\epsilon_0 r^2} \\
 &= K \frac{q}{r^2}; \quad \left(K = \frac{1}{4\pi\epsilon_0} \right)
 \end{aligned}$$

The directions of the electric intensities are shown in Fig. 12.18

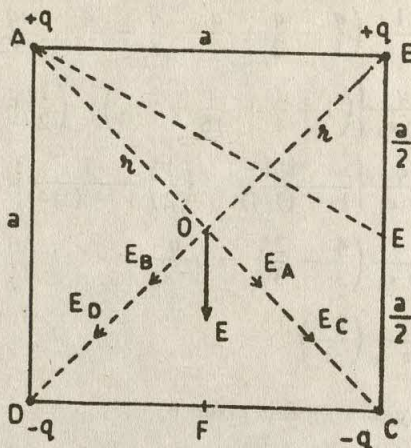


Fig. 12.18

Let E' be the resultant of E_A and E_C , then

$$E' = E_A + E_C = K \frac{2q}{r^2}$$

Similarly E'' , the resultant of E_B and E_D is

$$E'' = E_B + E_D = K \frac{2q}{r^2}$$

The resultant E is then given by

$$E^2 = E'^2 + E''^2 = K^2 \left\{ \left(\frac{2q}{r^2} \right)^2 + \left(\frac{2q}{r^2} \right)^2 \right\} = 2 K^2 \left(\frac{2q}{r^2} \right)^2$$

$$E = \sqrt{2} K \left(\frac{2q}{r^2} \right)$$

But

$$r = \frac{a}{\sqrt{2}} \quad \text{or} \quad r^2 = \frac{a^2}{2}$$

$$E = 4\sqrt{2} K \frac{q}{a^2}$$

The direction of E is parallel to AD or BC . Now

$$(AE)^2 = a^2 + \left(\frac{a}{2} \right)^2 = \frac{5a^2}{4}, \quad AE = \sqrt{5} \frac{a}{2}$$

Similarly $DE = \sqrt{5} \frac{a}{2}$ and $AF = BF = \sqrt{5} \frac{a}{2}$

Therefore, the total potential at E is given by

$$V_E = K \left(\frac{q}{AE} - \frac{q}{DE} + \frac{q}{BE} - \frac{q}{CE} \right) = 0$$

Similarly

$$\begin{aligned}
 V_F &= K \left(\frac{q}{AF} + \frac{q}{BF} - \frac{q}{DF} - \frac{q}{CF} \right) \\
 &= K \left(\frac{q}{\sqrt{5} \frac{a}{2}} + \frac{q}{\sqrt{5} \frac{a}{2}} - \frac{q}{\frac{a}{2}} - \frac{q}{\frac{a}{2}} \right) \\
 &= K \left(\frac{4q}{\sqrt{5}a} - \frac{4q}{a} \right) = \frac{4qK}{a} \left(\frac{1}{\sqrt{5}} - 1 \right)
 \end{aligned}$$

Work done in carrying charge e from O to E is

$$e(V_E - V_0) = e(0 - 0) = 0$$

Work done in carrying charge e from O to F is

$$e(V_F - V_0) = eV_F = K \frac{4qe}{a} \left(\frac{1}{\sqrt{5}} - 1 \right)$$

Example 10: A rigid insulated wire frame in the form of a right-angled triangle ABC , is set in a vertical plane as shown in Fig. 12.19. Two beads of equal masses m each and carrying charges q_1 and q_2 are connected by a cord of length l and can slide without friction on the wires.

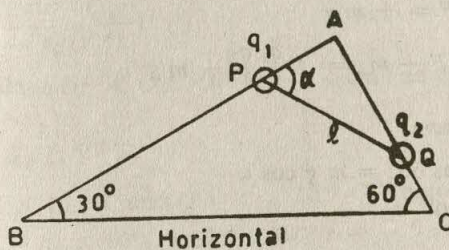


Fig. 12.19

Considering the case when the beads are stationary, determine

- the angle α
- the tension in the cord and
- the normal reaction on the beads.

If the cord is now cut, what are the values of the charges for which the beads continue to remain stationary? (IIT 1978)

Solution: The forces acting on the bead P are shown in Fig. 12.20.

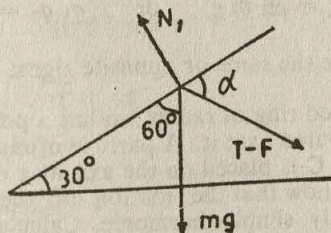


Fig. 12.20

Assuming that the charges are similar, the repulsive force F is

$$F = K \frac{q_1 q_2}{l^2}$$

where K is a constant. Let T be the tension in the string and N_1 , the normal reaction on the bead at P . Considering the components of forces perpendicular and parallel to AB , we have, for equilibrium,

$$m g \cos 60^\circ = (T - F) \cos \alpha \quad (1)$$

$$N_1 = m g \cos 30^\circ + (T - F) \sin \alpha \quad (2)$$

For the bead at Q , we have

$$m g \sin 60^\circ = (T - F) \sin \alpha \quad (3)$$

$$N_2 = m g \cos 60^\circ + (T - F) \cos \alpha \quad (4)$$

Squaring and adding Eqs (1) and (3) we get

$$m^2 g^2 = (T - F)^2$$

$$T - F = \pm m g \quad (5)$$

Assuming that the charges are similar, we have

$$T - F = + m g$$

$$T = F + m g = K \frac{q_1 q_2}{l^2} + m g \quad (6)$$

From Eq. (1) we have

$$m g \cos 60^\circ = m g \cos \alpha$$

$$\therefore \alpha = 60^\circ$$

Equation (2) gives

$$\begin{aligned} N_1 &= m g \cos 30^\circ + m g \sin 60^\circ \\ &= 2 m g \cos 30^\circ = m g \sqrt{3} \end{aligned}$$

From Eq. (4) we have

$$N_2 = m g \cos 60^\circ + m g \cos 60^\circ = m g$$

If the string is cut, $T = 0$ and we get, from Eq. (5)

$$F = \pm m g$$

$$F = K \frac{q_1 q_2}{l^2} = \pm m g \quad \text{or} \quad q_1 q_2 = \frac{\pm m g l^2}{K}$$

Thus q_1 and q_2 may have the same or opposite signs.

Example 11: A thin fixed ring of radius 1m has a positive charge of 1×10^{-5} C uniformly distributed over it. A particle of mass 0.9 g and having a negative charge of 1μ C is placed on the axis at a distance of 1 cm from the centre of the ring. Show that the motion of the negatively charged particle is approximately simple harmonic. Calculate the time period of oscillations.

(IIT 1982)

Solution: Consider two small and equal elements of charge Δq on the opposite sides of a diameter of the ring of radius R as shown in Fig. 12.21. P is a point on the axis of the ring at a distance r from the centre. The force on a negative charge $(-q)$ placed at P is given by

$$\Delta F = \frac{1}{4\pi\epsilon_0} \frac{q \Delta q}{(AP)^2}$$

and is directed towards A along PA . ΔF can be resolved into two components $\Delta F \cos \theta$ along the normal to OP and $\Delta F \sin \theta$ along PO . The element of charge Δq at B gives rise to a force equal in magnitude to ΔF and directed along PB . On resolving this force into horizontal and vertical components, the vertical components $\Delta F \cos \theta$ cancel out and we have a resultant force along PO alone. Summing over all such elements on the ring, the net force along PO is given by

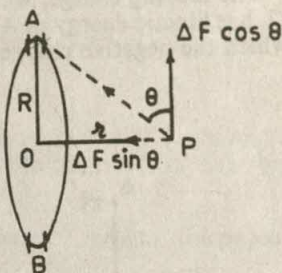


Fig. 12.21

$$F = \frac{1}{4\pi\epsilon_0} \frac{Qq \sin \theta}{(PA)^2}$$

where Q is the total charge on the ring. But $\sin \theta = OP/PA$. Therefore

$$F = \frac{1}{4\pi\epsilon_0} \frac{Qq}{(PA)^3} (OP)$$

Here $PA \approx OA$, since $OP \ll OA$. $OP = r = 1$ cm and $OA = 1$ m. Hence

$$F = \frac{1}{4\pi\epsilon_0} \frac{Qq}{R^3} r$$

where $q = -1 \times 10^{-6}$ C and $Q = 1 \times 10^{-5}$ C. Therefore, we have

$$F = -(9 \times 10^9) \times 10^{-6} \times 10^{-5} r = -9 \times 10^{-2} r$$

Acceleration a is given by

$$a = \frac{F}{m} = -\frac{9 \times 10^{-2}}{m} r = -\frac{k}{m} r$$

Thus the acceleration is proportional to the displacement r along the axis and directed towards the centre of the ring. The negatively charged particle, therefore, executes simple harmonic motion. The time period of this motion is given by

$$T = 2\pi \sqrt{\frac{m}{k}}$$

Here $m = 0.9$ g = 0.9×10^{-3} kg, $k = 9 \times 10^{-2}$ N m $^{-1}$. Therefore,

$$\begin{aligned} T &= 2\pi \sqrt{\frac{0.9 \times 10^{-3}}{9 \times 10^{-2}}} \\ &= 2\pi \sqrt{10^{-2}} \\ &= \frac{2\pi}{10} = 0.628 \text{ s} \end{aligned}$$

Example 12: Two fixed, equal, positive charges, each of magnitude 5×10^{-5} C, are located at points A and B separated by a distance of 6 m. An

equal and opposite charge moves towards them along the line COD , the perpendicular bisector of the line AB (Fig. 12.22).

The moving charge, when it reaches the point C at a distance of 4 m from O , has kinetic energy of 4 J. Calculate the distance of the farthest point D which the negative charge will reach before returning towards C .

(IIT 1985)

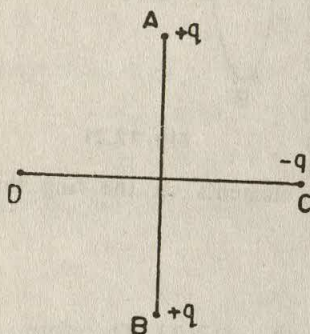


Fig. 12.22

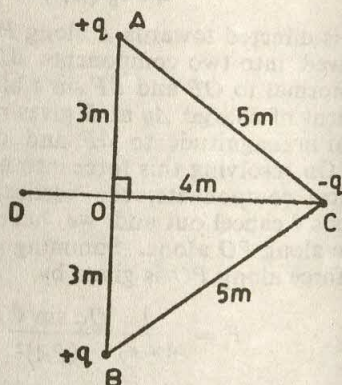


Fig. 12.23

Solution: From Fig. 12.23, $AC = BC$, $AO = 3$ m, $OC = 4$ m,

$$AC = BC = 5 \text{ m}$$

The potential energy of the charge at C due to the charges at A and B is given by

$$\begin{aligned} \text{PE} &= -\frac{q^2}{4\pi\epsilon_0 r} - \frac{q^2}{4\pi\epsilon_0 r} \\ &= -\frac{2q^2}{4\pi\epsilon_0 r} \\ &= -\frac{2 \times (5 \times 10^{-5})^2 \times 9 \times 10^9}{5.0} \\ &= -9.0 \text{ J} \end{aligned}$$

The kinetic energy at C is

$$\text{KE} = 4 \text{ J}$$

$$\text{Total energy} = \text{PE} + \text{KE} = 4 - 9 = -5 \text{ J}$$

The negative charge will reach the farthest point D such that its total energy is equal to -5 J.

Total energy at D due to charges at A and B

$$= -\frac{q^2}{4\pi\epsilon_0 (AD)} - \frac{q^2}{4\pi\epsilon_0 (AD)}$$

Thus we have

$$-5 = -\left(\frac{1}{4\pi\epsilon_0}\right) \frac{2q^2}{AD}$$

$$\begin{aligned}\text{or } AD &= \frac{1}{4\pi\epsilon_0} \frac{2q^2}{5} \\ &= \frac{9 \times 10^9 \times 2 \times (5 \times 10^{-5})^2}{5} = 9 \text{ m}\end{aligned}$$

$$\begin{aligned}\text{But } OD &= \sqrt{(AD)^2 - (AO)^2} = \sqrt{81 - 9} = \sqrt{72} \\ OD &= 8.485 \text{ m}\end{aligned}$$

Example 13: Eight charged water drops each with a radius of 1 mm and a charge of 10^{-10} C merge into a single drop. Find the potential of the big drop.

Solution: Considering a drop to be a perfect sphere of radius r , the potential due to a charge q is given by

$$V = \frac{1}{4\pi\epsilon_0} \frac{q}{r}$$

The charge of n drops $Q_0 = nQ = 8 \times 10^{-10}$ C.

The radius of the big drop (R) can be found from the condition

$$\begin{aligned}n\left(\frac{4}{3}\pi r^3 \rho\right) &= \frac{4}{3}\pi R^3 \rho \\ R &= r (n)^{1/3}\end{aligned}$$

Here $n = 8$ and $r = 1 \text{ mm} = 1 \times 10^{-3} \text{ m}$. Therefore,

$$R = 2 \times 10^{-3} \text{ m}$$

$$\text{Hence } V = \frac{9 \times 10^9 \times 8 \times 10^{-10}}{2 \times 10^{-3}} = 3600 \text{ V}$$

Example 14: A tiny nucleus has a charge $+50e$. (a) Find the potential V at a radius of 10^{-12} m from the nucleus. (b) If a proton is released from this point, how fast will it be moving when it is 1 m from the nucleus? ($+e = 1.6 \times 10^{-19}$ C, mass of proton $= 1.67 \times 10^{-27}$ kg)

$$\begin{aligned}\text{Solution: (a) } V &= \frac{q}{4\pi\epsilon_0 r} \\ &= \frac{9 \times 10^9 \times 50 \times 1.6 \times 10^{-19}}{10^{-12}} \\ &= 72,000 \text{ V} = 72 \text{ kV}\end{aligned}$$

(b) Potential V_1 at $r = 1 \text{ m}$ is

$$\begin{aligned}V_1 &= \frac{1}{4\pi\epsilon_0} \frac{q}{r} \\ &= \frac{9 \times 10^9 \times 1.6 \times 10^{-19} \times 50}{1} \\ &= 7.2 \times 10^{-8} \text{ V}\end{aligned}$$

which is almost negligible in comparison with 72 kV. Thus we can say that 1 m is far enough from the nucleus to consider it to be at infinity.

The proton falls through a potential difference of 72 kV. Therefore,

$$\text{Gain in KE} = \text{Loss in PE}$$

$$\frac{1}{2} m v_f^2 - \frac{1}{2} m v_0^2 = q V$$

Taking the initial velocity v_0 to be zero, we have

$$v_f^2 = \frac{2qV}{m} = \frac{2 \times 1.6 \times 10^{-19} \times 72 \times 10^3}{1.67 \times 10^{-27}} = 1.38 \times 10^{13}$$

$$v_f = 3.7 \times 10^6 \text{ ms}^{-1}$$

Example 15: A parallel plate condenser consists of two plates, each with area 200 cm^2 , separated by a 0.5 cm air gap. (a) Find its capacitance (b) If the capacitor is connected to a 500 V supply, find the charge on it, the energy stored in it, and the value of E between the plates. (c) If a liquid with dielectric constant 2.5 is poured between the plates so as to fill the air gap, how much additional charge will flow to the capacitor from the 500 V supply? Take $\epsilon_0 = 8.85 \times 10^{-12} \text{ Fm}^{-1}$.

Solution: (a) Capacitance of a parallel plate condenser is

$$C = K \epsilon_0 \frac{A}{d}$$

Here

$$A = 200 \text{ cm}^2 = 2 \times 10^{-2} \text{ m}^2$$

$$d = 0.5 \text{ cm} = 0.5 \times 10^{-2} \text{ m} \quad \text{and} \quad K = 1 \text{ for air.}$$

$$C = \frac{8.85 \times 10^{-12} \times 2 \times 10^{-2}}{5 \times 10^{-3}} \\ = 3.54 \times 10^{-11} \text{ F} = 35.4 \text{ pF}$$

$$(b) \quad q = CV = 3.54 \times 10^{-11} \times 500 = 1.77 \times 10^{-8} \text{ C}$$

$$\text{Energy} = \frac{1}{2} qV = 1.77 \times 10^{-8} \times 500 = 4.425 \times 10^{-6} \text{ J}$$

$$E = \frac{V}{d} = \frac{500}{5 \times 10^{-3}} = 1.0 \times 10^5 \text{ V m}^{-1}$$

(c) The capacity of the condenser now becomes

$$C' = 2.5 \times 3.54 \times 10^{-11} = 8.85 \times 10^{-11} \text{ F}$$

Therefore, now the charge will be

$$q' = C' V = 8.85 \times 10^{-11} \times 500 = 4.425 \times 10^{-8} \text{ C}$$

The additional charge which flows from the source is

$$q' - q = (4.425 - 1.77) \times 10^{-8} \text{ C} \\ = 2.655 \times 10^{-8} \text{ C}$$

Example 16: A capacitor of capacitance C is fully charged by a 200 V supply. It is then discharged through a small coil of resistance wire embedded in a thermally insulated block of specific heat $2.5 \times 10^2 \text{ J kg}^{-1} \text{ K}^{-1}$ and of mass 0.1 kg . If the temperature of the block rises by 0.4 K , find the value of C .

Solution: Energy stored in the capacitor is

$$\frac{1}{2} CV^2 = \frac{1}{2} \times C \times (200)^2 = 2 \times 10^4 \times C \text{ joule}$$

Energy appearing as heat in the block is

$$mc \theta = 0.1 \times 2.5 \times 10^2 \times 0.4 = 10 \text{ J}$$

Therefore

$$2 \times 10^4 \times C = 10$$

or

$$C = 5 \times 10^{-4} \text{ F} = 500 \mu\text{F}$$

Example 17: The plates of a parallel plate air capacitor consisting of two circular plates, each of 10 cm radius, placed 2 mm apart, are connected to the terminals of an electrostatic voltmeter. The system is charged to give a reading of 100 on the voltmeter scale. The space between the plates is then filled with oil of dielectric constant 4.5 and the voltmeter reading falls to 25. Calculate the capacitance of the voltmeter. Assume that the voltage recorded by the voltmeter is proportional to the scale reading.

Solution: Let V be the initial potential difference across the air capacitor and the voltmeter. If C is the capacitance of the condenser, and C_1 that of the voltmeter, then

$$\text{Total charge } Q = CV + C_1V = (C + C_1)V \quad (1)$$

When the space between the plates is filled with oil, the potential difference falls to V_1 and the capacitance C increases to $4.5C$. But the total charge remains the same. Hence,

$$4.5 CV_1 + C_1V_1 = Q = (C + C_1)V$$

$$(4.5C + C_1)V_1 = (C + C_1)V$$

$$\text{or} \quad \frac{4.5C + C_1}{C + C_1} = \frac{V}{V_1} = \frac{100}{25} = 4$$

(V and V_1 are proportional to the deflection).

$$4.5C + C_1 = 4C + 4C_1$$

$$C_1 = \frac{0.5C}{3} = \frac{1}{6}C$$

$$\text{Now} \quad C = \epsilon_0 \frac{A}{d}, \quad A = \pi r^2$$

$$C = \epsilon_0 \frac{\pi r^2}{d} = (4\pi \epsilon_0) \frac{r^2}{4d}$$

$$\text{Here} \quad r = 10 \text{ cm} = 0.1 \text{ m}, d = 2 \text{ mm} = 2 \times 10^{-3} \text{ m. Thus}$$

$$C = \frac{(0.1)^2}{9 \times 10^9 \times 4 \times 2 \times 10^{-3}} = 1.39 \times 10^{-10} \text{ F}$$

$$\text{Hence} \quad C_1 = \frac{1.39 \times 10^{-10}}{6} = 2.3 \times 10^{-11} \text{ F} = 23 \text{ pF}$$

Example 18: A capacitor of capacitance $4 \mu\text{F}$ is charged to a potential of 100 V and another of capacitance $6 \mu\text{F}$ is charged to a potential of 200 V .

These capacitors are now joined, with plates of like charges connected together. Calculate

- the potential across each after joining
- the total electrical energy stored before joining
- the total electrical energy stored after joining.

Explain why the energies calculated in (b) and (c) are different.

Solution: $V_1 = 100 \text{ V}$, $C_1 = 4 \mu\text{F}$
 $Q_1 = C_1 V_1 = 4 \times 10^{-6} \times 100 = 4 \times 10^{-4} \text{ C}$
 $V_2 = 200 \text{ V}$, $C_2 = 6 \mu\text{F}$
 $Q_2 = 6 \times 10^{-6} \times 200 = 12 \times 10^{-4} \text{ C}$

When the plates with like charges are connected together, as shown in Fig. 12.24, both capacitors have the same potential after the redistribution of charge.

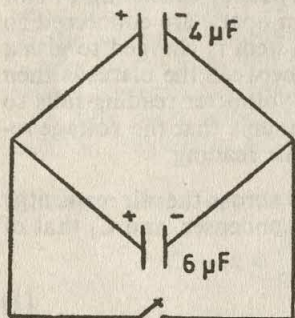


Fig. 12.24

$$\begin{aligned} \text{(a) Total charge } Q &= Q_1 + Q_2 \\ &= (4 + 12) \times 10^{-4} \\ &= 16 \times 10^{-4} \text{ C} \end{aligned}$$

Capacitance of the combination is

$$\begin{aligned} C &= C_1 + C_2 = 4 + 6 = 10 \mu\text{F} \\ &= 10 \times 10^{-6} \text{ F} \end{aligned}$$

Potential across each capacitor after joining is

$$V = \frac{Q}{C} = \frac{16 \times 10^{-4}}{10 \times 10^{-6}} = 160 \text{ V}$$

(b) Electrical energies W_1 and W_2 before joining are

$$W_1 = \frac{1}{2} C_1 V_1^2 = \frac{1}{2} (4 \times 10^{-6}) \times (100)^2 = 2 \times 10^{-2} \text{ J}$$

$$W_2 = \frac{1}{2} C_2 V_2^2 = \frac{1}{2} (6 \times 10^{-6}) \times (200)^2 = 12 \times 10^{-2} \text{ J}$$

$$W = W_1 + W_2 = (12 + 2) \times 10^{-2} = 0.14 \text{ J}$$

(c) Electrical energy after the capacitors are joined is

$$\frac{1}{2} \times 10 \times 10^{-6} \times (160)^2 = 0.128 \text{ J}$$

The difference in energies in cases (b) and (c) is dissipated as heat as the current flows through the connecting wires.

Example 19: From Fig. 12.25, find the value of the capacitance C if the equivalent capacitance between points A and B is to be $1 \mu\text{F}$. All the capacitances are in μF . (IIT 1977)

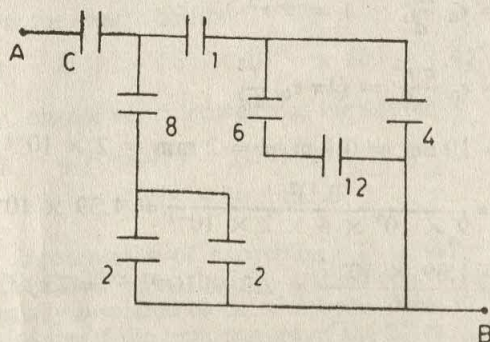


Fig. 12.25

Solution: The series combination of 6 and 12 is equivalent to 4, and the parallel combination of 2 and 2 is also equal to 4. Therefore, the combination reduces to the one shown in Fig. 12.26.

The parallel combination of 4 and 4 is equivalent to 8 and the series combination of 8 and 4 is $8/3$. Thus, we have the combination as shown in Fig. 12.27.

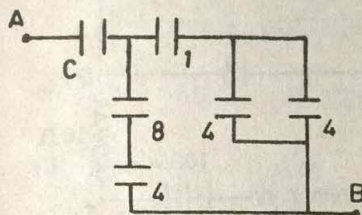


Fig. 12.26

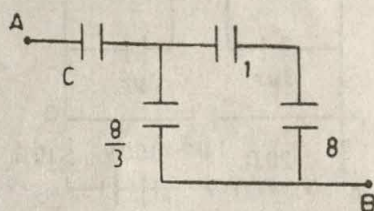


Fig. 12.27

The series combination of 1 and 8 yields $8/9$ as shown in Fig. 12.28. Now $8/3$ and $8/9$ are in parallel and their equivalent is $32/9$. Since the total capacitance between A and B is to be 1, we have

$$1 = \frac{1}{C} + \frac{9}{32}$$

$$C = \frac{32}{23} \mu\text{F} = 1.39 \mu\text{F}$$

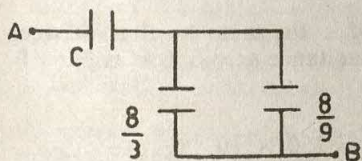


Fig. 12.28

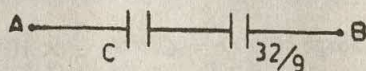


Fig. 12.29

Example 20: In Fig 12.30 shown below, find the potential difference between A and B and between the points B and C in the steady state.

(IIT 1979)

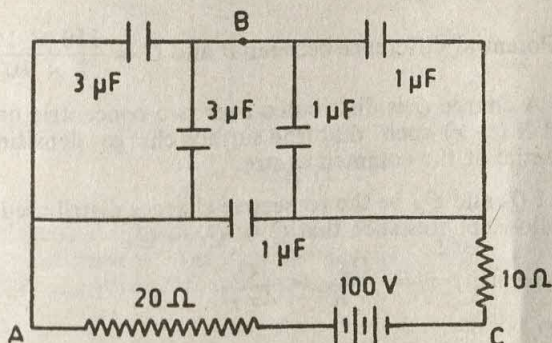


Fig. 12.30

Solution: The combination can be redrawn as shown in Fig 12.31 (a) which is equivalent to that shown in Fig. 12.31 (b). In the steady state, there is no current in the circuit. Hence the points P and Q are at the same potential as points A and C .

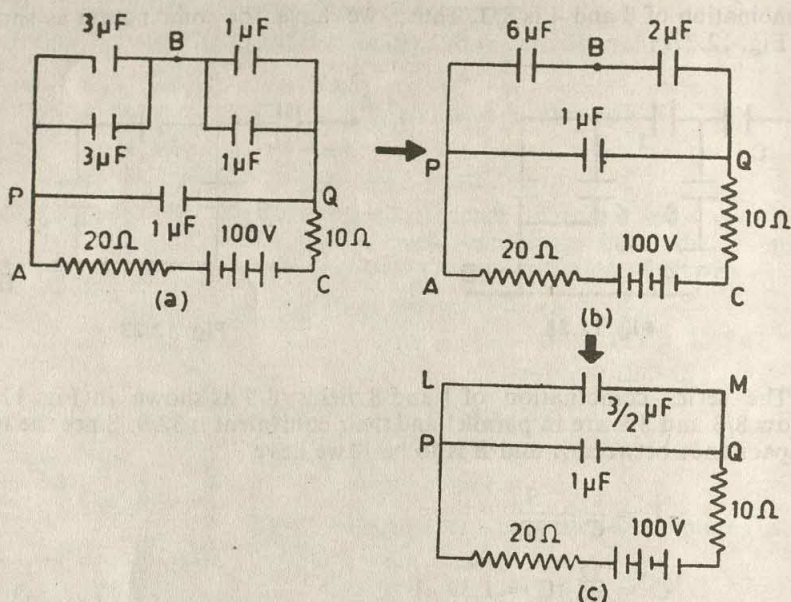


Fig. 12.31

Similarly, the potential between L and M is the same as that between points P and Q (see Fig. 12.31 (c)). The capacitance across LM is $3/2\mu\text{F}$. Therefore

$$\text{Charge } Q = CV = \frac{3}{2} \times 10^{-6} \times 100 = 150 \times 10^{-6} \text{ C}$$

Since $6\mu\text{F}$ and $2\mu\text{F}$ are connected in series, the charge on each condenser is the same and equals $150 \times 10^{-6} \text{ C}$. Thus, the potential difference between A and B is equal to that between the two ends of $6\mu\text{F}$ capacitance and is given by

$$\frac{Q}{C} = \frac{150 \times 10^{-6}}{6 \times 10^{-6}} = 25 \text{ V}$$

$$\text{Potential difference between } B \text{ and } C = \frac{150 \times 10^{-6}}{2 \times 10^{-6}} = 75 \text{ V}$$

Example 21: A charge Q is distributed over two concentric hollow spheres of radii r and R ($> r$) such that the surface charge densities are equal. Find the potential at the common centre. (IIT 1981)

Solution: Let Q_r and Q_R be the respective charges distributed over the two concentric hollow spheres such that $Q = Q_r + Q_R$.

$$\text{Surface charge density } \sigma = \frac{Q_R}{4\pi R^2} = \frac{Q_r}{4\pi r^2}$$

$$\therefore \frac{Q_r}{Q_R} = \frac{r^2}{R^2}$$

$$\text{or } \frac{Q_r + Q_R}{Q_R} = \frac{r^2 + R^2}{R^2} \quad (\text{adding 1 to both sides})$$

$$\text{or } \frac{Q}{Q_R} = \frac{r^2 + R^2}{R^2}$$

$$\therefore Q_R = \left(\frac{R^2}{r^2 + R^2} \right) Q$$

$$\text{and } Q_r = \left(\frac{r^2}{r^2 + R^2} \right) Q$$

$$\text{Potentials } V_R = \frac{Q_R}{4\pi\epsilon_0 R}, \quad V_r = \frac{Q_r}{4\pi\epsilon_0 r}$$

where V_R and V_r refer to the potentials due to Q_R and Q_r respectively.

$$\text{Thus } V_R = \left(\frac{R^2}{r^2 + R^2} \right) \frac{Q}{4\pi\epsilon_0 R}$$

$$\text{and } V_r = \left(\frac{r^2}{r^2 + R^2} \right) \frac{Q}{4\pi\epsilon_0 r}$$

The required potential V is given by

$$V = V_R + V_r = \frac{Q}{4\pi\epsilon_0} \left(\frac{R + r}{R^2 + r^2} \right)$$

Example 22: Figure 12.32 shows two identical parallel plate capacitors connected to a battery with the switch S closed. The switch is opened and the free space between the plates of the capacitors filled with a dielectric of dielectric constant 3. Find the ratio of the total electrostatic energy stored in both capacitors before and after the introduction of the dielectric.

Solution: When the switch S is closed, both capacitors have the same potential drop across them. The initial energy of both capacitors is

$$\frac{1}{2} C V^2 + \frac{1}{2} C V^2 = C V^2$$

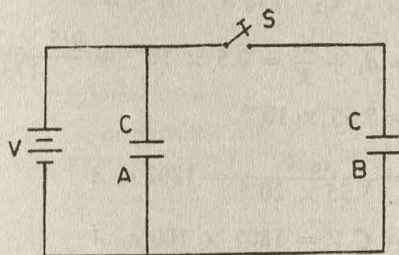


Fig. 12.32

When the dielectric is introduced between the plates, the capacitance of each capacitor becomes $3C$. Let V' be the new potential difference between the plates of the capacitor B (capacitance $3C$). Then

$$3C V' = C V \quad (\text{the total charge left on capacitor } B)$$

$$V' = \frac{V}{3}$$

$$\begin{aligned}\text{Electrostatic energy stored in capacitor } B &= \frac{1}{2} 3 C \left(\frac{V}{3} \right)^2 \\ &= \frac{1}{6} C V^2\end{aligned}$$

$$\begin{aligned}\text{Electrostatic energy stored in capacitor } A &= \frac{1}{2} 3 C V^2 \\ &= \frac{3}{2} C V^2\end{aligned}$$

Total electrostatic energy stored in both capacitors

$$= \frac{1}{6} C V^2 + \frac{3}{2} C V^2 = \frac{5}{3} C V^2$$

$$\text{The required ratio} = \frac{C V^2}{\frac{5}{3} C V^2} = \frac{3}{5}$$

Example 23: A potential difference of 100 V is applied between the plates of a parallel plate capacitor, which are 1 cm apart. One of the plates is in contact with a plane parallel plate of crystalline thallium bromide ($K = 173$) 9.5 mm thick. After the capacitor is disconnected, from the source of power, the crystalline plate is removed. What will be the potential difference between the plates after this is done?

Solution: Let A be the area of the plates. Initially the capacitor is equivalent to two capacitors C_1 and C_2 in series.

$$C_1 = \epsilon_0 \frac{A}{d_1}, \quad C_2 = K \epsilon_0 \frac{A}{d_2}$$

$$\text{Here} \quad d_1 = 0.5 \text{ mm} = 0.5 \times 10^{-3} \text{ m}, \quad d_2 = 9.5 \times 10^{-3} \text{ m}.$$

The resultant capacitance C is given by

$$\begin{aligned}\frac{1}{C} &= \frac{1}{C_1} + \frac{1}{C_2} = \frac{d_1}{\epsilon_0 A} + \frac{d_2}{K \epsilon_0 A} \\ \frac{\epsilon_0 A}{C} &= d_1 + \frac{d_2}{K} = 0.5 \times 10^{-3} + \frac{9.5 \times 10^{-3}}{173} \\ &= 5.55 \times 10^{-4}\end{aligned}$$

$$\therefore C = \frac{\epsilon_0 A}{5.55 \times 10^{-4}} = 1802 \epsilon_0 A$$

$$\text{Charge on plates} = C V = 1802 \times 100 \epsilon_0 A$$

The capacitance after the crystalline plate is removed is

$$C' = \frac{\epsilon_0 A}{d} = \frac{\epsilon_0 A}{1 \times 10^{-2}} = 100 \epsilon_0 A$$

Let V' be the new potential difference between the plates. Then

$$C V = C' V'$$

$$\text{or} \quad 1802 \times 100 \epsilon_0 A = 10^2 \epsilon_0 A V'$$

$$V' = 1802 \text{ V}$$

Example 24: C_1, C_2, C_3 and C_4 are four capacitors connected to a battery of constant emf equal to 12 V, as shown in Fig. 12.33. Find the charge on each capacitor when (a) switch S_1 is closed, (b) the switch S_2 is also closed. Given $C_1 = 1 \mu\text{F}$, $C_2 = 2 \mu\text{F}$, $C_3 = 3 \mu\text{F}$ and $C_4 = 4 \mu\text{F}$. What is the total charge drawn from the battery in each case?

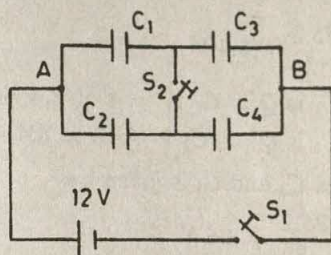


Fig. 12.33

Solution: (a) S_1 is closed and S_2 is open

C_1 and C_3 are in series with a total capacitance of $\frac{3}{4} \mu\text{F}$. Similarly, C_2 and C_4 are in series with a total capacitance of $\frac{4}{3} \mu\text{F}$. Hence, the total capacitance between A and B is

$$\frac{3}{4} + \frac{4}{3} = \frac{25}{12} \mu\text{F}$$

Since the voltage across AB is 12 V, the charge on C_1 and C_3 is

$$Q_1 = \frac{3}{4} \times 12 = 9 \mu\text{C} = Q_3$$

Similarly, the charge on C_2 and C_4 is

$$Q_2 = \frac{4}{3} \times 12 = 16 \mu\text{C} = Q_4$$

Thus, total charge drawn from the battery is $(9 + 16) = 25 \mu\text{C}$

(b) When S_1 and S_2 are both closed

Capacitors C_1 and C_2 are in parallel. Also C_3 and C_4 are in parallel. The two combinations are in series with the battery. Therefore

$$C' = C_1 + C_2 = 3 \mu\text{F}$$

$$C'' = C_3 + C_4 = 7 \mu\text{F}$$

Total capacitance between A and B is given by

$$\begin{aligned} \frac{1}{C} &= \frac{1}{C'} + \frac{1}{C''} \\ &= \frac{1}{3} + \frac{1}{7} \end{aligned}$$

or

$$C = \frac{21}{10} \mu\text{F}$$

Total charge drawn from the battery is

$$Q = \frac{21}{10} \times 12 = 25.2 \mu\text{C}$$

The potential difference across C_1 and C_2 is the same, since they are in parallel and is given by

$$V' = \frac{Q}{C'} = \frac{25.2}{3} = 8.4 \text{ V}$$

$$\text{Charge on } C_1 \text{ is } Q'_1 = C_1 V' = 1 \times 8.4 = 8.4 \mu\text{C}$$

$$\text{Charge on } C_2 \text{ is } Q'_2 = C_2 V' = 2 \times 8.4 = 16.8 \mu\text{C}$$

Potential difference across C_3 and C_4 is given by

$$V'' = \frac{Q}{C''} = \frac{25.2}{7} = 3.6 \text{ V}$$

$$\text{Charge on } C_3 \text{ is } Q'_3 = 3 \times 3.6 = 10.8 \mu\text{C}$$

$$\text{Charge on } C_4 \text{ is } Q'_4 = 4 \times 3.6 = 14.4 \mu\text{C}$$

Example 25: A slab of thickness t and dielectric constant K is introduced between the plates of a parallel plate condenser such that its faces are parallel to the plates and have the same surface area A as that of the plates of the condenser. What is the capacity of the system if d is the separation between the plates? If $K = 3$, what should be the value of t/d such that the capacitance of the system becomes twice that of the air condenser alone? Compute the change in the values of energy.

Solution: A partially filled condenser acts like two condensers in series, as shown in Fig. 12.34.

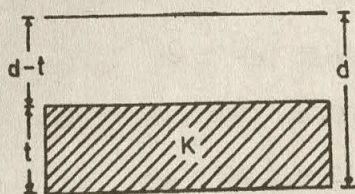


Fig. 12.34

The capacity of the air condenser is

$$C_1 = \frac{\epsilon_0 A}{d - t}$$

and that of the condenser with the dielectric is

$$C_2 = \frac{K \epsilon_0 A}{t}$$

The capacitance of the combination is given by

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{\epsilon_0 A} \left(d - t + \frac{t}{K} \right)$$

$$\therefore C = \frac{\epsilon_0 A}{d - t + \frac{t}{K}}$$

The capacity of the air condenser alone is

$$C_a = \frac{\epsilon_0 A}{d}$$

But $C = 2 C_a$, i.e.

$$\frac{\epsilon_0 A}{d - t + \frac{t}{K}} = \frac{2 \epsilon_0 A}{d}$$

or
$$\frac{d}{2} = d - t + \frac{t}{K}$$

or
$$t - \frac{t}{K} = \frac{d}{2}$$

$\therefore \frac{t}{d} = \frac{1}{2} \left(\frac{K-1}{K} \right)$, but $K = 3$.

Therefore
$$\left(\frac{t}{d} \right) = \frac{1}{3}$$

Energy of air condenser

$$= \frac{1}{2} \frac{Q^2}{C_a} = E_1$$

Energy of the condenser with the dielectric

$$= \frac{1}{2} \frac{Q^2}{C} = E_2$$

Then
$$\frac{E_2}{E_1} = \frac{C_a}{C} = \frac{1}{2} \quad \text{or} \quad E_2 = \frac{1}{2} E_1$$

Thus the potential energy of the condenser is reduced by the introduction of a dielectric. This energy is used up in polarising the dielectric (i.e. in bringing about a relative displacement of the positive and negative charges from their mean positions in the presence of an electric field).

Example 26: In Fig 12.35 the capacitors C_1 , C_3 , C_4 and C_5 have a capacitance of $4 \mu\text{F}$ each and the capacitor C_2 has a capacitance of $10 \mu\text{F}$. Find the effective capacitance between the points A and B .

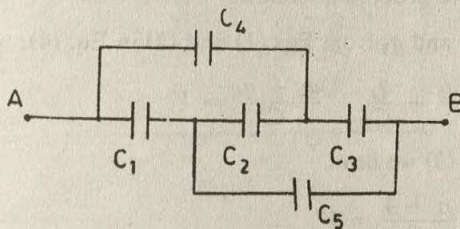


Fig. 12.35

Solution: Let q_1 , q_2 , q_3 , q_4 and q_5 be the respective charges on the condensers when a potential difference V is applied between the points A and B , as shown in Fig. 12.35.

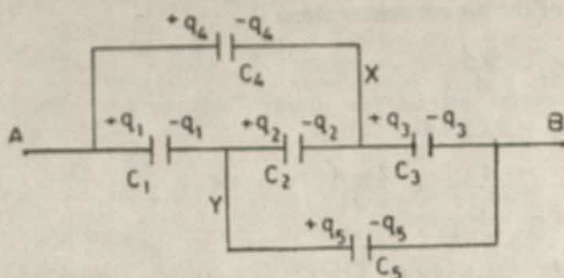


Fig. 12.36

Applying the principle of conservation of charge to the conductors X and Y , we have

$$-q_4 - q_2 + q_3 = 0 \quad (1)$$

and

$$-q_1 + q_2 + q_5 = 0 \quad (2)$$

Since the potential difference V between A and B is the same across any path chosen, we have

$$\frac{q_4}{C_4} + \frac{q_3}{C_3} = V = \frac{q_4}{C} + \frac{q_3}{C} \quad (3)$$

$$\frac{q_1}{C_1} + \frac{q_2}{C_2} + \frac{q_3}{C_3} = V = \frac{q_1}{C} + \frac{q_2}{C_2} + \frac{q_3}{C} \quad (4)$$

$$\frac{q_1}{C_1} + \frac{q_5}{C_5} = V = \frac{q_1}{C} + \frac{q_5}{C} \quad (5)$$

$$(\because C_1 = C_3 = C_4 = C_5 = C)$$

Equations (1) to (5) can be solved to determine the q 's in terms of the C 's and V . Because of the charge conservation, the charge on either terminal A or B should be equal. Hence the effective capacitance C_e equals the magnitude of the charge on either terminal divided by the potential difference between the terminals. Hence

$$C_e = \frac{q_1 + q_4}{V} = \frac{q_3 + q_5}{V}$$

To determine C_e , we proceed as follows

Substituting for q_1 and q_3 from Eqs (1) and (2) in Eq. (4), we get

$$\frac{q_2 + q_5}{C} + \frac{q_2}{C_2} + \frac{q_2 + q_4}{C} = V \quad (6)$$

From Eqs (1) and (3) we have

$$\frac{q_4}{C} + \frac{q_2 + q_4}{C} = V \quad (7)$$

Subtracting Eq. (7) from Eq. (6), we have

$$\frac{q_2}{C} + \frac{q_5}{C} + \frac{q_2}{C_2} - \frac{q_4}{C} = 0 \quad (8)$$

From Eqs (2) and (5), we have

$$\frac{q_2}{C} + \frac{2q_5}{C} = V \quad (9)$$

Multiplying Eq (8) by 2 and then subtracting Eq. (9) from it, we have

$$\frac{2q_2}{C_2} + \frac{q_2}{C} - \frac{2q_4}{C} = -V \quad (10)$$

From Eqs (1) and (3) we have

$$\frac{q_2}{C} + \frac{2q_4}{C} = V \quad (11)$$

On adding Eqs (10) and (11) we obtain

$$q_2 = 0, \quad q_4 = CV$$

and hence $q_3 = q_4$

Now Eq. (4) gives

$$(q_1 + q_4) = CV$$

$$\text{Therefore } C_v = \frac{q_1 + q_4}{V} = \frac{CV}{V} = C = 4 \mu\text{F}$$

Thus C_v is independent of C_2 .

12.5 TRUE-FALSE STATEMENTS WITH REASONS

We give below some statements. We have to decide whether they are true or false, giving in brief the reasons in support of our answers.

1. Potential difference may be expressed in watts per ampere.
2. The work needed to move a charge from one point to another depends on the path it follows between the two points.
3. Two identical metallic spheres of exactly equal masses are taken. One of them is given a positive charge Q coulombs and the other an equal negative charge. Their masses after charging are different.
4. Two protons A and B are placed between the two plates of a parallel plate capacitor charged to a potential difference V as shown in Fig. 12.37. The forces on the two protons are identical.

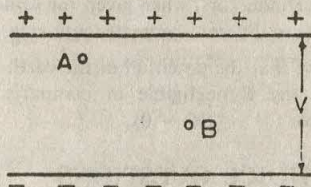


Fig. 12.37

5. Electrons move away from a region of higher potential to a region of lower potential.
6. A point in space can have a non-zero potential even when the electric field in the space is zero.

7. The dielectric constant of a conductor is zero.
8. When a dielectric is introduced between the plates of a condenser at a constant potential difference, the charge on the plates remains unchanged.
9. Two adjacent conductors that carry the same positive charge can have a potential difference between them.
10. It is a normal practice to assume the potential of the earth to be zero. This means that the earth has no net charge.

ANSWERS

1. *True* Because one watt equals one volt $\times$ one ampere.
2. *False* The work done to move a charge between two points depends on the position of the two points only, otherwise the principle of conservation of energy would be violated.
3. *True* The process of giving a positive charge involves removal of electrons, whereas a negative charge is given by adding electrons. Hence the masses of the two spheres will be different depending on the total mass of the electrons lost or gained by the respective spheres.
4. *True* The electric field $E (= V/d)$ is constant between the two plates, except at the edges. Therefore, the force experienced (eE) by a proton placed anywhere between the plates will be the same.
5. *False* Since an electron has a negative charge, it has less potential energy at a point where the potential is higher. It has a higher potential energy at a point where the potential is lower. Hence, electrons in an electric field will move away from a lower potential towards a higher potential region.
6. *True* Electric intensity at a given point is given by $E = -\frac{dV}{dx}$. If V is constant in space, the potential gradient in any direction is zero, hence the intensity is zero. Thus potential can exist where there is no electric field.
7. *False* The dielectric constant of a medium $= |E|/|E'|$, where the electric field E is reduced to E' in the presence of the dielectric. If a conductor is placed in the electric field, the intensity inside the conductor is zero. Therefore, the dielectric constant of the conductor is infinite.
8. *False* The introduction of a dielectric increases the capacity of the condenser. Since the plates are maintained at a constant potential difference, the charge on the plates increases ($Q = CV$). The extra charge flows to the plates from the source to which the latter are connected.
9. *True* The potential to which a conductor is raised not only depends on the amount of charge but also on the geometry and the size of the conductor. Therefore, the two conductors, when given the same charge, may be at different potentials if they have different geometry and size.
10. *False* By assuming that the potential of the earth is zero, we only mean that the net charge, if any, is negligible in comparison to the capacitance of the earth as a conductor ($V = Q/C \rightarrow 0$).

12.6 MULTIPLE CHOICE QUESTIONS

Select the correct answer from the given alternatives.

1. A hollow metal sphere of radius 5 cm is charged such that the potential on its surface is 10 V. The potential at the centre of the sphere is
 - (a) zero
 - (b) 10 V
 - (c) the same as that at a point 5 cm away from the surface
 - (d) the same as that at a point 25 cm away from the surface

(IIT 1983)

2. Two equal negative charges $-q$ are fixed at points $(0, a)$ and $(0, -a)$ on the y -axis. A positive charge Q is released from rest at the point $(2a, 0)$ on the x -axis. The charge Q will
- execute simple harmonic motion about the origin
 - move to the origin and remain at rest there
 - move to infinity
 - execute oscillatory but not simple harmonic motion
- (IIT 1984)
3. A $4 \mu\text{F}$ condenser is charged to 400 V and then its plates are joined through a resistance. The heat produced in the resistance is
- 0.64 J
 - 0.32 J
 - 0.16 J
 - 1.28 J
4. Each of the four capacitors in Fig. 12.38 is rated $50 \mu\text{F}$. The DC voltmeter reads 100 V. The charge on each plate of each capacitor is
- $2 \times 10^{-3} \text{ C}$
 - $5 \times 10^{-3} \text{ C}$
 - 0.2 C
 - 0.5 C

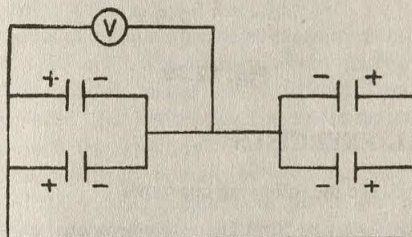


Fig. 12.38

5. An air-filled parallel plate capacitor has a capacitance of 10^{-12} F . The separation of the plates is doubled and wax is inserted between them, which increases the capacitance to $2 \times 10^{-12} \text{ F}$. The dielectric constant of wax is
- 2.0
 - 313
 - 4.0
 - 8.0
6. Three point charges $4q$, Q and q are placed in a straight line of length l at points distant 0, $l/2$ and l respectively. The net force on charge q is zero. The value of q is
- $-q$
 - $-2q$
 - $-\frac{1}{2}q$
 - $4q$
7. Two positive point charges of 12 and 8 microcoulombs respectively are placed 10 cm apart in air. The work done to bring them 4 cm closer is
- zero
 - 3.8 J
 - 4.8 J
 - 5.8 J
8. The work done in carrying a charge q once round a circle of radius r with a charge Q at the centre is
- $\frac{qQ}{4\pi\epsilon_0 r}$
 - $\frac{qQ}{4\pi\epsilon_0} \frac{1}{\pi r}$
 - $\frac{qQ}{4\pi\epsilon_0} \left(\frac{1}{2\pi r} \right)$
 - zero
9. When a capacitor is connected to a battery
- no current flows in the circuit at all
 - a current flows in the circuit for some time then decreases to zero

- (c) the current keeps on increasing, reaching a maximum value when the capacitor is charged to the voltage of the battery
 (d) an alternating current flows in the circuit.
10. A capacitor of capacitance $2 \mu\text{F}$ is connected as shown in Fig. 12.39. The internal resistance of the cell is 0.5Ω . The amount of charge on the capacitor plates is
- (a) zero (b) $2 \mu\text{C}$
 (c) $4 \mu\text{C}$ (d) $6 \mu\text{C}$

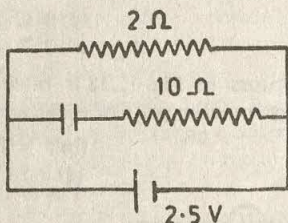


Fig. 12.39

12.7 NUMERICAL EXERCISES

Unless otherwise stated, use the following constants

$$\text{Electron mass} = 9.1 \times 10^{-31} \text{ kg}, \quad \text{electron charge} = 1.6 \times 10^{-19} \text{ C}$$

$$\epsilon_0 = 8.85 \times 10^{-12} \text{ F m}^{-1}$$

$$1/(4\pi\epsilon_0) = 9 \times 10^9 \text{ m F}^{-1}$$

- Two identically charged spheres are suspended by strings of equal lengths. The strings make an angle of 30° with each other. When suspended in a liquid of density 800 kg m^{-3} , the angle remains the same. What is the dielectric constant of the liquid? The density of the material of the spheres is 1600 kg m^{-3} .
(IIT 1976)
- A charge of $+3 \mu\text{C}$ is placed at $x = 0$ and a charge of $-5 \mu\text{C}$ at $x = 40 \text{ cm}$. Where must a third charge q be placed on the x -axis such that it experiences no force?
- Equal charges of $+4 \mu\text{C}$ are placed at the three corners of an equilateral triangle 2 m on a side. Calculate the magnitude and direction of the force on one of charges.
- Two plane parallel conducting plates 15 mm apart are held horizontal, one above the other, in air. The upper plate is maintained at a positive potential of 1500 V while the lower plate is earthed. Calculate the number of electrons which must be attached to a small oil drop of mass $4.90 \times 10^{-15} \text{ kg}$, if it remains stationary in the air between the plates. (Assume that the density of air is negligible in comparison with that of the oil). If the potential of the upper plate is suddenly changed to -1500 V , what is the initial acceleration of the charged drop?
- Two charged conducting plates in vacuum are 15 cm apart as shown in Fig. 12.40. The electric field between the plates is uniform and equals 3000 N C^{-1} . An electron is released from rest at a point P close to the negative plate. How long will it take to reach the other plate? How fast will it be going just before it hits the positive plate?

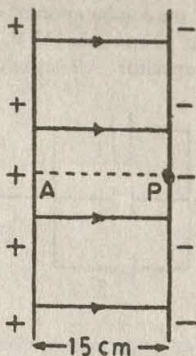


Fig. 12.40

6. A ball with a mass of 1 g and a charge of 10^{-8} C moves from point A whose potential is 600 V to a point B whose potential is zero. What was the velocity of the ball at point A if at point B it is 20 cm s^{-1} ?
7. An electron moves in a plane horizontal capacitor parallel to its plates with a velocity of $3.6 \times 10^7 \text{ ms}^{-1}$. The intensity of the electric field inside the capacitor is 3700 V m^{-1} and its plates are 20 cm long. Over what distance will the electron be displaced in a vertical direction under the action of the electric field during its motion in the capacitor?
8. A parallel plate capacitor with air between the plates is charged to a potential difference of 500 V, after which both plates are left insulated. A plastic plate is inserted, practically filling the gap and causing the potential difference across the capacitor to decrease to 75 V. What is the dielectric constant of plastic?
9. Two positive charges of 2×10^{-7} C and 3×10^{-7} C are separated by a distance of 0.1 m. Calculate the resultant electric field and electric potential
 - (a) at a midpoint between them
 - (b) at a point 0.04 m from the first and on the line joining the charges.
 - (c) at a point 0.04 m from the first and on the line joining the charges, but outside them
 - (d) at a point 0.1 m from each charge.
 - (e) Calculate the work required to move a charge of 4×10^{-7} C from the point in (c) to the point in (d).
10. A small sphere of mass 2×10^{-4} kg hangs by a thread between two vertical plates 5 cm apart. The charge on the sphere is 6×10^{-9} C. What is the potential difference between the plates if the thread assumes an angle of 10° with the vertical?
11. A capacitor is made of 25 thin metal sheets, each having an area of $6 \times 10^{-2} \text{ m}^2$ and separated from each other by paraffin paper $6 \times 10^{-4} \text{ m}$ thick ($K = 2.6$). Find the capacitance of the system.
12. A parallel plate capacitor with air as dielectric has plates of area $4 \times 10^{-2} \text{ m}^2$ which are 2 mm apart. It is charged by connecting it to a 100 V battery. It is then disconnected from the battery and connected in parallel with a similar uncharged capacitor with plates of half the area which are twice the distance apart as compared to the first capacitor. Calculate the final charge on each capacitor. Permittivity of air = $8.8 \times 10^{-12} \text{ F m}^{-1}$.
13. A plane air capacitor in which the plates 100 cm^2 in area are 1 mm apart is charged to 100 V. The plates are then moved apart to a distance of 25 mm. Find the energy of the capacitor before and after the plates are moved apart, if the power source before the plates are moved apart is (a) not disconnected and (b) disconnected.

14. Determine (a) the capacity of the combination of capacitors shown in Fig. 12.41. If the voltage applied between A and B is 120 V, find (b) the charge and (c) the potential difference on each capacitor. All capacitances are in microfarads.

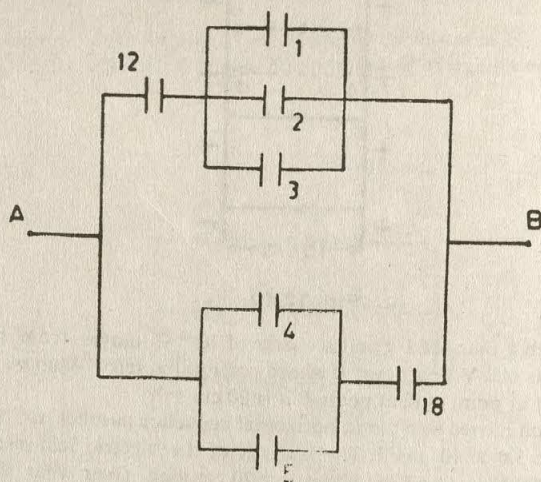


Fig. 12.41

15. Given the capacitor arrangement shown in Fig. 12.42, show that the relation between C_1 and C_2 must be $C_2 = 0.618 C_1$ in order that the capacitance of the system be equal to C_2 .

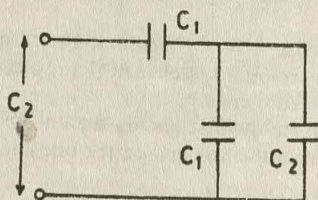


Fig. 12.42

16. Calculate the magnitude and direction of the electric field at the centre of a square 2 m on each side, if three charges of $+8 \mu\text{C}$ each are placed at three of the corners of the square.
17. A capacitor of capacitance $9 \mu\text{F}$ is charged from a source of emf 200 V. The capacitor is then disconnected from the source and connected in parallel with a second capacitor of capacitance $3 \mu\text{F}$. The second capacitor is now removed and discharged. What charge remains on the $9 \mu\text{F}$ capacitor? How many times would the process have to be performed in order to reduce the charge on the $9 \mu\text{F}$ capacitor to below 50% of its initial value? What would be the potential difference between its plates now?
18. A $20 \mu\text{F}$ capacitor is charged to a potential difference of 1000 V. The terminals of the charged capacitor are then connected to those of an uncharged $5 \mu\text{F}$ capacitor. Calculate (a) the original charge of the system, (b) the final potential difference across each capacitor, (c) the final energy of the system and (d) the decrease in energy when the capacitors are connected.

ANSWERS

MULTIPLE CHOICE QUESTIONS

- | | | | | |
|--------|--------|--------|--------|---------|
| 1. (b) | 2. (d) | 3. (b) | 4. (b) | 5. (c) |
| 6. (a) | 7. (d) | 8. (d) | 9. (b) | 10. (c) |

NUMERICAL EXERCISES

- $K = 2$
- 1.375 m
- 0.0624 N, away from triangle perpendicular to opposite side
- 3 electrons, 19.6 m s^{-2} (2 g)
- $2.4 \times 10^{-8} \text{ s}$, $1.30 \times 10^7 \text{ m s}^{-1}$
- 0.167 m s^{-1}
- over 0.01 m
- $K = 6.67$
- $3.6 \times 10^5 \text{ N C}^{-1}$ towards $0.2 \mu\text{C}$, $9 \times 10^4 \text{ V}$
 - $3.75 \times 10^6 \text{ N C}^{-1}$ towards $0.3 \mu\text{C}$, $9 \times 10^4 \text{ V}$
 - $1.26 \times 10^6 \text{ N C}^{-1}$ away from $0.2 \mu\text{C}$, $6.43 \times 10^4 \text{ V}$
 - $3.92 \times 10^5 \text{ N C}^{-1}$ at 6.6° from vertical, $4.5 \times 10^4 \text{ V}$
 - $-7.72 \times 10^{-8} \text{ J}$
- 2880 V
- $9.6 \times 10^{-11} \text{ F}$
- $14.1 \times 10^{-9} \text{ C}$, $3.5 \times 10^{-9} \text{ C}$
- $4.43 \times 10^{-7} \text{ J}$, $1.78 \times 10^{-8} \text{ J}$
 - $4.43 \times 10^{-7} \text{ J}$, $1.11 \times 10^{-8} \text{ J}$
- $10 \mu\text{F}$
 - $12 \mu\text{F} : 480 \mu\text{C}$, $18 \mu\text{F} : 720 \mu\text{C}$, $4 \mu\text{F} : 320 \mu\text{C}$, $5 \mu\text{F} : 400 \mu\text{C}$
 $80 \mu\text{C}$, $2 \mu\text{F} : 160 \mu\text{C}$, $3 \mu\text{F} : 240 \mu\text{C}$
 - $12 \mu\text{F}$ and $18 \mu\text{F} : 40 \text{ V}$; all others : 80 V
- $3.6 \times 10^4 \text{ N C}^{-1}$, towards the empty corner
- $1.35 \times 10^{-3} \text{ C}$, three times, 84 V
- (a) 0.02 C , (b) 800 V , (c) 8 J (d) 2 J

Current Electricity (EMF and Simple Circuits)

13.1 REVIEW OF BASIC CONCEPTS

13.1.1 Electric Current The flow of charge constitutes electric current. In metals and vacuum tubes, the carriers of charge are the electrons, which have a negative charge. In many high energy nuclear accelerators, the current consists of a flow of positive charges (protons). Some types of transistors make use of the flow of 'holes' that behave like positive charges. A positive charge moving in one direction behaves (in nearly all external effects) like a negative charge moving in the opposite direction. For consistency, therefore, the direction of current is taken to be the direction in which the positive charges move. The direction of this conventional current is opposite to the direction of electron flow. Thus the term *current* means the flow of an equivalent positive charge.

The flow of current in a conductor is due to free electrons. Under the action of an electric field, the free electrons move with a certain average drift velocity in the direction of the field. This constitutes an electric current'. If I is the electric current flowing through a conductor, then

$$I = \frac{dQ}{dt} \quad (13.1)$$

where dQ is the net charge transported across the cross-section of the conductor in time dt . In SI units, I is in amperes (A) when Q is measured in coulombs and t in seconds.

13.1.2 Ohm's Law The ratio of the potential difference V between any two points in a conductor and the current I flowing through it is constant and equals the resistance R between the two points. Thus

$$R = \frac{V}{I} \quad (13.2)$$

Equation (13.2) expresses the well-known Ohm's law. If V is expressed in volts (V) and current I in amperes (A), the unit of resistance R is the *ohm*, denoted by the symbol Ω . Ohm's law is valid only for metallic resistors at a definite temperature and for steady currents. The law is approximately valid for other types of resistors but fails for semiconductors and gases. We shall, however, be concerned only with those resistors for which Ohm's law holds. The inverse of resistance is called conductance.

13.1.3 Resistivity For a conductor of length l and uniform cross-sectional area A ,

$$R \propto \frac{l}{A}$$

$$R = \rho \frac{l}{A} \quad (13.3)$$

where ρ is a constant for the material of the conductor at the given temperature and is known as the resistivity of the conductor. It is also called the specific resistance of the material. The unit for resistivity is ohm-metre ($\Omega \text{ m}$).

The resistivity of a conductor depends upon its temperature. For most conductors the resistivity increases with the increase in temperature; these conductors have a positive temperature coefficient of resistance. But, there are a few exceptions for which resistivity decreases with increase in temperature; these materials are said to have a negative temperature coefficient of resistance. Some examples are carbon and semi-conductors.

For metallic conductors, we have, for small temperature changes,

$$\rho = \rho_0 (1 + \alpha t) \quad (13.4)$$

where ρ is the resistivity at $t^\circ \text{C}$, ρ_0 is the resistivity at 0°C and α is the temperature coefficient of resistance in $(^\circ \text{C})^{-1}$, based on a reference temperature of 0°C . If ρ_2, ρ_1 are the resistivities at temperatures t_1 and t_2 , then

$$\rho_2 = \rho_1 [1 + \alpha (t_2 - t_1)] \quad (13.5)$$

For any given conductor, the resistance R is proportional to its resistivity ρ , so Eqs (13.4) and (13.5) are equivalent to

$$R = R_0 (1 + \alpha t) \quad (13.6)$$

and
$$R_2 = R_1 [1 + \alpha (t_2 - t_1)] \quad (13.7)$$

13.1.4 Series and Parallel Resistors

(a) *Series Resistors* When resistors are connected in series, as shown in Fig. 13.1, the total resistance R is equal to the sum of the individual resistances, i.e.

$$R = R_1 + R_2 + R_3 + \dots \quad (13.8)$$

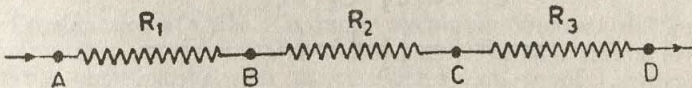


Fig. 13.1

In a series arrangement, the current is the same in all resistors. The total potential difference between A and B is equal to the sum of the potential differences across R_1, R_2 and R_3 .

(b) *Parallel resistors* In Fig. 13.2 three resistors are shown connected in parallel between the points A and B . The effective resistance R of the combination is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \quad (13.9)$$

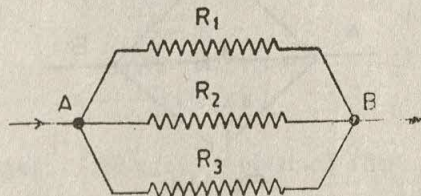


Fig. 13.2

In a parallel arrangement, the potential difference is the same across each resistor and the total current is the sum of the individual currents in the resistors. The current in each resistor is inversely proportional to the resistance.

13.1.5 Electromotive Force There must be a source of energy in a circuit in which a continuous current exists. The source may be a battery or an electrical generator which provide energy and power. A battery is said to be in 'open circuit' when practically no current flows from the battery (for example, when a high resistance voltmeter is connected across its terminals). The electromotive force E of the battery (or the generator) is the potential difference across its terminals on an open circuit.

The *terminal voltage* of a battery or a generator when it delivers a current I is equal to the total electromotive force E minus the potential drop in its internal resistance r . This potential drop cannot be observed using a voltmeter (which reads the terminal potential difference). Thus

$$\text{Terminal voltage} = E - I r$$

where $I r$ is the voltage drop across internal resistance.

The terminal potential difference (pd) increases as the external resistance R increases (correspondingly the current in the circuit decreases). When R has an infinitely large value, the terminal pd becomes equal to the emf E .

$$E = I R + I r$$

$$\text{Terminal pd } V = I R = E - I r \quad (13.10)$$

13.1.6 Cells in Series and Parallel The total emf E when cells are connected in series (Fig. 13.3) is given by

$$E = E_1 + E_2 + E_3 + E_4 + \dots \quad (13.11)$$

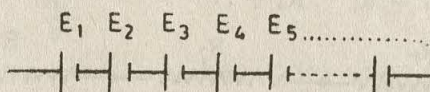


Fig. 13.3

The total internal resistance is equal to the sum of the individual internal resistances.

When identical cells are connected in parallel (Fig. 13.4), the total emf = E , the emf of any one of the cells. Here the sum of the reciprocals of the individual internal resistances is equal to the reciprocal of the total internal resistance. The calculation of the total emf of a parallel combination of different cells is more complicated. This problem can be tackled by using Kirchhoff's laws, described later in this chapter.

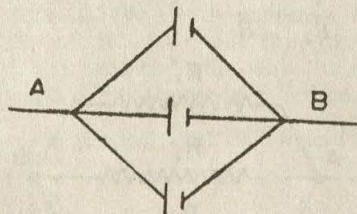


Fig. 13.4

Electromotive force (emf) is measured in joules per coulomb or volts.

13.1.7 Energy, Power and Heating Effect When a current I flows for a time t from a source of emf E , then the amount of charge that flows in time t is

$$Q = I t$$

$$\text{Electrical energy delivered } W = Q E = E I t \quad (13.12)$$

$$\text{Electrical power generated } P = \frac{W}{t} = E I \quad (13.13)$$

If I is in amperes and E in volts, power P will be in watts. If t is in seconds, energy W is in joules.

The power delivered to the external resistor is called the output power and is given by

$$P_{\text{out}} = V I = I^2 R = \frac{V^2}{R} \quad (13.14)$$

$$E I = I^2 R + I^2 r$$

where $I^2 r$ is the power consumed by the internal resistance. Thus

$$P_{\text{out}} = \frac{E^2 R}{(R + r)^2} \quad (13.15)$$

R is called the load resistor. The power output is maximum when $R = r$.

An electrical current flowing through a conductor produces heat in it. This is known as the Joule effect. The heat developed in Joules is given by

$$H = I^2 R t \quad (13.16)$$

To get heat in calories, H is divided by 4.2. $J = 4.2$ joule per calorie is called the mechanical equivalent of heat.

13.1.8 Combination of Cells To obtain maximum potential difference or maximum current in an external circuit, a number of cells can be arranged in a suitable combination. Let us consider a number of identical cells arranged in n rows, each row consisting of m cells as shown in Fig. 13.5.

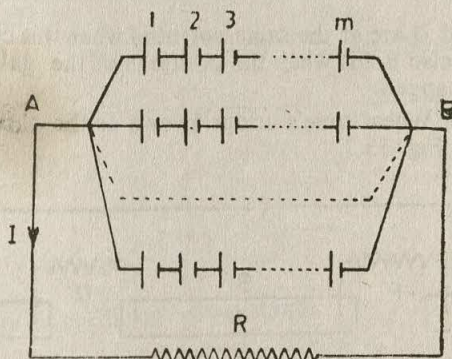


Fig. 13.5

The maximum current in the circuit is obtained when

$$n R = m r$$

where R is the resistance of the external circuit and r is the internal resistance of each cell. If E is the emf of each cell, the maximum current is given by

$$I_{\max} = \frac{m n E}{n R + m r} = \frac{m E}{2 R} = \frac{n E}{2 r} \quad (13.17)$$

When the current is maximum, the power in the external circuit is also maximum.

13.1.9 Wheatstone's Bridge and its Applications The network of resistances shown in Fig. 13.6 connected through a galvanometer is known as the Wheatstone's bridge.

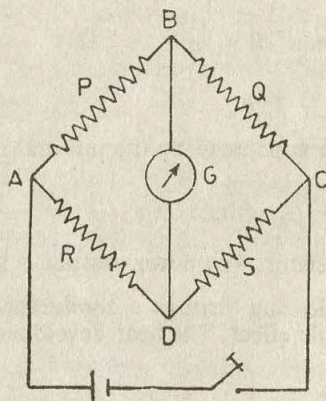


Fig. 13.6

The resistances P , Q , R and S can be adjusted to have no current in the galvanometer, i.e. to have no deflection in the galvanometer. The bridge is then said to be balanced. For a balanced bridge, we have

$$\frac{P}{Q} = \frac{R}{S} \quad (13.18)$$

The points B and D are at the same potential when this condition is satisfied. This relation also holds when the positions of the galvanometer and battery are interchanged.

A simple form of Wheatstone's bridge known as the slide-wire on metre bridge is shown in Fig. 13.7.

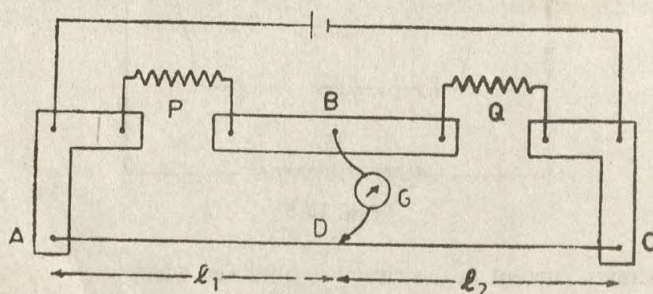


Fig. 13.7

Here resistances R and S are replaced by a uniform wire of length l m on which contact can be made by means of a slider. The point of contact on the wire corresponds to the point D . Since the wire is uniform, the resistances of the two segments of the wire are proportional to their lengths. Thus, when the bridge is balanced, we have

$$\frac{P}{Q} = \frac{R}{S} = \frac{l_1}{l_2}$$

If one of the resistances P or Q is known, the other can be determined.

13.1.10 Kirchhoff's Laws Kirchhoff's first law is based upon the law of conservation of charge and states that the algebraic sum of the currents entering any junction point in a circuit is zero. In a given time, as much charge flows into any point as flows away from that point.

Kirchhoff's second law (the loop theorem) is based on the law of conservation of energy. According to this law, the algebraic sum of the changes in potential around any closed path is zero.

Kirchhoff's laws are useful for the systematic application of Ohm's law to electrical circuits.

13.2 SI UNITS

Table 13.1

<i>Physical Quantity</i>	<i>SI Unit</i>	<i>Symbol</i>
Current	ampere	A
Charge	coulomb	C
Potential difference	volt	V
Electromotive force	volt	V
Resistance	ohm	Ω
Resistivity or specific resistance	ohm metre	$\Omega \text{ m}$
Temperature coefficient of resistance	per kelvin	K^{-1}
Electrical energy	joule	J
Heat energy	joule	J
Power	joule per second or watt	J s^{-1} or W

13.3 FORMULAE

1. Ohm's law $V = IR$

2. Resistivity $\rho = \left(\frac{A}{l}\right) R$

3. Temperature coefficient of resistance

$$\alpha = \frac{\rho - \rho_0}{t} = \frac{R - R_0}{t}$$

4. Resistances in series $R = R_1 + R_2 + R_3 + \dots$

5. Resistances in parallel $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \dots$

6. Terminal pd $V = IR = E - Ir$
7. Cells in series $E = E_1 + E_2 + E_3 + \dots$
8. Electrical energy $W = EIt$
9. Heat produced in a resistor $H = I^2 R t = VIt$
10. Power dissipated in a resistor $P = VI = I^2 R$
11. Maximum current in n rows of m cells each

$$I_{\max} = \frac{mE}{2R} = \frac{nE}{2r}$$

12. Wheatstone's bridge, when balanced: $\frac{P}{Q} = \frac{R}{S}$
13. Kirchhoff's law
 - (i) For a junction: $I_1 + I_2 + I_3 + \dots = 0$
 - (ii) For a loop: $I_1 R_1 + I_2 R_2 + I_3 R_3 + \dots = \text{sum of potential drops.}$

13.4 NUMERICAL EXAMPLES

Example 1: What is the net resistance between points A and F in the circuit shown in Fig. 13.8(a) ?

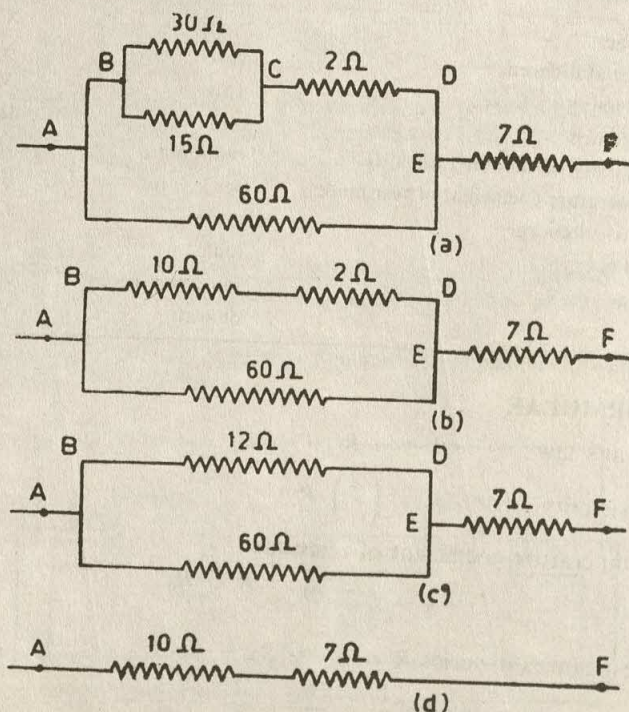


Fig. 13.8

Solution: First consider the parallel combination BC . The resistance of this combination is given by

$$\frac{1}{R} = \frac{1}{30} + \frac{1}{15} = \frac{3}{30} \quad \text{or} \quad R = 10 \, \Omega$$

Substitute $10 \, \Omega$ for the combination and redraw the circuit as shown in Fig. 13.8 (b). Now the series combination BD can be replaced by a single resistor of resistance

$$R = 10 + 2 = 12 \, \Omega$$

Now the circuit can be redrawn as in Fig. 13.8 (c). Combination AE in the parallel arrangement can be replaced by a single resistor of resistance given by

$$\frac{1}{R} = \frac{1}{12} + \frac{1}{60} = \frac{6}{60} \quad \text{or} \quad R = 10 \, \Omega$$

The circuit can again be redrawn as shown in Fig. 13.8 (d). The resistance between points A and F is then given by

$$R = 10 + 7 = 17 \, \Omega$$

Example 2: What are the currents through the $3 \, \Omega$, $5 \, \Omega$ and $6 \, \Omega$ resistors in the circuit shown in Fig. 13.9 (a)? Also find the terminal voltage of each battery.

Solution: To find the total current, the circuit should first be simplified. The combination of $3 \, \Omega$ and $6 \, \Omega$ resistors gives

$$\frac{1}{R} = \frac{1}{6} + \frac{1}{3} = \frac{3}{6} \quad \text{or} \quad R = 2 \, \Omega$$

The original circuit now reduces to that shown in Fig. 13.9 (b). Now the entire circuit has a series combination which gives

$$R = 1 + 5 + 1 + 2 = 9 \, \Omega$$

Since the two batteries would send charges in opposite directions, the net emf is $(60 - 6) = 54 \, \text{V}$. Applying Ohm's law to the entire circuit, the current through each battery and the $5 \, \Omega$ resistor is given by

$$I = \frac{54}{9} = 6 \, \text{A}$$

Now we shall calculate the currents through the $3 \, \Omega$ and $6 \, \Omega$ resistors. In Fig. 13.9(b), the net resistance between A and B is $2 \, \Omega$ and a current of $6 \, \text{A}$ flows. Hence the pd across each resistor in Fig. 13.9(a) is

$$V = IR = 6 \times 2 = 12 \, \text{V}$$

The current through each resistor is

$$I_1 = \frac{12}{6} = 2 \, \text{A}$$

$$I_2 = \frac{12}{3} = 4 \, \text{A}$$

Since I_1 and I_2 add up to $6 \, \text{A}$, which is the total current entering through point A (Kirchhoff's law).

In Fig. 13.9(c), we calculate the (Ir) drops through the internal resistances, keeping in mind the signs. The terminal voltage of the 60 V battery is

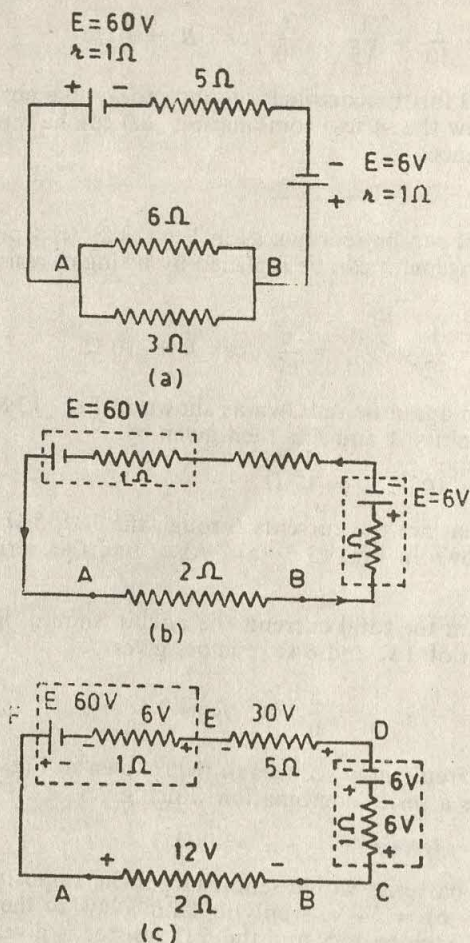


Fig. 13.9

found by going from E to F through the battery; a drop of 6 V is followed by a rise of 60 V, so the net pd is $(-6 + 60) = +54$ V, with F positive relative to E . Similarly, in going through the other battery from D to C , there is a rise of 6 V followed by a rise of 6 V and, therefore, the net pd is $(+6 + 6) = +12$ V, the terminal voltage of the battery with C positive relative to D .

Example 3: When a resistor of 20Ω is connected in series with a battery, the current is 0.5 A. When a resistor of 10Ω is connected, the current becomes 0.8 A. Calculate the emf and the internal resistance of the battery.

Solution: Let E be the emf of the battery and r its internal resistance.

Then $E = 0.5(20 + r)$ (1)

Also $E = 0.8(10 + r)$ (2)

$$0.5(20 + r) = 0.8(10 + r)$$

or $10 + 0.5r = 8 + 0.8r$

or $0.3r = 2, \quad r = 6.67 \, \Omega$

Hence $E = 0.5 \times 26.67 = 13.34 \, \text{V}$

Example 4: Determine the total resistance in the circuit shown in Fig. 13.10. Also determine the current in each resistor and the potential drop across each resistor.

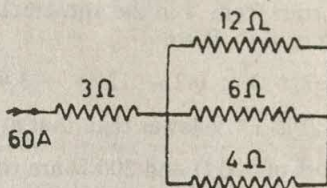


Fig. 13.10

Solution: The total resistance of the parallel combination is given by

$$\frac{1}{R} = \frac{1}{12} + \frac{1}{6} + \frac{1}{4}$$

$\therefore R = 2 \, \Omega$

Total resistance of the circuit is given by

$$R = 2 + 3 = 5 \, \Omega$$

Current through the $3 \, \Omega$ resistor = $60 \, \text{A}$

Potential difference across the $3 \, \Omega$ resistor = $60 \times 3 = 180 \, \text{V}$

The potential difference across the parallel combination will be the same for each resistor and is equal to $60 \times 2 = 120 \, \text{V}$. Using Ohm's law, the currents in the three resistors are as follows.

Current in the 12 resistor = $\frac{120}{12} = 10 \, \text{A}$

Current in the 6 resistor = $\frac{120}{6} = 20 \, \text{A}$

Current in the 4 resistor = $\frac{120}{4} = 30 \, \text{A}$

Example 5: What is the potential difference between points A and D in the circuit shown in Fig. 13.11? Which point is at a higher potential?

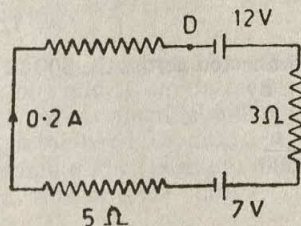


Fig. 13.11

Solution: The current flows in a resistor from a point at a higher potential to a point at a lower potential. The current is the same in all resistors in a series circuit and flows in a direction as driven by the battery of a higher emf. Starting from point D in the clockwise direction, the potential changes up to A are

$$12 - 3 \times 0.2 - 7 - 5 \times 0.2 = 12 - 0.6 - 7 - 1.0 = 3.4 \text{ V}$$

Thus the potential at A is higher by 3.4 V than that at D .

Alternatively, we can start from A in the anti-clockwise direction. Then the potential changes from A to D are

$$5 \times 0.2 + 7 + 3 \times 0.2 - 12 = -3.4 \text{ V}$$

Thus the potential at D is 3.4 V lower than that at A .

Example 6: Two resistors of 500Ω and 300Ω are connected in series with a battery of emf 20 V and of negligible internal resistance. A voltmeter with a resistance of 500Ω is used to measure the potential difference across the terminals of the resistors. Find the error in the measurements.

Solution: Total resistance in series with the battery when the voltmeter is not connected is (see Fig. 13.12)

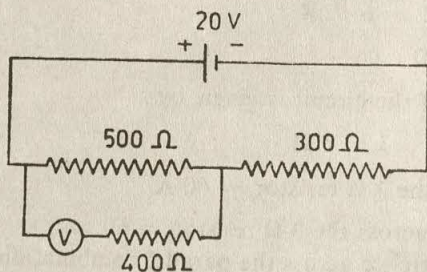


Fig. 13.12

$$R = 300 + 500 = 800 \Omega$$

$$\text{Current } I = \frac{20}{800} = 0.025 \text{ A}$$

The same current flows through a series circuit. Hence

$$\begin{aligned} \text{Potential difference across the } 500 \Omega \text{ resistor} &= 500 \times 0.025 \\ &= 12.5 \text{ V} \end{aligned}$$

$$\begin{aligned} \text{Potential difference across the } 300 \Omega \text{ resistor} &= 300 \times 0.025 \\ &= 7.5 \text{ V} \end{aligned}$$

When the voltmeter is connected across the 500Ω resistor, the resistance of the combination is given by

$$\frac{1}{R} = \frac{1}{500} + \frac{1}{500}$$

or

$$R = 250 \Omega$$

$$\text{Total resistance} = 250 + 300 = 550 \Omega$$

$$\text{Voltmeter reading} = \frac{250}{550} \times 20 = 9.1 \text{ V}$$

$$\text{Error in the value} = 12.5 - 9.1 = 3.4 \text{ V}$$

When the voltmeter is connected across the 300Ω resistor, the combined resistance is given by

$$\frac{1}{R} = \frac{1}{500} + \frac{1}{300} = \frac{8}{1500}$$

or
$$R = \frac{1500}{8} \Omega$$

$$\text{Total resistance} = 500 + \frac{1500}{8} = \frac{5500}{8} \Omega$$

$$\text{Voltmeter reading} = \frac{1500}{8} \times \frac{8}{5500} \times 20 = 5.5 \text{ V}$$

$$\text{Error in the value} = 7.5 - 5.5 = 2.0 \text{ V}$$

Example 7: What will be the error in the voltmeter readings in Ex. 6 above if the voltmeter has a resistance of 1500Ω ?

Solution: When the voltmeter is connected across the 500Ω resistor, the combined resistance is

$$\frac{500 \times 1500}{500 + 1500} = \frac{750,000}{2000} = 375 \Omega$$

$$\text{Total resistance in the circuit} = 375 + 300 = 675 \Omega$$

$$\text{Voltmeter reading} = \frac{375}{675} \times 20 = 11.1 \text{ V}$$

$$\text{Error} = 12.5 - 11.1 = 1.4 \text{ V}$$

Similarly, when the voltmeter is connected across the 300Ω resistor, the combined resistance is

$$\frac{300 \times 1500}{300 + 1500} = 250 \Omega$$

$$\text{Total resistance} = 500 + 250 = 750 \Omega$$

$$\text{Voltmeter reading} = \frac{250}{750} \times 20 = 6.7 \text{ V}$$

$$\text{Error} = 7.5 - 6.7 = 0.8 \text{ V}$$

Example 8: In a power station, a copper bar designed to carry many amperes of current is 2 m long and 10 cm^2 in cross-section. Determine the resistance of the bar at 0°C . What potential difference is needed to cause a current of 5000 A through the bar? The resistivity of copper at 0°C is $1.59 \times 10^{-8} \Omega \text{ m}$. Also compute the resistance of the bar if it is stretched to form a long and uniform wire of 1 mm^2 cross-section.

Solution: The cross-section of the bar $= 10 \text{ cm}^2 = 10 \times 10^{-4} = 10^{-3} \text{ m}^2$

The resistance of the bar is

$$R = \frac{\rho l}{A} = \frac{1.59 \times 10^{-8} \times 2}{10^{-3}} = 3.18 \times 10^{-5} \Omega$$

Using Ohm's law, the potential difference across its ends is

$$V = IR = 5000 \times 3.18 \times 10^{-5} = 0.159 \text{ V}$$

$$\text{Volume of the bar} = 2 \times 10^{-3} \text{ m}^3$$

$$\text{Area of cross-section of the wire} = 1 \text{ mm}^2 = 10^{-6} \text{ m}^2$$

$$\text{Length of the wire } l = \frac{\text{volume}}{\text{area}} = \frac{2 \times 10^{-3}}{1 \times 10^{-6}} = 2 \times 10^3 \text{ m}$$

The wire has the same amount of copper as the bar, but it is 2000 m long. The resistance of the wire is

$$R = \frac{\rho l}{A} = \frac{1.59 \times 10^{-8} \times 2000}{10^{-6}} = 31.8 \Omega$$

Example 9: All resistances in Fig. 13.13 below are in ohms. Find the effective resistance between the points *A* and *B*. (IIT 1979)

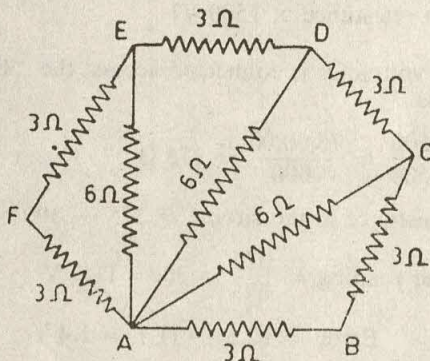


Fig. 13.13

Solution: Resistors *AF* and *FE* are in series with each other. Therefore, network *AEF* reduces to a parallel combination of two resistors of 6 Ω each. Thus, the resistance between *A* and *E* is given by

$$\frac{1}{R} = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} \quad \text{or} \quad R = 3 \Omega$$

Similarly, the resistance between *A* and *D* is given by

$$\frac{1}{\frac{1}{6} + \frac{1}{6}} = 3 \Omega$$

Now, resistor *AC* is in parallel with the series combination of *AD* and *DC*. Therefore, the resistance between *A* and *C* is

$$\frac{1}{\frac{1}{6} + \frac{1}{6}} = 3 \Omega$$

$AC + CB = 3 + 3 = 6 \Omega$. Since they are in series, the resistance between A and B is given by

$$\frac{1}{R} = \frac{1}{6} + \frac{1}{3} = \frac{3}{6} \quad \text{or} \quad R = 2 \Omega$$

Example 10: In Fig. 13.14 below, find the potential difference between the points A and B and between the points B and C in the steady state.

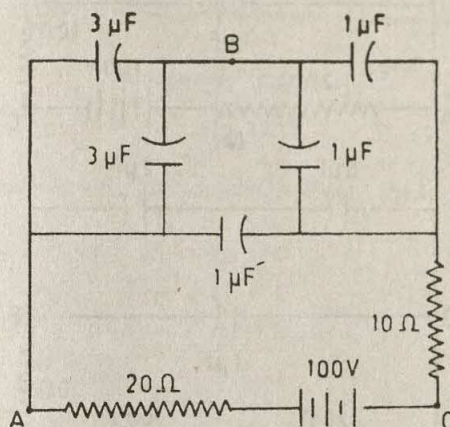


Fig 13.14

Solution: Figure 13.14 can be redrawn as shown in Fig. 13.15 (a). The parallel combination of capacitors results in the effective capacitance values shown in Fig. 13.15 (b) which is further simplified to Fig. 13.15 (c) by considering the series capacitors.

The potential difference between D and E is equal to that between F and G .

$$V = 100 \text{ volts}$$

$$\text{Capacitance } C = 1.5 \mu\text{F} = 1.5 \times 10^{-6} \text{ F}$$

$$\text{Charge } Q = CV = 1.5 \times 10^{-6} \times 100 = 1.5 \times 10^{-4} \text{ C}$$

The charge on the $6 \mu\text{F}$ and $2 \mu\text{F}$ capacitors is the same; since they are connected in series and equals $1.5 \times 10^{-4} \text{ C}$.

The potential difference between A and B is equal to that between the two ends of the $6 \mu\text{F}$ capacitor.

$$V_{AB} = \frac{Q}{C} = \frac{1.5 \times 10^{-4}}{6 \times 10^{-6}} = 25 \text{ V}$$

is the potential difference between A and B . The potential difference between B and C is

$$V_{BC} = \frac{1.5 \times 10^{-4}}{2 \times 10^{-6}} = 75 \text{ V}$$

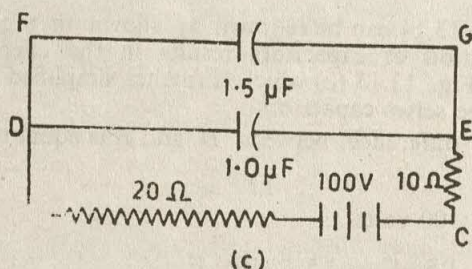
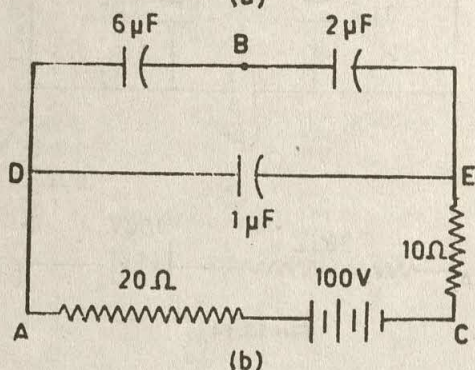
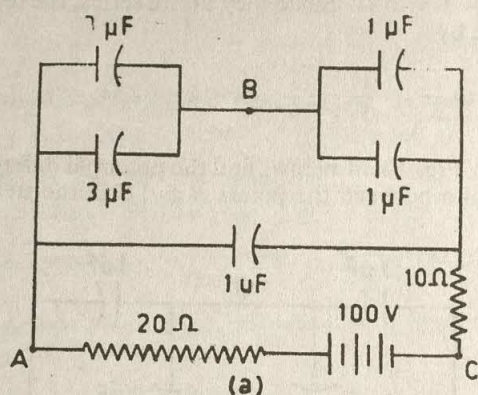


Fig. 13.15

Example 11: In the circuit shown in Fig. 13.16, $E_1 = 3\text{ V}$, $E_2 = 2\text{ V}$, $E_3 = 1\text{ V}$ and $R = r_1 = r_2 = r_3 = 1\text{ }\Omega$.

- (i) Find the potential difference between the points A and B , and the current through each branch.

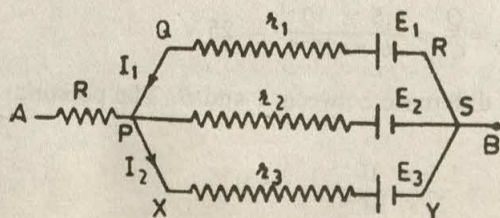


Fig. 13.16

- (ii) If r_2 is short-circuited and the point A is connected to point B find the currents through E_1 , E_2 , E_3 and the resistor R . (IIT 1981)

Solution: (i) Let I_1 , I_2 and I_3 be the currents flowing through the resistances r_1 , r_2 and r_3 respectively as indicated in Fig. 13.17.

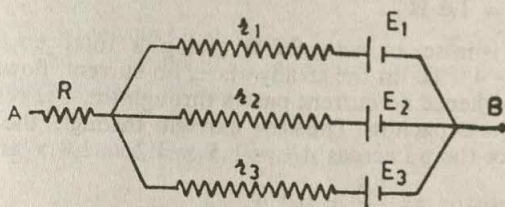


Fig. 13.17

Using Kirchhoff's law for the loop $PQRSP$ and the loop $PQRSYXP$, we have

$$I_1 r_1 + I_2 r_2 + E_2 - E_1 = 0$$

$$I_3 r_3 + E_3 - E_1 + I_1 r_1 = 0$$

$$E_2 + I_2 r_2 = E_1 - I_1 r_1 = E_3 + I_3 r_3$$

At point P , $I_1 = I_2 + I_3$

$$E_1 - (I_2 + I_3) r_1 = E_3 + I_3 r_3$$

$$3 - I_2 - I_3 = 1 + I_3$$

$$2 I_3 + I_2 = 2$$

(i)

Also $E_2 + I_2 r_2 = E_3 + I_3 r_3$

$$2 + I_2 = 1 + I_3$$

or $I_3 - I_2 = 1$

(ii)

From Eqs (i) and (ii) we have

$I_1 = 1$ A, $I_2 = 0$ and $I_3 = 1$ A. Hence, the potential difference between A and B is equal to 2 V.

(ii) When r_2 is short-circuited, the potential difference across PS is still 2 V and the current through R is 2 A. The current through E_2 is 2 A and the currents through E_1 and E_3 remain unchanged.

Example 12: Calculate the steady-state current in the $2\ \Omega$ resistor shown in Fig. 13.18. The internal resistance of the battery is negligible and the capacitance of the condenser is $0.2\ \mu\text{F}$. (IIT 1982)

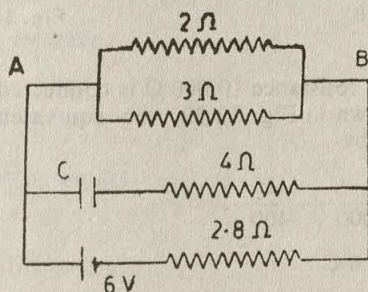


Fig. 13.18

Solution: The resistance of the parallel combination of $2\ \Omega$ and $3\ \Omega$ resistors is given by

$$\frac{1}{R} = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$

which gives $R = 1.2\ \Omega$

This resistance is in series with $2.8\ \Omega$ giving a total effective resistance $= 1.2 + 2.8 = 4.0\ \Omega$. In the steady state, no current flows through the capacitor C and hence no current passes through the $4\ \Omega$ resistor which is in series with the capacitor. Thus the current through the circuit $= 6/4 = 1.5\ \text{A}$. Hence the pd across $AB = 1.5 \times 1.2 = 1.8\ \text{V}$ and the current through $2\ \Omega$ resistor $= \frac{1.8}{2} = 0.9\ \text{A}$.

Example 13: Two resistors $400\ \Omega$ and $800\ \Omega$ are connected in series with a $6\ \text{V}$ battery. It is desired to measure the current in the circuit. An ammeter of $10\ \Omega$ resistance is used for this purpose. What will be the reading of the ammeter? Similarly, if a voltmeter of $10,000\ \Omega$ resistance is used to measure the potential difference across the $400\ \Omega$ resistor, what will be the reading of the voltmeter? (IIT 1982)

Solution: When an ammeter of resistance $10\ \Omega$ is connected in series, as shown in Fig. 13.19, the total resistance in the circuit is

$$800 + 400 + 10 = 1210\ \Omega$$

The ammeter will indicate a current of $= \frac{6}{1210}$
 $= 0.00496\ \text{A}$
 $= 4.96\ \text{mA}$

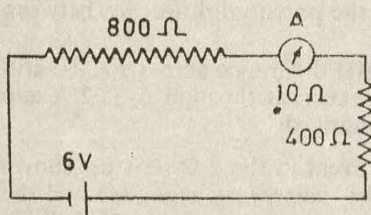


Fig. 13.19

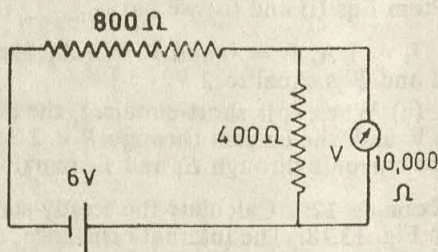


Fig. 13.20

If a voltmeter of resistance $10,000\ \Omega$ is connected in parallel with the $400\ \Omega$ resistor, as shown in Fig. 13.20, the equivalent resistance of the combination is given by

$$\frac{1}{R} = \frac{1}{10,000} + \frac{1}{400}$$

which gives $R = 384.6\ \Omega$

Total resistance in the circuit $= 800 + 384.6 = 1184.6\ \Omega$

The current in the circuit = $\frac{6}{1184.6} = 5.065 \times 10^{-3} \text{ A}$

The potential difference indicated by the voltmeter is

$$384.6 \times 5.065 \times 10^{-3} = 1.95 \text{ V}$$

Example 14: In the series circuit shown in Fig. 13.21, E , F , G and H are cells of emf 2, 1, 3 and 1 V respectively, and their internal resistances are 2, 1, 3 and 1 Ω respectively. Calculate

- the potential difference between B and D and
 - the potential difference across the terminals of each of the cells G and H .
- (IIT 1984)

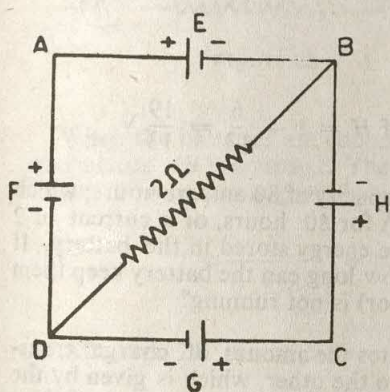


Fig. 13.21

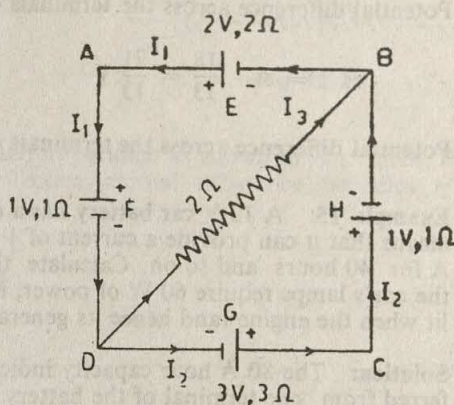


Fig. 13.22

Solution: Let I_1 and I_2 be the currents flowing along BAD and DCB respectively, as shown in Fig. 13.22. Let I_3 be the current flowing through the 2 Ω resistor along DB .

Applying Kirchhoff's law at the junction D , we have

$$I_1 = I_2 + I_3$$

or
$$I_3 = I_1 - I_2$$

Applying Kirchhoff's law to the loop BAD , we have

$$2I_1 + I_1 + 2(I_1 - I_2) = 2 - 1 = 1$$

or
$$5I_1 - 2I_2 = 1 \quad \text{(i)}$$

For the loop DCB , we have

$$3I_2 + I_2 - 2(I_1 - I_2) = 3 - 1 = 2$$

or
$$6I_2 - 2I_1 = 2 \quad \text{(ii)}$$

Multiplying Eq. (i) by 3, we get

$$15I_1 - 6I_2 = 3 \quad \text{(iii)}$$

Adding Eqs (ii) and (iii) we have

$$13I_1 = 5 \quad \text{or} \quad I_1 = \frac{5}{13} \text{ A}$$

Using this value of I_1 in Eq. (ii), we get

$$I_2 = \frac{6}{13} \text{ A}$$

Therefore $I_3 = I_1 - I_2 = \frac{5}{13} - \frac{6}{13} = -\frac{1}{13} \text{ A}$

Thus the current through the 2Ω resistor flows along BD . The potential difference across B and $D = 2 \times \frac{1}{13} = \frac{2}{13} \text{ V}$. Therefore,

$$\begin{aligned} \text{Potential difference across the terminals of } G &= 3 - \frac{6}{13} \times 3 \\ &= 3 - \frac{18}{13} = \frac{21}{13} \text{ V} \end{aligned}$$

$$\text{Potential difference across the terminals of } H = 1 + \frac{6}{13} = \frac{19}{13} \text{ V.}$$

Example 15: A 12 V car battery has a capacity of 80 ampere hours, which means that it can provide a current of 1 A for 80 hours, or a current of 2 A for 40 hours and so on. Calculate the energy stored in the battery. If the car's lamps require 60 W of power, how long can the battery keep them lit when the engine (and hence its generator) is not running?

Solution: The 80 A hour capacity indicates the amount of charge transferred from one terminal of the battery to the other, which is given by the product of current (in A) and the time (in Seconds).

$$Q = I \times t = 80 \times 60 \times 60 = 2.88 \times 10^5 \text{ C}$$

The energy provided by the battery is

$$E = QV = 2.88 \times 10^5 \times 12 = 3.46 \times 10^6 \text{ J}$$

Since power $P = \frac{E}{t}$, the time for which the battery will last is

$$\begin{aligned} t = \frac{E}{P} &= \frac{3.46 \times 10^6}{60} = 5.8 \times 10^5 \text{ s} \\ &= 16 \text{ hours} \end{aligned}$$

Example 16: A galvanometer together with an unknown resistance in series is connected across two identical batteries each of 1.5 V. When the batteries are connected in series, the galvanometer records a current of 1 A, and when the batteries are in parallel, the current is 0.6 A. What is the internal resistance of the battery? (IIT 1973)

Solution: Let r be the internal resistance of each battery. If R is the total resistance of the galvanometer and the unknown resistor, then the current I in the circuit is given by

$$I = \frac{2E}{R + 2r}$$

$$1.0 = \frac{2 \times 1.5}{R + 2r} \quad \text{or} \quad R + 2r = 3.0 \quad (i)$$

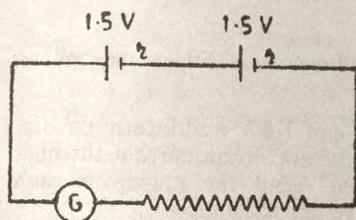


Fig. 13.23

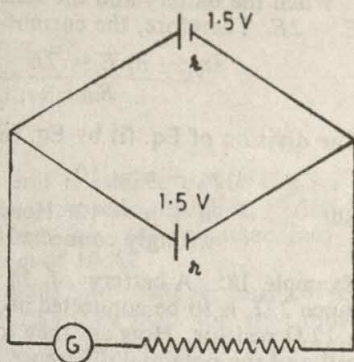


Fig. 13.24

When the batteries are connected in parallel, as shown in Fig. 13.24, the emf across AB is constant. The effective internal resistance becomes $r/2$. Hence the current is given by

$$I = \frac{E}{R + \left(\frac{r}{2}\right)} = 0.6$$

$$\text{which gives } 2R + r = 5.0 \quad \text{(ii)}$$

Multiplying Eq. (i) by 2, we have

$$2R + 4r = 6.0 \quad \text{(iii)}$$

Equations (ii) and (iii) give

$$r = \frac{1}{3} \Omega$$

Example 17: Twelve cells each having the same emf are connected in series and are kept in a closed box. Some of the cells are wrongly connected. This battery of cells is connected in series with an ammeter and two cells identical with the others. The current is 3 A when the cells and the battery aid each other and is 2 A when the cells and the battery oppose each other. How many cells in the battery are wrongly connected? (IIT 1976)

Solution: If m cells are connected correctly and n cells are connected wrongly, we have

$$m + n = 12$$

If E is the emf of each cell, the total emf of the battery is $(m - n)E$.

When the battery and the cells aid each other, the net emf

$$= (m - n)E + 2E$$

If R is the total resistance of the circuit, the current is given by

$$I = \frac{(m - n)E + 2E}{R} = 3 \quad \text{(i)}$$

When the battery and the cells oppose each other, the net emf is $(m - n)E - 2E$. Therefore, the current is

$$\frac{(m - n)E - 2E}{R} = 2 \quad (\text{ii})$$

The division of Eq. (i) by Eq. (ii) gives

$$m - n = 10$$

But $m + n = 12$. Hence $m = 11$ and $n = 1$. Thus, one cell is wrongly connected.

Example 18: A battery of 24 cells, each of emf 1.5 V and internal resistance $2\ \Omega$, is to be connected in order to send the maximum current through a $12\ \Omega$ resistor. How are they to be connected? Find the current in each cell and the potential difference across the external resistance.

Solution: Let x be the number of cells in each row and let there be y rows in parallel.

$$\text{Total number of cells} = xy = 24$$

$$\text{Resistance of each row} = 2x \text{ ohms}$$

$$\text{Total internal resistance} = \frac{2x}{y} \text{ ohms}$$

(because there are y rows in parallel).

The maximum current passes through the circuit when the internal resistance of the battery of cells equals the external resistance.

$$\text{Thus } \frac{2x}{y} = 12 \quad \text{or} \quad \frac{x}{y} = 6$$

$$\text{But } xy = 24.$$

$$\text{Hence } x = 12 \text{ and } y = 2$$

i.e. there should be two rows of 12 cells in series (see Fig. 13.25).

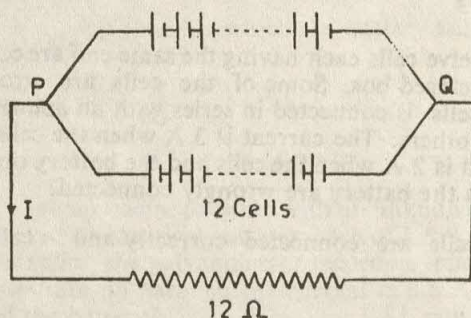


Fig. 13.25

The current in the circuit is

$$\begin{aligned} I &= \frac{\text{Total emf}}{\text{Total resistance}} = \frac{1.5 \times 12}{12 + 12} \\ &= \frac{18}{24} = 0.75 \text{ A} \end{aligned}$$

Because the two rows have the same resistance, the current in each arm must be $= 0.375 \text{ A}$

Therefore, current through each cell $= 0.375 \text{ A}$

The potential difference across the external resistance is

$$= 12 \times 0.75 = 9 \text{ V}$$

Example 19: A galvanometer having a coil resistance of 100Ω gives a full scale deflection when a current of 1 mA is passed through it. What is the value of the resistance which can convert this galvanometer into a meter giving full scale deflection for a current of 10 A ?

A resistance of the required value is available but it will get burnt, if the energy dissipated in it is greater than 1 W . Can it be used for the above described conversion of the galvanometer? When this modified galvanometer is connected across the terminals of a battery, it reads a current of 4 A . The current drops to 1.0 A when a resistance of 1.5Ω is connected in series with the modified galvanometer. Find the emf and the internal resistance of the battery. (IIT 1972)

Solution: Out of 10 A , the current passing through the galvanometer coil is $1 \text{ mA} = 0.001 \text{ A}$. The remaining current of $(10 - 0.001) = 9.999 \text{ A}$ must pass through a shunt of suitable resistance, say, S ohms, as shown in Fig. 13.26.

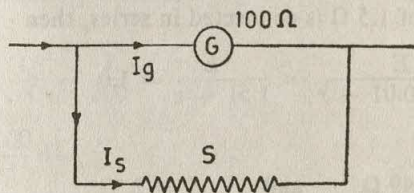


Fig. 13.26

To calculate the value of S , let the currents through the galvanometer and the shunt S be I_g and I_s respectively. Since S is connected in parallel with the galvanometer of resistance G , the potential difference across the two must be the same. Thus

$$I_g G = I_s S$$

Here $I_g = 0.001 \text{ A}$, $I_s = 9.999 \text{ A}$, $G = 100 \Omega$

$$S = \frac{I_g G}{I_s} = \frac{0.001 \times 100}{9.999} = \frac{1}{99.99} \Omega$$

$$\begin{aligned} \text{Power dissipated in the shunt} &= I^2 R = (9.999)^2 \times \frac{1}{99.99} \\ &= 0.9999 \text{ W} \end{aligned}$$

Since this is less than 1 W , the given resistance can be used as shunt. The shunt resistance is

$$S = \frac{1}{99.99} = 0.01 \Omega$$

The combined resistance of the galvanometer and the shunt is given by

$$\frac{1}{X} = \frac{1}{G} + \frac{1}{S} = \frac{G + S}{GS}$$

or
$$X = \frac{GS}{G + S} = \frac{100 \times 0.01}{100.01} = 0.01 \Omega$$

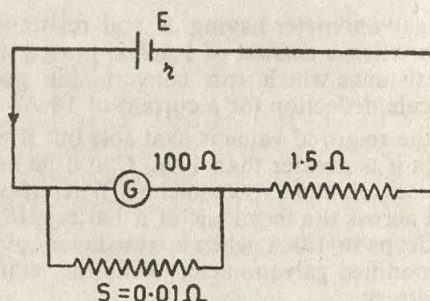


Fig. 13.27

Current
$$I = \frac{E}{0.01 + r} = 4 \quad (i)$$

When a resistance of 1.5Ω is connected in series, then

$$\frac{E}{1.5 + 0.01 + r} = \frac{E}{1.51 + r} = 1.0 \quad (ii)$$

Eqs (i) and (ii) give

$$r = 0.49 \Omega \quad \text{and} \quad E = 2 \text{ V},$$

Example 20: A battery of emf 1.4 V and internal resistance 2Ω is connected to a 100Ω resistor through an ammeter. The resistance of the ammeter is $4/3 \Omega$. A voltmeter is also connected to find the potential difference across the resistor.

- (i) Draw the circuit diagram.
- (ii) The ammeter reads 0.02 A . What is the resistance of the voltmeter?
- (iii) The voltmeter reads 1.10 V . What is the error in the reading?

(IIT 1975)

Solution: (i) Figure 13.28 shows the circuit diagram.

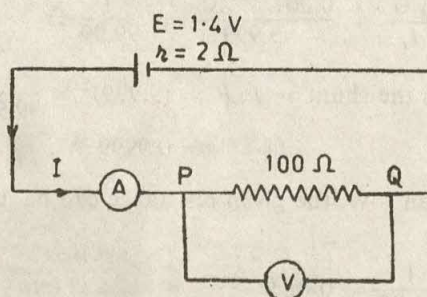


Fig. 13.28

(ii) Let R be the resistance of the voltmeter. Then the total resistance in the circuit is

$$(Y + 4/3 + 2)$$

where Y is given by

$$\frac{1}{R} + \frac{1}{100} = \frac{1}{Y} \quad \text{or} \quad Y = \frac{100 R}{100 + R}$$

$$\begin{aligned} \text{Therefore, } Y + \frac{4}{3} + 2 &= \frac{100 R}{100 + R} + \frac{10}{3} \\ &= \frac{310 R + 1000}{3(100 + R)} \end{aligned}$$

The current I is given by

$$I = \frac{\text{emf}}{\text{total resistance}} = \frac{1.4 \times 3(100 + R)}{310 R + 1000} = 0.02 \text{ A}$$

Therefore,

$$6.2 R + 20 = 4.2(100 + R)$$

which gives

$$R = 200 \Omega$$

(iii) The total resistance X of the 100Ω resistor and the voltmeter is given by

$$\begin{aligned} \frac{1}{X} &= \frac{1}{100} + \frac{1}{200} = \frac{3}{200} \\ X &= \frac{200}{3} \Omega \end{aligned}$$

$$\begin{aligned} \text{Potential difference across the voltmeter} &= \frac{200}{3} \times 0.02 \\ &= 1.33 \text{ V} \end{aligned}$$

$$\text{Voltmeter reading} = 1.10 \text{ V}$$

$$\text{Error} = 1.33 - 1.10 = 0.23 \text{ V}$$

Example 21: If each of the resistances in the network, shown in Fig. 13.29, is R , what is the resistance between the terminals A and B ? (IIT 1978)

Solution: The circuit diagram can be redrawn as shown in Fig. 13.30.

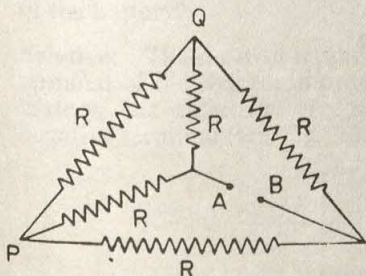


Fig. 13.29

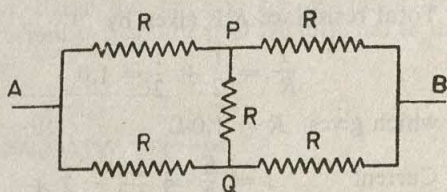


Fig. 13.30

This is equivalent to a balanced Wheatstone's bridge. Since points P and Q are at the same potential, no current flows in the resistance R connecting them. Resistance X between A and B is, therefore, given by

$$\frac{1}{X} = \frac{1}{2R} + \frac{1}{2R} = \frac{1}{R} \quad \text{or} \quad X = R$$

Example 22: A wire of resistance $0.1 \Omega \text{ cm}^{-1}$ is bent to form a square $ABCD$ of side 10 cm . A similar wire is connected between the corners B and D to form the diagonal BD . Find the effective resistance of this combination between corners A and C . If a 2 V battery of negligible internal resistance is connected across A and C , calculate the total power dissipated. (IIT 1971)

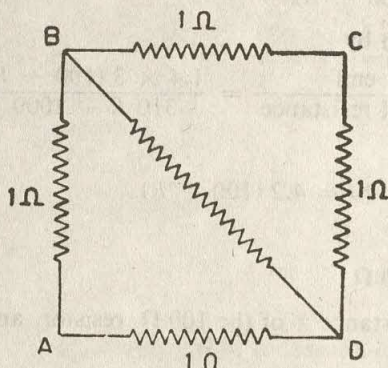


Fig. 13.31

Solution: Resistance of each side of the square $= 10 \times 0.1 = 1 \Omega$. As shown in Fig. 13.31, the square forms a Wheatstone's bridge which satisfies the balancing condition. Thus, no current flows along the diagonal BD . The combination, therefore, reduces to the one shown in Fig 13.32.

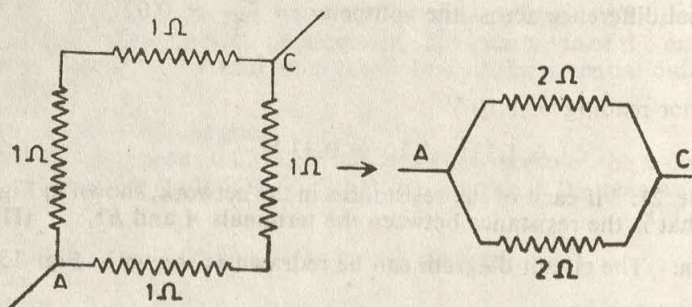


Fig. 13.32

Total resistance R is given by

$$\frac{1}{R} = \frac{1}{2} + \frac{1}{2} = 1.0$$

which gives $R = 1.0 \Omega$

Current
$$I = \frac{E}{R} = \frac{2.0}{1.0} = 2 \text{ A}$$

Power dissipated
$$P = EI = 2 \times 2 = 4 \text{ W}$$

Example 23: A potentiometer wire of length 100 cm has a resistance of $10\ \Omega$. It is connected in series with a resistance and an accumulator of emf 2 V and of negligible internal resistance. A source of emf 10 mV is balanced against a length of 40 cm of the potentiometer wire. What is the value of the external resistance? (IIT 1976)

Solution: Let XY be the potentiometer wire and R, the external resistance, as shown in Fig. 13.33.

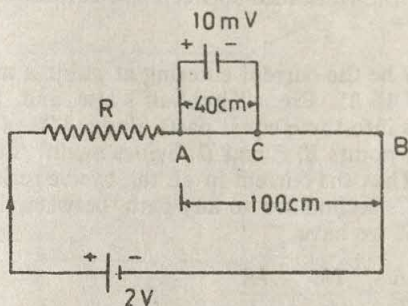


Fig. 13.33

Potential drop across the wire XY = current $\times$ resistance

$$= \left(\frac{2}{R + 10} \right) \times 10 = \frac{20}{R + 10}$$

Therefore, the potential drop per cm of the wire is

$$= \frac{20}{100 (R + 10)} \text{ V cm}^{-1}$$

The fall of potential across 40 cm of the wire is

$$\frac{40 \times 20}{100 (R + 10)} = \frac{8}{R + 10} \text{ V}$$

which must be equal to the emf of the source when the balance is achieved. Thus

$$\frac{8}{R + 10} = 10 \times 10^{-3} = \frac{1}{100}$$

$$R + 10 = 800 \quad \text{or} \quad R = 790\ \Omega$$

Example 24: A battery of emf 2 V and internal resistance $0.1\ \Omega$ is being charged with a current of 5 A. In what direction will the current flow inside the battery? What is the potential difference between the two terminals of the battery? (IIT 1980)

Solution: The positive terminal of the battery is connected to the positive terminal of the charger in order to charge the battery. Hence, inside the battery, the direction of the current is from the positive terminal to the negative terminal (see Fig. 13.34).

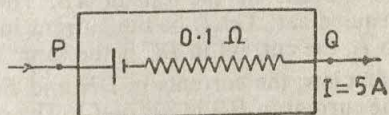


Fig. 13.34

The emf between the terminals of the battery is $(V_P - V_Q)$.

$$\text{Now } V_P - 2.0 - (0.1 \times 5) = V_Q$$

$$V_P - V_Q = 2 + 0.5 = 2.5 \text{ V}$$

Example 25: Twelve identical wires, each of resistance R , are joined to form a cube. Find (a) the resistance between diagonally opposite corners of the cube and (b) the resistance between two corners on the same edge of the cube.

Solution: (a) Let I be the current entering at point A and leaving at point G , as shown in Fig. 13.35. From Kirchhoff's law and by symmetry, the current at A divides into three equal parts along AD , AB and AE . Similarly, the current at points B , E and D divides again into two equal parts, each equal to $I/6$. Thus the current in all the twelve resistances are known. Applying Kirchhoff's second law to any path between the points A and G , such as path $AEFG$, we have

$$V = \frac{IR}{3} + \frac{IR}{6} + \frac{IR}{3}$$

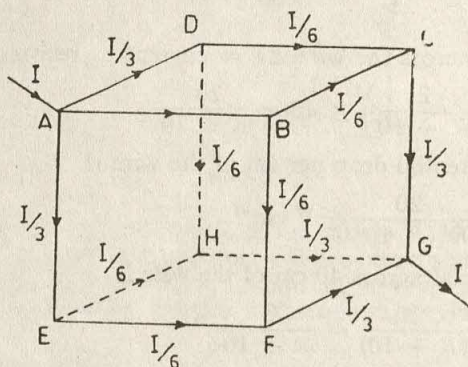


Fig. 13.35

It is clear from the diagram that the same result is obtained for all such paths. If the effective resistance of the network between points A and G is R_E , then

$$V = IR_E = \left(\frac{1}{3} + \frac{1}{6} + \frac{1}{3} \right) IR$$

$$R_E = \frac{5}{6} R$$

(b) Let V be the potential difference between the points A and B . Let the current I enter at A and leave at point B as shown in Fig. 13.36.

Suppose I_1 is the current in the resistor AB . The paths to B from A through E or D are equivalent. Let I_2 be the current in AE , which is the same as in AD . Also I_3 , the current in DC is the same as in EF .

From Kirchhoff's first law, the currents in DH and EH are each equal to $(I_2 - I_3)$. Similarly, the current in HG is $2(I_2 - I_3)$. The currents in resistors CB and BF are each equal to I_2 , because the points A and B are symmetri-

cally situated, except that current I leaves at B while it enters at A . The current in CG is $(I_3 - I_2)$ and that in GF is

$$(I_3 - I_2) + 2(I_2 - I_3) = I_2 - I_3$$

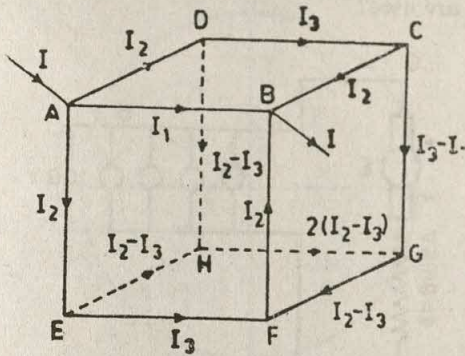


Fig. 13.36

The currents in all the twelve resistances are now specified. Applying Kirchhoff's second law, we obtain three equations which can be solved for the three unknowns I_1 , I_2 and I_3 .

For the mesh $ADCBA$, we have

$$V = I_2 R + I_3 R + I_3 R = R(2I_2 + I_3) \quad (i)$$

For mesh $EFGHE$, we have

$$0 = 4(I_2 - I_3) R - I_3 R = 4I_2 R - 5I_3 R \quad (ii)$$

Also, for resistor AB , we have

$$V = I_1 R \quad (iii)$$

From Eq. (ii), $4I_2 = 5I_3$

Substituting in Eq. (i), we have

$$V = R \left(2I_2 + \frac{4}{5} I_2 \right) = \frac{14}{5} I_2 R \quad (iv)$$

Substituting for V from Eq. (iii) in Eq. (iv) we have

$$I_1 = \frac{14}{5} I_2$$

The total current is

$$I = I_1 + 2I_2 = I_1 + \frac{10}{14} I_1 = \frac{12}{7} I_1$$

If R_E is the equivalent resistance of the whole network between A and B , we have

$$V = IR_E = \frac{12}{7} I_1 R_E$$

But

$$V = I_1 R$$

$$R_E = \frac{7}{12} R$$

Example 26: A portable generator is delivering 80 A to a string of lamps at a carnival (see Fig. 13.37). The terminal voltage of the generator is 100 V and its internal resistance is 0.05Ω . (a) What is the emf of the generator? (b) If the load consists of a number of 50 W lamps connected in parallel, how many lamps are used?

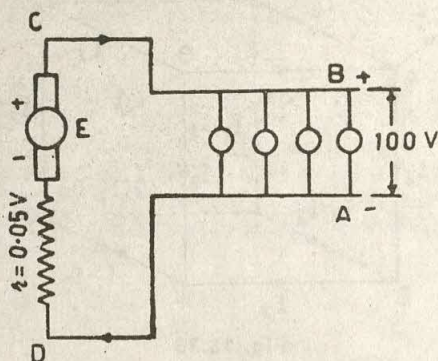


Fig. 13.37

Solution: In Fig. 13.37 the generator is represented as a source of emf with a series resistor of 0.05Ω .

(a) The potential drop inside the generator is

$$Ir = 80 \times 0.05 = 4 \text{ V}$$

Applying Kirchhoff's second law to the loop ABCDA, we have

$$+100 - E + 4 = 0 \quad \text{giving} \quad E = 104 \text{ V}$$

(b) For each lamp, $P = VI$; hence the current in each lamp is

$$I = \frac{P}{V} = \frac{50}{100} = 0.5 \text{ A}$$

Since the total current is 80 A, the number of lamps is

$$\frac{80}{0.5} = 160$$

This can be checked as follows

$$\text{Resistance of each lamp} = \frac{100}{0.5} = 200 \Omega$$

If 160 lamps are in parallel, the total resistance R_p is given by

$$\frac{1}{R_p} = \frac{1}{200} + \frac{1}{200} + \dots = \frac{160}{200}$$

which gives $R_p = 1.25 \Omega$

$$\text{Total current through the system} = \frac{V}{R} = \frac{100}{1.25} = 80 \text{ A}$$

Example 27: ABCD is a uniform circular wire of resistance 2Ω . AOC and BOD are two wires (each of resistance 1Ω) forming diameters at right angles to each other. Show that the resistance of the network is $15/14 \Omega$, if the battery is placed in AD.

Solution: As shown in Fig. 13.38, the uniform wire $ABCD$ of resistance $2\ \Omega$ has been divided into four equal parts and the two wires AOC and BOD are of resistance $1\ \Omega$ each. Therefore, the resistance of each part, viz. $AB, BC, CD, DA, OA, OB, OC$ and OD is $R = 0.5\ \Omega$.

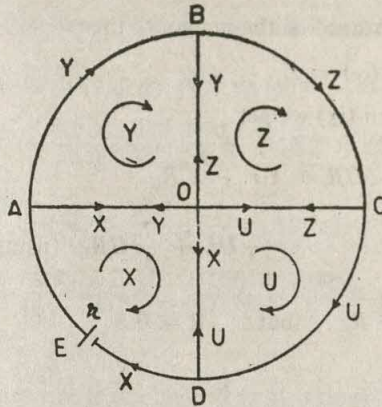


Fig. 13.38

Let the clockwise cyclic currents be X, Y, Z, U flowing through the various parts AD, AB, BC and CD of the circular wire respectively. Let E be the emf of the battery and r its internal resistance.

Applying Kirchhoff's second law to the loop $DEAOD$, we have

$$(R + r)X + R(X - Y) + R(X - U) = E$$

$$\text{or } rX + 3RX - RY - RU = E \quad (i)$$

For the loop $ABOA$, we have

$$RY + R(Y - Z) + R(Y - X) = 0$$

$$\text{or } 3Y - Z - X = 0 \quad (ii)$$

For mesh $BCOB$, we have

$$RZ + R(Z - U) + R(Z - Y) = 0$$

$$\text{or } 3Z - U - Y = 0 \quad (iii)$$

For mesh $CDOC$, we have

$$RU + R(U - X) + R(U - Z) = 0$$

$$\text{or } 3U - X - Z = 0 \quad (iv)$$

Subtracting Eq. (ii) from Eq. (iv) we get

$$Y = U \quad (v)$$

Combining this with Eq. (iii) we have

$$Z = \frac{2}{3}U \quad (vi)$$

Substituting this value of Z in Eq. (iv) we obtain

$$X = \frac{7}{3}U \quad (vii)$$

Using Eqs (v) and (vii) in Eq. (i) we get

$$\frac{7}{3} Ur + 5UR = E \quad (\text{viii})$$

Let R_E be the resistance of the network, then

$$X(r + R_E) = E \quad (\text{ix})$$

From Eqs (viii) and (ix) we get

$$\begin{aligned} \frac{7}{3} Ur + 5UR &= Xr + XR_E \\ &= \frac{7}{3} Ur + \frac{7}{3} UR_E \quad (\text{using Eq. (vii)}) \end{aligned}$$

Hence $R_E = \frac{15}{7} R$, but $R = 0.5$

Therefore, $R_E = \frac{15}{14} \Omega$

Example 28: A line having a total resistance of 0.2Ω delivers 10 kW at 220 V to a small factory. What is the efficiency of transmission?

Solution: Power loss in the line

$$\begin{aligned} &= I^2 R = \left(\frac{VI}{I} \right)^2 R \\ &= \left(\frac{P}{V} \right)^2 R \\ &= \left(\frac{10,000}{220} \right)^2 \times 0.2 \\ &= 413 \text{ W} = 0.413 \text{ kW} \end{aligned}$$

$$\begin{aligned} \text{Efficiency} &= \frac{\text{power delivered by line}}{\text{power supplied to line}} \\ &= \frac{10}{10 + 0.413} = 0.96 \quad \text{or} \quad 96\% \end{aligned}$$

Example 29: A copper wire having a cross-sectional area of 0.5 mm^2 and a length of 0.1 m is initially at 25°C , and is thermally insulated from the surroundings. If a current of 10 A is set up in this wire, (a) find the time in which the wire will start melting. The change of resistance of the wire with temperature may be neglected. (b) What will this time be, if the length of the wire is doubled? Density of copper $= 9 \times 10^3 \text{ kg m}^{-3}$, specific heat of copper $= 9 \times 10^{-2} \text{ cal kg}^{-1} (^\circ \text{C})^{-1}$, melting point $= 1075^\circ \text{C}$ and specific resistance $= 1.6 \times 10^{-8} \Omega \text{ m}$. (IIT 1979)

Solution:

(a) Mass of copper wire $= \text{volume} \times \text{density}$

$$= 0.5 \times 10^{-6} \times 0.1 \times 9 \times 10^3 = 45 \times 10^{-5} \text{ kg}$$

Rise in temperature of the wire $= \theta = 1075^\circ - 25^\circ = 1050^\circ \text{C}$

Specific heat $S = 0.09 \text{ cal kg}^{-1} (^\circ \text{C})^{-1}$

Heat required to melt the copper wire is

$$H = \frac{I^2 R t}{4.2} = \text{mass of wire} \times \theta \times S$$

where I is the current in the wire, R its resistance and t the time for which the current flows.

If the resistivity of the wire is ρ and L its length, we have

$$R = \frac{\rho L}{A} = \frac{1.6 \times 10^{-8} \times 0.1}{0.5 \times 10^{-6}} = 3.2 \times 10^{-3} \Omega$$

Thus
$$\frac{10 \times 10 \times 3.2 \times 10^{-3} \times t}{4.2} = 45 \times 10^{-5} \times 1050 \times 0.09$$

or
$$t = \frac{45 \times 4.2 \times 10^{-5} \times 1050 \times 0.09}{10 \times 10 \times 3.2 \times 10^{-3}} = 558 \text{ s}$$

(b) Electric energy = $I^2 R t$

When the length of the wire is doubled, R is also doubled. However, since the mass of the wire is also doubled, H is also doubled. Hence the wire will start melting in the same time.

Example 30: The walls of a closed cubical box of edge 50 cm are made of a material of thickness 1 mm and thermal conductivity $4 \times 10^{-4} \text{ cal s}^{-1} \text{ cm}^{-1} (^\circ\text{C})^{-1}$. The interior of the box is maintained at 100°C above the outside temperature by a heater placed inside the box and connected across a 400 V DC source. Calculate the resistance of the heater. (IIT 1971)

Solution: The heat transmitted per second through the walls of the closed box is given by

$$\frac{Q}{t} = \frac{KA(\theta_2 - \theta_1)}{d}$$

where A is the total surface area of the box.

$$A = 6 \times 50 \times 50 \text{ cm}^2$$

$$\begin{aligned} \frac{Q}{t} &= \frac{4 \times 10^{-4} \times 6 \times 50 \times 50 \times 100}{0.1} \\ &= 6000 \text{ cal s}^{-1} \end{aligned}$$

In order to maintain the temperature difference between the inside and the outside, the heat lost must be compensated by the production of heat through electric current in the coil. The heat produced per second is given by

$$H = \frac{V^2}{4.2 R} = 6000$$

$$R = \frac{V^2}{4.2 \times 6000} = \frac{400 \times 400}{4.2 \times 6000} = 6.35 \Omega$$

13.5 TRUE-FALSE STATEMENTS WITH REASONS

We give below some statements. We have to decide whether they are true or false giving, in brief, the reason for our answer.

1. The resistance of a copper wire varies directly as the length and inversely as the diameter.
2. In a simple battery circuit, the point of the lowest potential is the negative terminal of the battery.
3. The resistance of an incandescent lamp is greater when the lamp is switched off.
4. In a simple battery circuit, the current decreases steadily as we go around the circuit from the positive terminal.
5. An ordinary 100 W lamp has less resistance than a 60 W lamp.
6. The kilowatt-hour is a unit of energy.
7. The heat energy produced in a current carrying wire is proportional to the current.
8. At a constant voltage, the heat developed in a uniform wire varies inversely as the length of the wire used.
9. The emf of a cell is greater than the potential difference between its terminals as measured by a voltmeter.
10. It is not possible to construct two wires of the same length, one of copper and the other of iron, such that they have the same resistance at the same temperature.

ANSWERS

1. *False* because the resistance varies inversely as the area of cross-section (i.e. square of the diameter).
2. *True* The current flows towards the point of the lowest potential, as it does in such a circuit, from the positive to the negative terminal.
3. *False* The resistance is greater when the lamp is on because then the filament is at a higher temperature.
4. *False* A constant current flows in such a circuit.
5. *True* $P = VI = I^2 R = V^2/R$. Therefore, $R = V^2/P$. Thus a lamp with a higher power rating will have a lower resistance.
6. *True* Watt is a unit of power (J s^{-1}). When multiplied by time, we get the total energy consumed.
7. *False* The heat developed in a current carrying wire is proportional to the square of the current.
8. *True* $P = I^2 R = V^2/R$. Since R varies directly as the length of the wire, the heat produced varies inversely as the length of the wire.
9. *True* When a current I is drawn from a cell, the potential difference across the internal resistance is Ir and the emf

$$E = V + Ir$$

where V is the voltmeter reading.

10. *False* Resistance can be adjusted by using a wire of a suitable thickness to match the resistance of the other wire. For a given length, the resistance of the two wires would be the same if

$$\frac{\rho_1}{\rho_2} = \frac{A_1}{A_2}$$

where ρ and A denote the resistivity and area of cross-section of the wires.

13.6 MULTIPLE CHOICE QUESTIONS

Choose the correct answer from the given alternatives.

- The terminal voltage of a battery is
 - always equal to its emf
 - always less than its emf
 - greater or less than its emf depending on the direction of the current through the battery
 - greater or less than its emf depending on the magnitude of its internal resistance.
- For accurate measurements, the resistance of a voltmeter should be
 - as large as possible
 - equal to the resistance across which the potential difference is to be measured
 - as small as possible
 - infinity.
- To convert a galvanometer into an ammeter, we should connect
 - a low resistance in series with it
 - a high resistance in series with it
 - a low resistance in parallel with it
 - a high resistance in parallel with it.
- A new flashlight cell with an emf of 1.5 V gives a current of 15 A when connected directly to an ammeter of resistance $0.04\ \Omega$. The internal resistance of the cell is

(a) 0.04	(b) 0.06
(c) 0.10	(d) 10
- N identical cells, each of emf E , are connected in parallel. The emf of the combination is

(a) NE	(b) E
(c) N^2E	(d) E/N
- A big dry cell A and a small dry cell B have the same emf. The internal resistance of A
 - is greater than that of B
 - is equal to that of B
 - is less than that of B
 - may have any value independent of its size.
- A nichrome wire 50 cm long and 1 mm^2 in cross-section carries a current of 4 A when connected to a 2V storage battery. The resistivity of nichrome is

(a) $1 \times 10^{-6}\ \Omega\text{ m}$	(b) $2 \times 10^{-7}\ \Omega\text{ m}$
(c) $4 \times 10^{-7}\ \Omega\text{ m}$	(d) $5 \times 10^{-7}\ \Omega\text{ m}$
- The resistance of a straight conductor does not depend on its

(a) temperature	(b) shape of cross-section
(c) material	(d) length
- In Fig. 13.39, galvanometer reads zero. The resistance X is

(a) 7
(b) 14
(c) 21
(d) 28

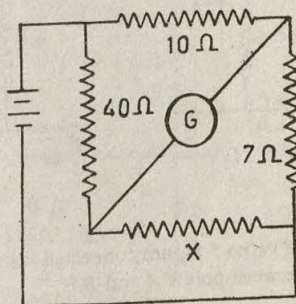


Fig. 13.39

10. A DC milliammeter has a resistance of $12\ \Omega$ and gives a full scale deflection for a current of 0.01 A . To convert it into a voltmeter giving a full-scale deflection for 3 V , the resistance required to be put in series with the instrument is
- (a) $102\ \Omega$ (b) $288\ \Omega$
 (c) $300\ \Omega$ (d) $412\ \Omega$
11. In Fig. 13.40, the ammeter A reads
- (a) 1.6 A (b) 2.4 A
 (c) 3.5 A (d) 4.3 A

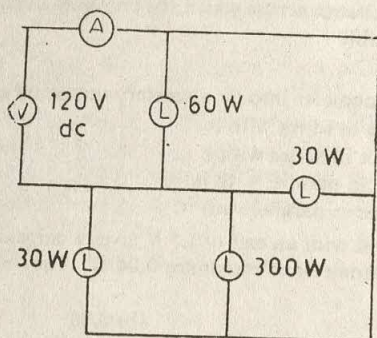


Fig. 13.40

12. A wire of resistance $4\ \Omega$ is stretched to twice its original length. What is the resistance of the wire now?
- (a) $1\ \Omega$ (b) $4\ \Omega$
 (c) $8\ \Omega$ (d) $16\ \Omega$
13. Four resistors each of value $4\ \Omega$, are connected as shown in Fig. 13.41. The equivalent resistance between points A and B is
- (a) $1\ \Omega$ (b) $3\ \Omega$
 (c) $4\ \Omega$ (d) $16\ \Omega$

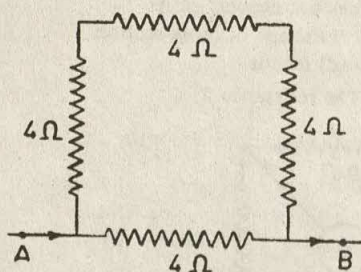


Fig. 13.41

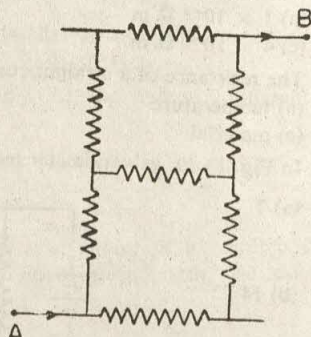


Fig. 13.42

14. Seven resistors, each of value $5\ \Omega$, are connected as shown in Fig. 13.42. The equivalent resistance between points A and B is
- (a) $5\ \Omega$ (b) $7\ \Omega$
 (c) $14\ \Omega$ (d) $35\ \Omega$

15. Four resistors of $4\ \Omega$ each are connected to a 2 V battery as shown in Fig. 13.43. The reading of the ammeter in the circuit is
- (a) $1/8\text{ A}$ (b) $3/8\text{ A}$
(c) $1/2\text{ A}$ (d) 2 A

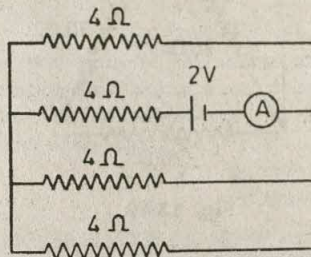


Fig. 13.43

16. The resistance between points A and B in the circuit shown in Fig. 13.44 is
- (a) $10\ \Omega$ (b) $20\ \Omega$
(c) $30\ \Omega$ (d) $40\ \Omega$

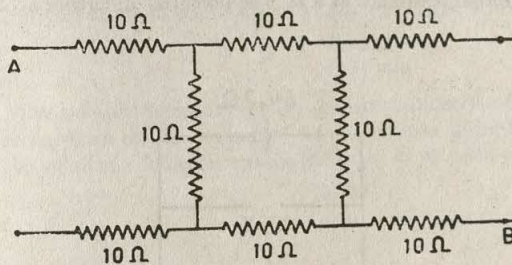


Fig. 13.44

17. The resistance between points A and B in the circuit shown in Fig. 13.45 is
- (a) $4\ \Omega$ (b) $6\ \Omega$
(c) $10\ \Omega$ (d) $8\ \Omega$

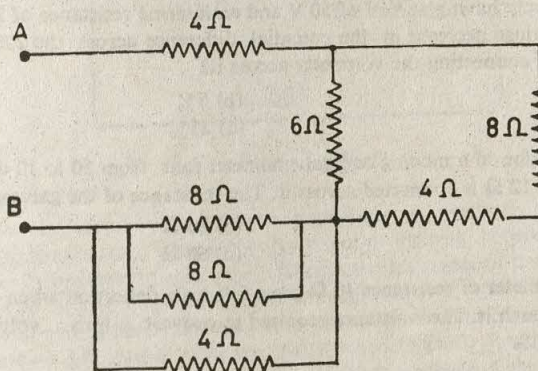


Fig. 13.45

18. The current I in the circuit shown in Fig. 13.46 is

- (a) $1/45$ A
 (b) $1/15$ A
 (c) $1/10$ A
 (d) $1/5$ A

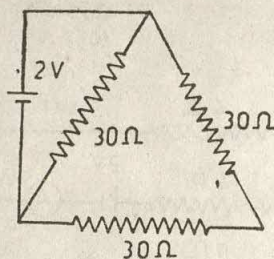


Fig. 13.46

19. A cell supplies a current of 0.9 A through a $2\ \Omega$ resistor and a current of 0.3 A through a $7\ \Omega$ resistor. What is the internal resistance of the cell?

- (a) $0.5\ \Omega$
 (b) $1.0\ \Omega$
 (c) $1.2\ \Omega$
 (d) $2.0\ \Omega$

20. Figure 13.47 shows a circuit with two cells in opposition to each other. One cell has an emf of 6 V and internal resistance of $2\ \Omega$ and the other cell has an emf of 4 V and internal resistance of $8\ \Omega$. The potential difference across the terminals X and Y is

- (a) 5.4 V
 (b) 5.6 V
 (c) 5.8 V
 (d) 6.0 V

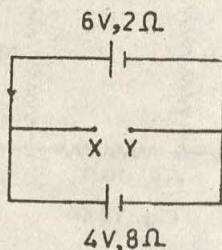


Fig. 13.47

21. A voltmeter having a resistance of $1800\ \Omega$ is employed to measure the potential difference across a $200\ \Omega$ resistor which is connected to the terminals of a DC power supply having an emf of 50 V and an internal resistance of $20\ \Omega$. What is the percentage decrease in the potential difference across the $200\ \Omega$ resistor as a result of connecting the voltmeter across it?

- (a) 1%
 (b) 5%
 (c) 10%
 (d) 25%

22. The deflection in a moving coil galvanometer falls from 50 to 10 divisions when a shunt of $12\ \Omega$ is connected across it. The resistance of the galvanometer coil is

- (a) $24\ \Omega$
 (b) $36\ \Omega$
 (c) $48\ \Omega$
 (d) $60\ \Omega$

23. A galvanometer of resistance $10\ \Omega$ gives full-scale deflection when 1 mA current passes through it. The resistance required to convert it into a voltmeter reading upto 2.5 V is

- (a) $24.9\ \Omega$
 (b) $249\ \Omega$
 (c) $2490\ \Omega$
 (d) $24900\ \Omega$

24. A copper wire (resistivity = $1.7 \times 10^{-8} \Omega \text{ m}$, density = 8900 kg m^{-3}) and an aluminium wire (resistivity = $2.8 \times 10^{-8} \Omega \text{ m}$, density = 2700 kg m^{-3}) have the same mass per unit length. The ratio of the resistance per unit length of aluminium and copper wire is
- (a) $1:\sqrt{2}$ (b) 1:2
(c) $2:\sqrt{3}$ (d) 2:3
25. The driver cell of a potentiometer has an emf of 2V and negligible internal resistance. The potentiometer wire has a resistance of 5Ω and is 1 m long. The resistance which must be connected in series with the wire so as to have a potential difference of 5 mV across the whole wire is
- (a) 1985 Ω (b) 1990 Ω
(c) 1995 Ω (d) 2000 Ω
26. Three bulbs of ratings 40 W, 60 W and 100 W are designed to work on 220 V mains. Which bulb will burn most brightly if they are connected in series across 220 V mains?
- (a) 40 W bulb (b) 60 W bulb
(c) 100 W bulb (d) all will burn equally brightly
27. An electric kettle has two coils. When one coil is connected to the a.c. mains, the water in the kettle boils in 10 minutes. When the other coil is used, the same quantity of water takes 45 minutes to boil. How long will it take for the same quantity of water to boil if the two coils are connected in parallel?
- (a) 6 min (b) 12 min
(c) 18 min (d) 24 min
28. A 12 V, 24 W tungsten filament bulb is supplied with current from n cells connected in series. Each cell has an emf of 1.5 V and internal resistance of 0.25Ω . What is the value of n in order that the bulb runs at its rated power?
- (a) 6 (b) 8
(c) 12 (d) 16

13.7 NUMERICAL EXERCISES

1. In the circuit shown in Fig. 13.48, a voltmeter reads 30 V when it is connected across the 400Ω resistor. Calculate what the same voltmeter will read when it is connected across the 300Ω resistor. (IIT 1980)

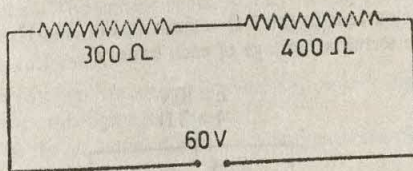


Fig. 13.48

2. A piece of nichrome wire is 1 m long and 1 mm in diameter. Its ends are joined so as to form a ring. What is the resistance of the ring between two points diametrically opposite to each other? What is the resistance of the ring between two points 90° apart? Specific resistance of nichrome = $1 \times 10^{-8} \Omega \text{ m}$.
3. A coil of wire is immersed in a cup of water and is connected to a source that delivers 4A of current. After the water starts to boil, it is found that 120g of water is vaporized in 10 minutes. What is the resistance of the coil?

4. A bird whose feet are 5 cm apart perches on a power line that carries 866 A of current. If the resistance of the wire is $12 \times 10^{-8} \Omega$, what is the potential difference between the bird's feet?
5. A motor connected to 120 V DC mains develops 320 W of useful mechanical power and 22 W is lost through friction. The current from the mains is 3 A. (a) What electrical power is being supplied to the motor? (b) What is the efficiency of the motor? What is the resistance of the motor?
6. Four batteries are connected in series as shown in Fig. 13.49. Compute the terminal voltage of each battery.

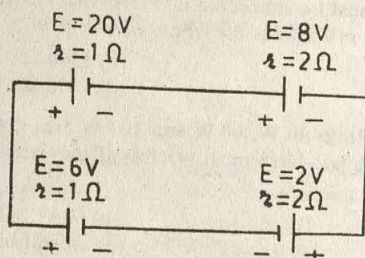


Fig. 13.49

7. Figure 13.50 shows a Wheatstone's bridge that is almost balanced, with point C grounded. (a) Calculate the potential of point A. (b) Calculate the potential of point B. (c) If a galvanometer is connected between A and B, what is the direction of the current through it? (d) For what value of the resistor BC would the bridge be in balance?

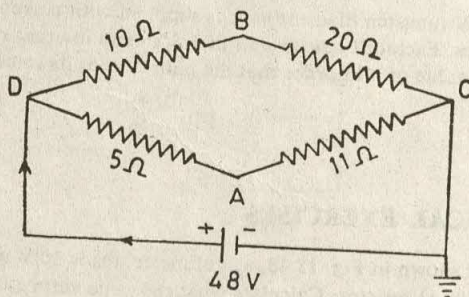


Fig. 13.50

8. (a) What is the current in the 2Ω resistor in Fig. 13.51?
(b) What is the terminal voltage of each battery?

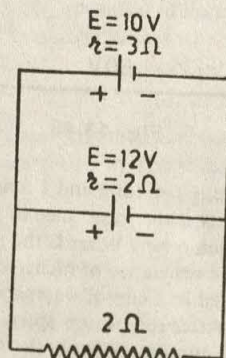


Fig. 13.51

9. Three equal resistors connected in series across a source of emf together dissipate 10 W of power. What would be the power dissipated if the same resistors were connected in parallel across the same source of emf? (IIT 1972)
10. An electric tea kettle has two heating coils. When one of the coils is switched on, the tea in the kettle begins to boil in six minutes. When the other coil is switched on, the boiling begins in eight minutes. In what time will the boiling begin, if both coils are switched on simultaneously when they are (i) in series and (ii) in parallel? (IIT 1975)
11. A battery of emf 1.4 V and internal resistance 2Ω is connected to a resistor of 100Ω resistance through an ammeter. The resistance of the ammeter is $4/3\Omega$. A voltmeter has also been connected to measure the potential difference across the resistor.
 - (a) Draw the circuit diagram.
 - (b) The ammeter reads 0.02 A. What is the resistance of the voltmeter?
 - (c) The voltmeter reads 1.10 V. What is the error in the reading? (IIT 1975)
12. A resistor is made by joining two wires of the same material. The radii of the two wires are 1 mm and 3 mm respectively while their lengths are 3 cm and 5 cm respectively. A battery of emf 16 V and negligible resistance is connected across the resistor. What is the potential drop across the shorter wire? (IIT 1970)
13. Three electric lamps designed for a voltage of 110 V are rated at 40, 40 and 80 W respectively. How should the three lamps be connected to produce a normal glow with a voltage of 220 V in the mains? Find the current flowing through the lamps with a normal glow. Draw the necessary circuit diagram.
14. In the circuit shown in Fig. 13.52, the batteries E_1 and E_2 have the same emf equal to 2V and internal resistances $r_1 = 1\Omega$ and $r_2 = 1.5\Omega$ respectively. Find the current through each battery and in the resistance R .

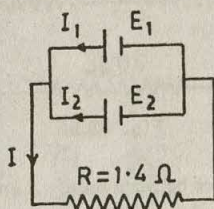


Fig. 13.52

15. A cell of negligible internal resistance is connected to a resistor of 1000Ω and a voltmeter in series. The voltmeter reads 30V. When another resistor of 1000Ω is added in series, the voltmeter reads 20V. Calculate the emf of the cell and the resistance of the voltmeter.
16. (a) Calculate the total resistance in the circuit shown in Fig. 13.53.
 (b) Find the current through each resistor.
 (c) Also determine the potential difference across each resistor.

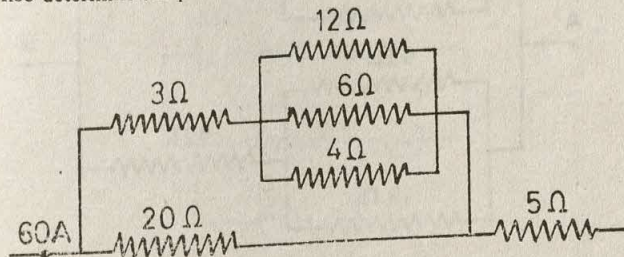


Fig. 13.53

17. (a) Determine the total resistance of the circuit shown in Fig. 13.54.
 (b) What is the current through each resistor.
 (c) Find the potential difference across each resistor.

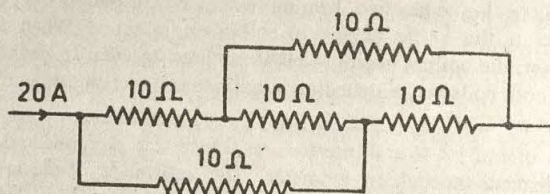


Fig. 13.54

18. Calculate the resistance between points *A* and *B* in the circuit shown in Fig. 13.55.

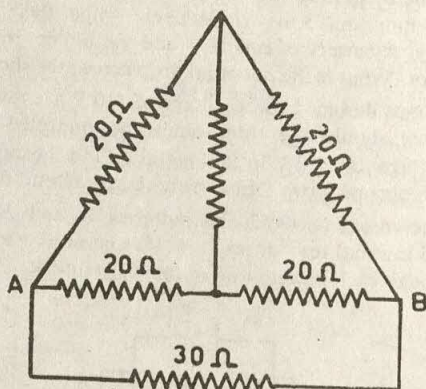


Fig. 13.55

19. (a) Find the total resistance between points *A* and *B* in the circuit shown in Fig. 13.56.
 (b) Calculate the potential difference between points *A* and *C* if the current in the $8\ \Omega$ resistor is 0.5 A .

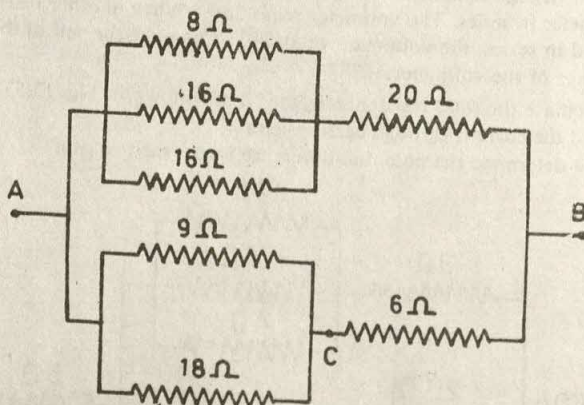


Fig. 13.56

20. Calculate the resistance between A and B in the circuit shown in Fig. 13.57.

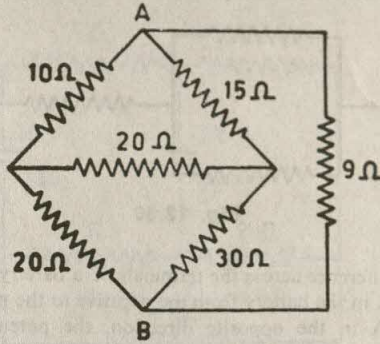


Fig. 13.57

21. The long resistor between A and B in Fig. 13.58 has a resistance of 300Ω and is tapped at the one-third points. (a) Find the equivalent resistance between P and Q . (b) Find the potential difference between B and C when the potential difference between P and Q is 320 V .

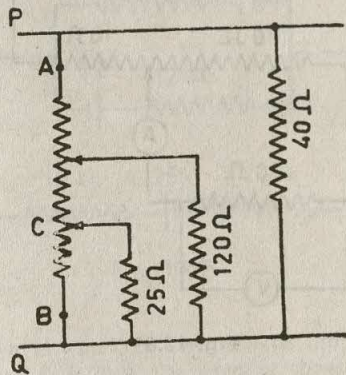


Fig. 13.58

22. Calculate the ammeter and voltmeter readings in the circuit shown in Fig. 13.59.

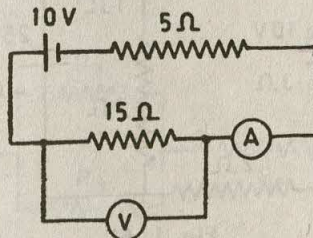


Fig. 13.59

23. Each of the three resistors in Fig. 13.60 has a resistance of $2\ \Omega$ and can dissipate a maximum of $18\ \text{W}$ without becoming excessively hot. What is the maximum power the circuit can dissipate?

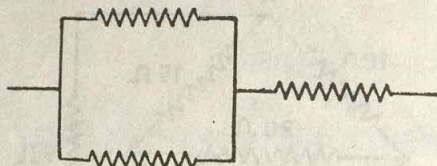


Fig. 13.60

24. The potential difference across the terminals of a battery is $8.5\ \text{V}$ when there is a current of $3\ \text{A}$ in the battery from the negative to the positive terminal. When the current is $2\ \text{A}$ in the opposite direction, the potential difference becomes $11\ \text{V}$.
- (a) What is the internal resistance of the battery?
- (b) What is its emf?
25. Calculate the ammeter and voltmeter readings in the circuit shown in Fig. 13.61.

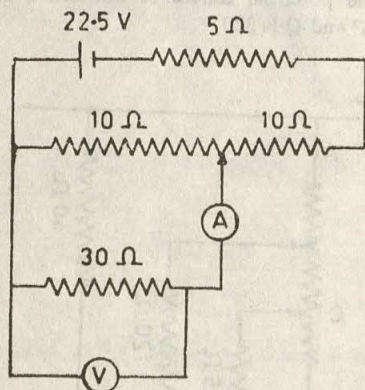


Fig. 13.61

26. Find the current in each resistor in the circuit shown in Fig. 13.62.

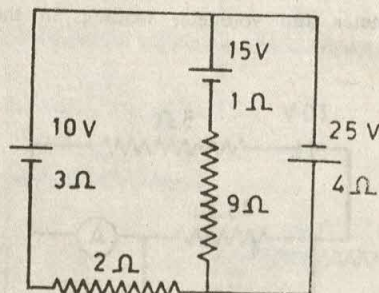


Fig. 13.62

27. Determine the charge on the plates of the capacitor in the circuit shown in Fig. 13.63.

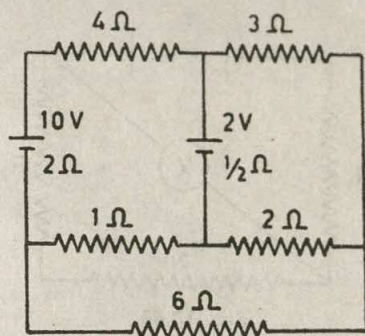


Fig. 13.63

28. Determine the current in each conductor in the circuit shown in Fig. 13.64.

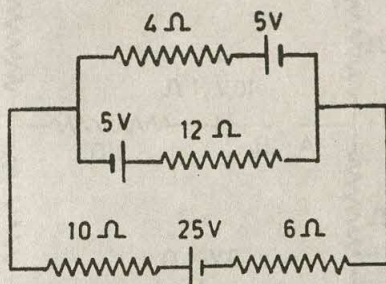


Fig. 13.64

29. In the circuit shown in Fig. 13.65, the potential drop across the resistor R_1 is 40 V. $R_2 = 20\ \Omega$ and $R_4 = 25\ \Omega$. The ammeter shows a current of 2A. Find the resistance R_2 , assuming that the resistance of the battery and of the ammeter is negligible.

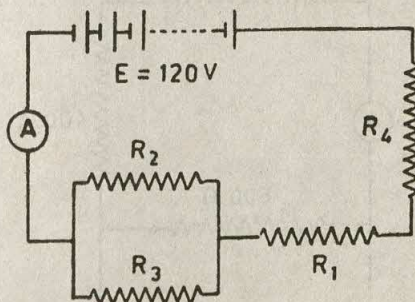


Fig. 13.65

30. In the circuit shown in Fig. 13.66, the resistors R_1 , R_2 and R_3 have a resistance of $200\ \Omega$ each. The voltmeter reads 100 V and its resistance is $1000\ \Omega$. Find the emf of the battery assuming that its internal resistance is negligible.

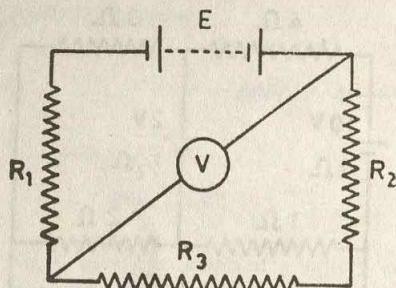


Fig. 13.66

31. Determine the potential difference between points A and B in the circuit shown in Fig. 13.67. If A and B are connected, calculate the current in the 12 V cell.

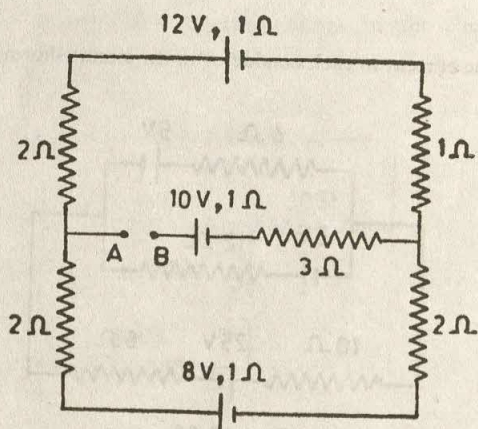


Fig. 13.67

32. Find the readings of the ammeter and the voltmeter in the circuit shown in Fig. 13.68. The resistance of the voltmeter is $1000\ \Omega$. Disregard the resistance of the battery and the ammeter.

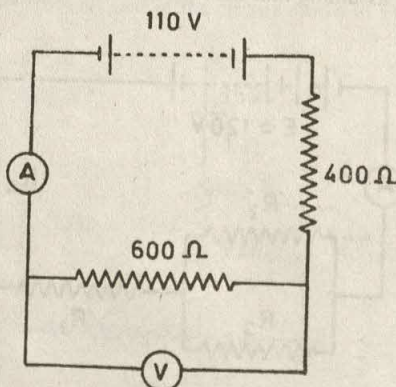


Fig. 13.68

33. In the circuit shown in Fig. 13.69, (a) find the potential difference between A and B when switch S is open. (b) What is the current through the switch when it is closed? (c) In Fig. 13.70, what is the potential difference when switch S is open? (d) What is the current through the switch when it is closed? What is the equivalent resistance of the circuit in Fig. 13.70? (e) When switch S is open and (f) when it is closed?

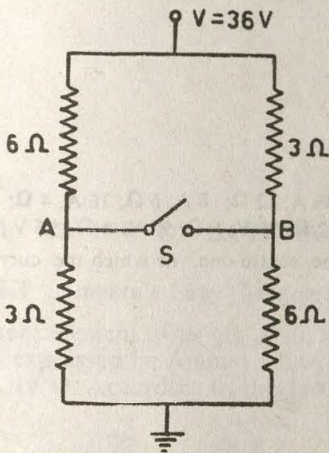


Fig. 13.69

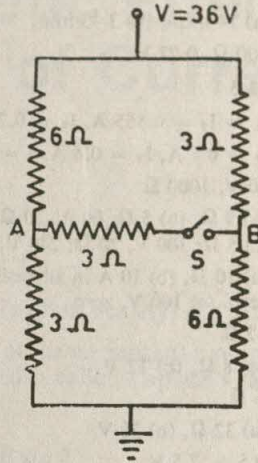


Fig. 13.70

34. An ammeter with resistance $0.18\ \Omega$ produces full-scale deflection when 10 A current passes through it. What resistance will be required so that the ammeter may be used to measure currents upto 100 A ? How should it be connected?
36. A voltmeter gives full scale deflection for a potential difference of 30 V . The resistance of the voltmeter is $2000\ \Omega$. What resistance should be used to measure potential differences upto 75 V with the help of this voltmeter? How should the resistance be connected?

ANSWERS

MULTIPLE CHOICE QUESTIONS

- | | | | |
|---------|---------|---------|---------|
| 1. (b) | 2. (a) | 3. (c) | 4. (b) |
| 5. (b) | 6. (c) | 7. (a) | 8. (b) |
| 9. (d) | 10. (b) | 11. (c) | 12. (d) |
| 13. (b) | 14. (b) | 15. (b) | 16. (c) |
| 17. (c) | 18. (c) | 19. (a) | 20. (b) |
| 21. (a) | 22. (c) | 23. (c) | 24. (b) |
| 25. (c) | 26. (a) | 27. (d) | 28. (c) |

NUMERICAL EXERCISES

- 22.5 V
- $0.318\ \Omega$, $0.218\ \Omega$
- $28.2\ \Omega$

4. $520 \mu\text{V}$
5. (a) 360 W, (b) 89%, (c) 2Ω
6. (a) 16 V, (b) 0, (c) 6 V, (d) 10 V
7. (a) +33 V, (b) +32 V, (c) A to B, (d) 22Ω
8. (a) 3.5 A, (b) 7 V
9. 90 W
10. (a) 14 min., (b) 3.43 min.
11. 200Ω , 0.23 V
12. 13.5 V
13. $I_1 = I_2 = 0.365\text{ A}$, $I_3 = 0.73\text{ A}$
14. $I_1 = 0.6\text{ A}$, $I_2 = 0.4\text{ A}$, $I = 1\text{ A}$
15. 60 V, 1000Ω
16. (a) 9Ω , (b) 5Ω ; 60 A, 20Ω ; 12 A, 3Ω ; 48 A, 12Ω ; 8 A, 6Ω ; 16 A, 4Ω ; 24 A, 5Ω ; 300 V, 20Ω ; 240 V, 3Ω ; 144 V, 12Ω ; 96 V, 6Ω ; 96 V, 4Ω ; 96 V
17. (a) 10Ω , (b) 10 A in all resistors except the centre one, in which the current is zero, (c) 100 V, zero
18. 12Ω
19. (a) 8Ω , (b) 12 V
20. 6Ω
21. (a) 32Ω , (b) 20 V
22. 0.5 A, 7.5 V
23. 27 W
24. (a) 0.5 Ω , (b) 10 V
25. 0.25 A, 7.5 V
26. 2Ω , 3 A; 9Ω , 2 A; 4Ω , 5 A
27. 4Ω ; 1.137 A, $1/2\Omega$; 0.58 A, 3Ω ; 0.558 A, 2Ω ; 0.308 A, 1Ω ; 0.887 A, 6Ω ; 0.251 A
28. 4Ω , $10/38\text{ A}$; 12Ω , $35/38\text{ A}$; 10Ω , $45/38\text{ A}$; 6Ω , $45/38\text{ A}$
29. $R_2 = 60\Omega$
30. $E = 170\text{ V}$
31. 0.22 V, $13/28\text{ A}$
32. 0.142 A, 53.2 V
33. (a) -12 V, (b) ∞ , (c) -12 V, (d) $12/7\text{ A}$, (e) 4.5 Ω , (f) 4.2 Ω
34. 0.02 Ω in parallel with the ammeter.
35. 3000Ω in series with the voltmeter.

Magnetic Effects of Current

14.1 REVIEW OF BASIC CONCEPTS

14.1.1 Ampere's Law Magnetic induction (or flux density) $\vec{dB}$ due to a current element of length $\vec{dl}$ (in the direction of the current) at a point near it is expressed by Ampere's law, sometimes also called Laplace's law (see Fig. 14.1). According to this law

$$\vec{dB} = ki \frac{\vec{dl} \times \vec{r}}{r^3}, \quad |\vec{dB}| = ki \frac{dl \sin \theta}{r^2} \quad (14.1)$$

where i is the current and $\vec{r}$ the position vector of the point P from the element, and θ is the angle between $\vec{dl}$ and $\vec{r}$. The proportionality constant k depends on the units used. In the SI units, k is assigned a value of 10^{-7} weber per ampere metre in free space. For convenience, constant k is replaced by $\mu_0/4\pi$, where μ_0 is called the permeability of free space. This substitution removes the factor 4π from many expressions in which it would, otherwise, appear. Thus

$$|\vec{dB}| = \frac{\mu_0}{4\pi} \frac{i dl \sin \theta}{r^2} \quad (14.2)$$

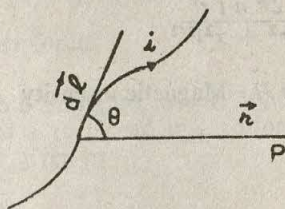


Fig. 14.1

The direction of $\vec{dB}$ is always perpendicular to the plane determined by the line tangential to the current element $\vec{dl}$ and the line joining $\vec{dl}$ to P and is given by the righthand rule. In general, $\vec{B}$ due to a current can be calculated by finding the vector sum of the contributions of all the elements that make up the current.

14.1.2 Biot—Savart Law The magnetic induction due to an infinitely long straight wire, carrying a current, at any point inside it, distant r from the wire is given by

$$|\vec{B}| = \left(\frac{\mu_0}{4\pi} \right) \frac{2i}{r} \quad (14.3)$$

The direction of the field is perpendicular to the plane determined by the point and the line of the conductor (see Fig. 14.2).

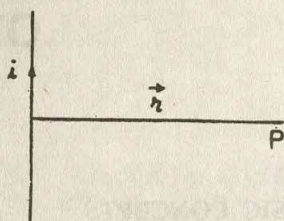


Fig. 14.2

The lines of induction are circles with their centres at the conductor. Relation (14.3) was originally developed from experimental observations by J Biot and F Savart. It is easily derived from Ampere's law.

14.1.3 Magnetic Induction at the Centre of a Circular Current The magnetic induction at the centre of a circular current i is given by

$$B = \left(\frac{\mu_0}{4\pi} \right) \frac{2\pi n i}{r} \quad (14.4)$$

where n is the number of turns in the circular coil and r is its mean radius. The direction of B is along the axis outwards from the face of the coil in which the flow of current is in the anti-clockwise direction.

14.1.4 Magnetic Induction on the Axis of a Circular Coil Magnetic induction B at a point distant x on the axis of a circular coil of radius r carrying a current i is given by

$$B = \left(\frac{\mu_0}{4\pi} \right) \frac{2\pi n i r^2}{(r^2 + x^2)^{3/2}} \quad (14.5)$$

14.1.5 Magnetic Intensity $\vec{H}$ Magnetic intensity $\vec{H}$ at a point in a magnetic field is defined as

$$\vec{H} = \frac{\vec{B}}{\mu_0} \quad (14.6)$$

where $\vec{B}$ is the magnetic induction at that point.

14.1.6 Units of $\vec{B}$ and $\vec{H}$ The SI unit of magnetic induction is called *tesla* (T) or weber per square metre (Wb m^{-2}). In CGS units, magnetic induction is expressed in *gauss* (G).

$$1 \text{ gauss} = 10^{-4} \text{ tesla}$$

One gauss is numerically equal to one oersted.

The unit of magnetic intensity is ampere per metre (A m^{-1}) in the SI system. The electromagnetic unit of current can be defined from Eq. (14.4), in which $\mu_0/4\pi = 1$ in the CGS system. Thus 1 emu current is defined as the current flowing through a circular coil of radius 1 cm and of 1 turn, which produces at its centre a magnetic field intensity equal to 2π oersted.

$$1 \text{ A} = 0.1 \text{ emu current}$$

14.1.7 Force on a Moving Charge in a Magnetic Field The force on a charge q moving with a velocity $\vec{v}$ in a magnetic field of magnetic induction $\vec{B}$ is given by (see Fig. 14.3)

$$\vec{F} = q (\vec{v} \times \vec{B}) \quad (14.7)$$

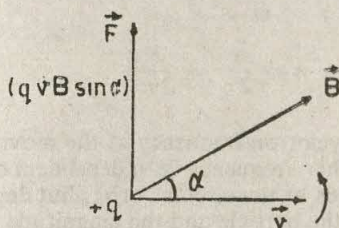


Fig. 14.3

The force $\vec{F}$ is directed along a line perpendicular to the plane containing $\vec{v}$ and $\vec{B}$. To determine its direction along the line, imagine rotating $\vec{v}$ into $\vec{B}$, as shown in Fig. 14.3. The direction of $\vec{F}$ is the direction of the advance of a right hand screw rotated in this direction.

14.1.8 Lorentz Force When a charged particle moves through space where both electric and magnetic fields are present, both fields exert forces on the particle and the total force is the vector sum of the forces due to electric and magnetic fields, i.e.

$$\vec{F} = q (\vec{E} + \vec{v} \times \vec{B}) \quad (14.8)$$

This force is called Lorentz force.

14.1.9 Magnetic Force on a Straight Current When a straight linear conductor carrying a current i is placed in a uniform magnetic field, it experiences a magnetic force given by

$$\vec{F} = i (\vec{l} \times \vec{B}) \quad (14.9)$$

where $\vec{l}$ is the length of the conductor in the direction of the current in it.

14.1.10 Circular Motion of a Charged Particle When a charged particle in a magnetic field of induction $\vec{B}$ is given a velocity $\vec{v}$ in a direction at right angles to the field, it experiences a force given by

$$|\vec{F}| = q v B \sin \theta = q v B$$

The force has a constant magnitude and is always perpendicular to $\vec{v}$. The resulting motion is a uniform circular motion. The magnetic force on the moving charge provides the centripetal force for the uniform circular motion. If m is the mass of the particle,

$$F = q v B = \frac{m v^2}{r} \quad (14.10)$$

where r is the radius of the circular orbit.

$$r = \frac{m v}{B q} \quad (14.11)$$

$$\text{Angular speed} = \omega = \frac{v}{r} = \frac{B q}{m} \quad (14.12)$$

$$\text{Frequency of revolution } n = \frac{\omega}{2\pi} = \frac{B q}{2\pi m} \quad (14.13)$$

This is known as the cyclotron frequency of the moving charge in a uniform magnetic field. This frequency is independent of the linear speed of the particle or the radius of the circular orbit, but depends on the charge to mass ratio (q/m) of the particle and the magnitude of the field B .

14.1.11 Force between Two Parallel Straight Currents When the currents in two neighbouring straight conductors flow in the same direction, there is a force of attraction between them. If the currents are in opposite directions, there is a repulsive force between them. The force per unit length on either conductor is given by

$$F = \left(\frac{\mu_0}{4\pi} \right) \frac{2 i_1 i_2}{r} \quad (14.14)$$

where r is the distance between the two parallel conductors. This relation is used to define an ampere in terms of the force between conductors. When $i_1 = i_2 = 1\text{ A}$ and $r = 1\text{ m}$, then $F = \frac{\mu_0}{2\pi}$. Thus 1 ampere current is that current, which flowing through each of the parallel infinitely long straight wires of negligible area of cross-section separated by a distance of 1 m in free space, produced a force between them of $2 \times 10^{-7}\text{ N m}^{-1}$ per unit length of either wire.

14.1.12 Magnetic Moment of a Current Loop A current loop orients itself in a magnetic field like a small magnet (magnetic dipole). The torque acting on a current loop in a magnetic field of induction B is given by

$$\tau = (N i A) B \sin \theta \quad (14.15)$$

where i is the current flowing in N turns of the loop with face area A and making an angle θ with the field. Comparing it with the electric dipole, $(N i A)$ can be regarded as the magnetic dipole moment M . The magnetic dipole moment (M) of the loop lies along the axis of the loop; its direction being given by the right-hand rule. Let the fingers of the right hand curl around the loop in the direction of the current, the extended right thumb will then point in the direction of $\vec{M}$.

A magnetic dipole (a short magnet) experiences a torque when placed in a uniform magnetic field, given by

$$\vec{\tau} = \vec{M} \times \vec{B} \quad (14.16)$$

where $\vec{M}$ is the magnetic moment of dipole.

14.1.13 Magnetic Flux The magnetic flux through a coil of area A in a uniform magnetic field of flux density B (induction) is defined as

$$\phi = \vec{B} \cdot \vec{A} = BA \cos \theta \quad (14.17)$$

where θ is the angle between the normal to the area A of the coil and the direction of $\vec{B}$ as shown in Fig. 14.4.

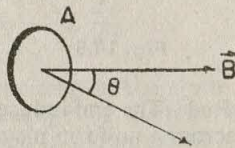


Fig. 14.4

A change in the field strength or a change in area (brought about by turning the coil to change θ) causes the flux through the coil to change. If the coil has N turns, the total flux linked with the coil is

$$\phi = N A B \cos \theta \quad (14.18)$$

The SI unit of magnetic flux is called *weber* (Wb). The unit of B , the flux density or flux per unit area is, therefore, weber per square metre (Wb m^{-2}) which is called *tesla* (T).

14.1.14 Faraday's Law Faraday's experiments showed that the change of magnetic flux in a circuit gives rise to an emf in the circuit, known as the induced emf. Faraday's law states that the magnitude of this induced emf e is proportional to the rate of change of magnetic flux linked with the circuit. The change in flux can arise due to the relative motion between the coil and the field.

14.1.15 Lenz's Law According to this law, the direction of the induced current (caused by induced emf) is always such as to oppose the change causing it. Hence

$$e = -k \frac{d\phi}{dt}$$

where k is a positive constant. The negative sign expresses Lenz's law. It means that the induced emf is such that, if the circuit is closed, the induced current opposes the change in flux. If the flux linkage ϕ is expressed in webers, t in seconds and e in volts, then $k = 1$. Hence, in the SI system

$$e = - \frac{d\phi}{dt}$$

14.1.16 Fleming's Right Hand Rule This simple rule gives the direction of the induced emf (and hence of the induced current) when a straight conductor moves at right angles to an external magnetic field. Let the thumb and the first two fingers of the right hand be stretched so that they are at right angles to one another. If the first finger points in the direction of the magnetic field and the thumb indicates the direction of the motion of the conductor, then the middle finger gives the direction of the induced emf (see Fig. 14.5).

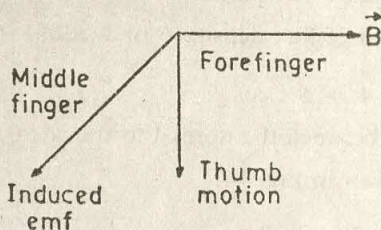


Fig. 14.5

14.1.17 Induced EMF in a Rod The emf induced between the ends of a straight conductor moving across a uniform magnetic field is given by (see Fig. 14.6)

$$e = - \frac{d\phi}{dt} = - B l v \quad (14.19)$$

where l is the length of the conductor and v its velocity.

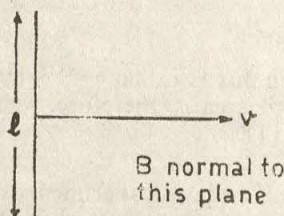


Fig. 14.6

14.1.18 Induced EMF in a Rotating Coil If a coil has an area $\vec{A}$ and its normal makes an angle θ with the uniform magnetic field $\vec{B}$, then the flux through the coil is

$$\phi = N A B \cos \theta$$

where N is the number of turns in the coil. If the coil is rotated with a uniform angular velocity ω rad s^{-1} about an axis in the plane of the coil and at right angles to the field, the induced emf is given by

$$e = - \frac{d\phi}{dt} = - \frac{d}{dt} (N A B \cos \omega t) \quad \left(\because \omega = \frac{\theta}{t} \right)$$

$$e = N A B \omega \sin \omega t = e_0 \sin \omega t \quad (14.20)$$

where $e = N A B \omega \sin \omega t$ is the maximum induced emf. Equation (14.20) shows that the induced emf varies sinusoidally with time.

14.1.19 Self Induction Whenever a current flows in a circuit, a magnetic field is set up (around the circuit) that links with that circuit and varies as the current varies. Hence, any circuit in which there is a varying current, has an induced emf in it due to the variation of its own magnetic field. This phenomenon is called self induction and the emf thus induced is called self induced emf.

If ϕ is the magnetic flux linked with the circuit due to a current i , then

$$\phi \propto i$$

$$\text{or} \quad \phi = Li \quad (14.21)$$

where L is called the coefficient of self induction for that circuit and is defined as the magnetic flux linked with the circuit when a unit current flows in it. Quantity L is also known as the self inductance of the circuit. If ϕ and i change with time, the self induced emf in the circuit is given by

$$e = - \frac{d\phi}{dt} = - L \frac{di}{dt} \quad (14.22)$$

The self inductance of a circuit is, therefore, also equal to the self induced emf per unit rate of change of current. If i is in ampere and t in seconds, then L is expressed in a unit called *henry* (H).

14.1.20 Mutual Induction Whenever a variation of current in one circuit produces a change in the magnetic flux in a neighbouring circuit, it produces an induced emf in the latter. This phenomenon is called mutual induction. The circuit in which the current is varied, is called the primary and the other circuit, in which the emf is induced, is called the secondary.

The coefficient of mutual induction (M) or simply, the mutual inductance between two such coupled circuits is the magnetic flux linked with one circuit when a unit current flows in the other.

Magnetic flux ϕ_s linked with the secondary coil due to a current i_p in the primary coil is given by

$$\phi_s = M i_p \quad (14.23)$$

and the induced emf in the secondary coil is given by

$$e_s = - \frac{d\phi_s}{dt} = - M \frac{di_p}{dt} \quad (14.24)$$

The SI unit of mutual inductance M is henry (H) which equals weber per ampere. Thus

$$1 \text{ H} = 1 \text{ Wb A}^{-1} = 1 \text{ V s A}^{-1}$$

14.1.21 L and M for a Solenoid A solenoid is a long wire wound closely over a cylindrical core. A uniform magnetic field is produced along the axis when a current flows in it. The self inductance of such a coil is given by

$$L = \mu_0 \frac{N^2 A}{l} \quad (14.25)$$

where l is the length over which the coil is wound and A its area of cross-section. Here L is in henry, A in m^2 and l in m. If the wire is wound over a circular core of mean radius r , then

$$L = \mu_0 \frac{N^2 A}{2 \pi r} \quad (14.26)$$

Such a coil is known as a toroidal solenoid.

If a coil of N_2 turns surrounds a long solenoid of length l and cross-sectional area A wound with N_1 turns of wire, the mutual inductance between the two coils is given by

$$M = \mu_0 \frac{A N_1 N_2}{l} \quad (14.27)$$

14.1.22 Transformers When a secondary coil and a primary coil are wound on the same core, the system is known as a transformer. In an ideal transformer, the entire magnetic flux is linked between the primary and the secondary, without any leakage of the magnetic lines of induction. An alternating emf applied to the primary, causes an induced emf in the secondary without any change in the frequency. It can be proved that, at any instant of time, the ratio of the number of turns in the two coils is equal to the ratio of the emfs across them. i.e.

$$\frac{N_s}{N_p} = \frac{e_s}{e_p} \quad (14.28)$$

or
$$e_s = e_p \left(\frac{N_s}{N_p} \right)$$

Thus, there is an increase in the voltage in the secondary if $N_s > N_p$. This is known as a step-up transformer. For $N_s < N_p$, $e_s < e_p$ and we call this a step-down transformer. The increase in voltage results in a smaller current and vice versa, thus satisfying the energy conservation.

In an ideal transformer, the entire power is transferred from the primary to the secondary. Hence, for an ideal transformer, we have

$$e_s i_s = e_p i_p$$

$$\therefore \frac{e_s}{e_p} = \frac{L_p}{L_s} = \frac{N_s}{N_p} \quad (14.29)$$

14.1.23 Earth's Magnetic Field A freely suspended magnetic needle at any point on earth assumes a definite direction. Except near the poles, the magnetic lines of earth's field lie in a roughly north-south direction, with some tilting away from the horizontal. The angle between the direction of a compass needle and the true north-south line is known as the angle of declination δ , as shown in Fig. 14.7.

The ordinary compass tells the direction of the horizontal component of the earth's field. There is also a vertical component, for the real direction of the field is everywhere inclined. If B is the total magnetic field at a point, the horizontal and vertical components are given by (see Fig. 14.7)

$$B_H = B \cos \theta$$

and

$$B_V = B \sin \theta$$

so that $\tan \theta = \frac{B_V}{B_H}$

Angle θ is called the magnetic dip at that point.

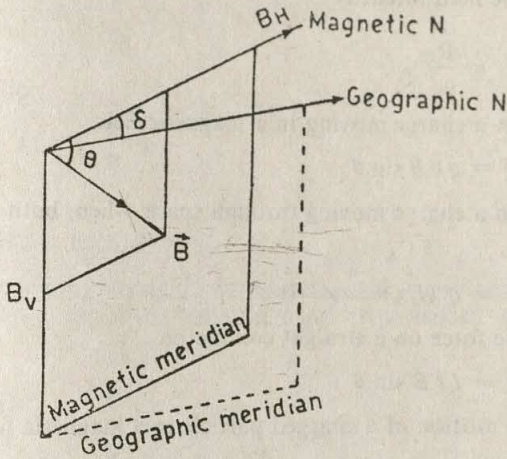


Fig. 14.7

14.1 SI UNITS

Table 14.1

Physical Quantity	SI Unit	Symbol
Magnetic flux	weber	Wb
Magnetic induction (or flux density)	tesla or weber per square metre	T or Wb m ⁻²
Magnetic field intensity	ampere per metre	A m ⁻¹
Magnetic moment of current carrying circuit	ampere square metre	A m ²
Inductance	henry	H
Magnetic permeability	henry per metre	H m ⁻¹

14.3 FORMULAE

1. Magnetic induction due to a long straight wire at a point outside.

$$|\vec{B}| = \left(\frac{\mu_0}{4\pi} \right) \frac{2i}{r} = 10^{-7} \left(\frac{2i}{r} \right)$$

2. Magnetic induction on the axis of a circular coil

$$B = \left(\frac{\mu_0}{4\pi} \right) \frac{2\pi n i r^2}{(x^2 + r^2)^{3/2}}$$

3. Magnetic induction at the centre of a circular coil

$$B = \left(\frac{\mu_0}{4\pi} \right) \frac{2\pi n i}{r}$$

4. Magnetic field intensity

$$\vec{H} = \frac{\vec{B}}{\mu_0}$$

5. Force on a charge moving in a magnetic field.

$$F = q v B \sin \theta$$

6. Force on a charge moving through space when both
- E
- and
- B
- are present

$$\vec{F} = q (\vec{E} + \vec{v} \times \vec{B})$$

7. Magnetic force on a straight conductor.

$$F = i l B \sin \theta$$

8. Circular motion of a charged particle in a magnetic field

$$\text{Radius } r = \frac{mv}{Bq}$$

$$\text{Angular speed } \omega = \frac{Bq}{m}$$

$$\text{Frequency } n = \frac{Bq}{(2\pi m)}$$

9. Force between two straight parallel wires

$$F = \left(\frac{\mu_0}{4\pi} \right) \frac{2 i_1 i_2}{r}$$

10. Torque acting on a current loop

$$\tau = N i A B \sin \theta = M B \sin \theta$$

where $M = N i A$ is the magnetic moment of the current loop.

11. Magnetic flux

$$\phi = N B A \cos \theta$$

12. Induced emf

$$e = - d\phi/dt = - L \left(\frac{di}{dt} \right)$$

13. Induced emf in a rod moving in a magnetic field

$$e = - B l v$$

14. Self inductance of a solenoid

$$L = \mu_0 N^2 \frac{A}{l}$$

15. Mutual inductance of a solenoid.

$$M = \mu_0 A N_1 \frac{N_2}{l}$$

16. Voltage in the secondary of a transformer.

$$e_s = \left(\frac{N_s}{N_p} \right) e_p = \left(\frac{i_p}{i_s} \right) e_p$$

17. Angle of dip

$$\theta = \tan^{-1} \left(\frac{B_V}{B_H} \right)$$

14.4 NUMERICAL EXAMPLES

Example 1: Three long, straight and parallel wires, carrying currents, are arranged as shown in Fig. 14.8. Find the force experienced by a 25 cm length of wire C.

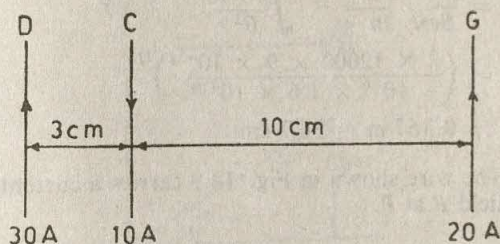


Fig. 14.8

Solution: The magnetic field due to wire D at wire C is

$$B_D = \left(\frac{\mu_0}{4\pi} \right) \frac{2i}{r} = \frac{10^{-7} \times 2 \times 30}{0.03} = 2 \times 10^{-4} \text{ T}$$

which is directed into the page.

Similarly, the field due to wire G at C is

$$B_G = \frac{10^{-7} \times 2 \times 20}{0.1} = 0.4 \times 10^{-4} \text{ T}$$

which is directed out of the page.

Therefore, the field at the position of the wire C is

$$B = B_D - B_G = 2 \times 10^{-4} - 0.4 \times 10^{-4} = 1.6 \times 10^{-4}$$

and is directed into the page.

The force on 25 cm of wire C is

$$F = B i l \sin 90^\circ = 1.6 \times 10^{-4} \times 10 \times 0.25 = 4 \times 10^{-4} \text{ N}$$

Using the right-hand rule, we find that the force on wire C is towards the right.

Example 2: An electron is accelerated by a potential difference of 12000 V. It then enters a uniform magnetic field of 10^{-3} T applied perpendicular to

its path. Find the radius of the path. Mass of an electron, $m_e = 9 \times 10^{-31}$ kg and charge of an electron, $e = 1.6 \times 10^{-19}$ C.

Solution: Force acting on the electron $= B e v$

where v is the velocity of the electron. If r is the radius of its path, we have

$$\frac{mv^2}{r} = B e v$$

or
$$r = \frac{m v}{B e}$$

The velocity v is given by

$$\frac{1}{2} m v^2 = e V$$

Hence

$$\begin{aligned} r &= \frac{m}{B e} \sqrt{\frac{2 e V}{m}} = \sqrt{\frac{2 V m}{B^2 e}} \\ &= \left(\frac{2 \times 12000 \times 9 \times 10^{-31}}{10^{-6} \times 1.6 \times 10^{-19}} \right)^{1/2} \\ &= 0.367 \text{ m} = 36.7 \text{ cm} \end{aligned}$$

Example 3: The wire shown in Fig. 14.9 carries a current of 60 A. Find the magnetic field B at P .

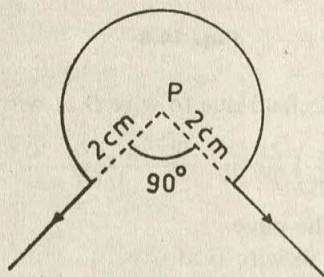


Fig. 14.9

Solution: The field at P arises from $3/4$ th of the circular loop only because P lies on the straight wires themselves. Thus

$$\begin{aligned} B &= \frac{3}{4} \left(\frac{\mu_0}{4 \pi} \right) \frac{2 \pi i}{r} \\ &= \frac{3 \times 10^{-7} \times 2 \pi \times 60}{4 \times 0.02} \\ &= 1.4 \times 10^{-3} \text{ T} \end{aligned}$$

which is directed out of the page.

Example 4: A proton (charge $q = 1.6 \times 10^{-19}$ C, mass $m = 1.67 \times 10^{-27}$ kg) is shot with a speed 8×10^6 m s $^{-1}$ at an angle of 30° with the x -axis.

A uniform magnetic field $B = 0.30 \text{ T}$ exists along the x -axis. Show that path of the proton is a helix. Find the radius and the pitch of the helix.

Solution: The x and y components of the velocity of the proton are (see Fig. 14.10)

$$v_x = v \cos 30^\circ = 8 \times 10^6 \times 0.866 = 6.93 \times 10^6 \text{ m s}^{-1}$$

$$v_y = v \sin 30^\circ = 8 \times 10^6 \times 0.5 = 4.0 \times 10^6 \text{ m s}^{-1}$$

The magnetic force due to v_x is zero ($F = q v B \sin 0^\circ = 0$) since B and v_x are along the same direction. The particle moves in a helical path, as shown in Fig. 14.10 due to the combined action of v_x , v_y and the magnetic field B . The magnetic force due to v_y has no x -component. Therefore, the motion along the x -axis is uniform, at speed v_x .

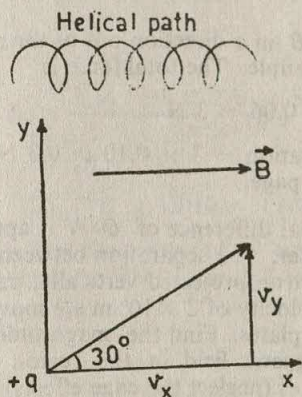


Fig. 14.10

The radius of the circular transverse motion is

$$r = \frac{m v_y}{qB} = \frac{1.67 \times 10^{-27} \times 4 \times 10^6}{1.6 \times 10^{-19} \times 0.3} = 0.139 \text{ m}$$

The proton will spiral along the x -axis; the radius of the spiral (or helix) will be 13.9 cm.

To find the pitch of the helix (the x -distance travelled during one revolution), the time taken to complete one circle is

$$T = \frac{2\pi r}{v_y} = \frac{2\pi \times 0.139}{4 \times 10^6} = 2.19 \times 10^{-7} \text{ s}$$

During this time, the proton will travel an x -distance (pitch) given by

$$\begin{aligned} \text{Pitch} &= v_x \times T \\ &= 6.93 \times 10^6 \times 2.19 \times 10^{-7} \\ &= 1.515 \text{ m} \end{aligned}$$

Example 5: A 50 turn coil shown in Fig. 14.11 carries a current of 2 A in a magnetic field $B = 0.25 \text{ Wb m}^{-2}$. Find the torque acting on the coil. In what direction will it rotate?

Solution: The sides AB and DC are along the field lines; hence the force on each side is zero. The force on each vertical wire is

$$f = i l B = 2 \times 0.12 \times 0.25 = 0.060 \text{ N}$$

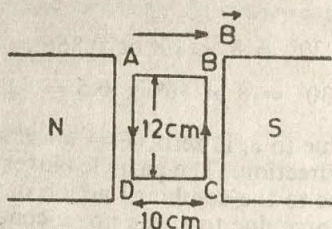


Fig. 14.11

The force f acts on AB in a direction out of the page and on BC into the page, thus forming a couple. The total force is

$$F = 50 \times 0.06 = 3 \text{ N}$$

Torque $\tau = F \times \text{separation} = 3 \times 0.10 = 0.3 \text{ N m}$ and tends to rotate the side AD out of the page.

Example 6: A potential difference of 600V is applied across the plates of a parallel plate condenser. The separation between the plates is 3 mm. An electron projected vertically, parallel to the plates, with a velocity of $2 \times 10^6 \text{ m s}^{-1}$ moves undeflected between the plates. Find the magnitude and direction of the magnetic field in the region between the condenser plates (neglect the edge effects) (IIT 1981)

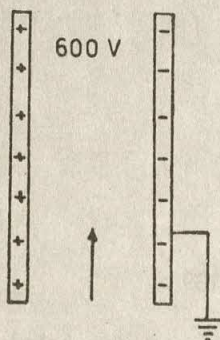


Fig. 14.12

Solution: Electric intensity $E = \frac{V}{d}$

where V is the potential difference between the plates and d , the separation between them.

$$d = 3 \text{ mm} = 3 \times 10^{-3} \text{ m}$$

$$E = \frac{V}{d} = \frac{600}{3 \times 10^{-3}} = 2 \times 10^5 \text{ V m}^{-1}$$

Since the electron moves undeflected between the plates, the force due to magnetic field must balance the force due to electric field. Thus

$$B e v = e E$$

$$\text{or } B = \frac{E}{v} = \frac{2 \times 10^5}{2 \times 10^6} = 0.1 \text{ T}$$

The magnetic field is perpendicular to the plane of the paper directed inwards.

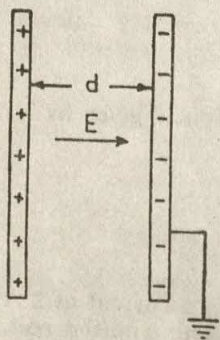


Fig. 14.13

Example 7: A coil of 50 turns and 10 cm diameter is made out of a wire of resistivity $2 \times 10^{-6} \Omega \text{ cm}$. The coil is connected to a source of emf 10 V and of negligible internal resistance.

- (a) Find the current through the coil.
 (b) What must be the potential difference across the coil so as to nullify the earth's magnetic field, $H = 0.314 \text{ G}$ at the centre of the coil. How should the coil be placed to achieve this result? (IIT 1964)

Solution:

(a) Radius of the coil, $r = 10/2 = 5 \text{ cm}$

$$\begin{aligned}\text{Length of the wire, } l &= (2 \pi r) \times 50 \\ &= 2 \pi \times 5 \times 50 \text{ cm}\end{aligned}$$

$$\text{Radius of the wire} = 0.1 \text{ mm} = 0.01 \text{ cm}$$

$$\begin{aligned}\text{Area of cross-section of the wire} &= \pi \times (0.01)^2 \\ &= \pi \times 10^{-4} \text{ cm}^2\end{aligned}$$

$$\text{Resistance of the wire } R = \frac{\rho l}{A}$$

$$\begin{aligned}\text{Therefore, } R &= \frac{2 \times 10^{-6} \times 2 \pi \times 5 \times 50}{\pi \times 10^{-4}} \\ &= 10 \Omega\end{aligned}$$

$$\text{Current in the wire } i = E/R = 10/10 = 1.0 \text{ A}$$

(b) Magnetic induction B at the centre of a circular coil of radius r having n turns and carrying a current of $i = 1 \text{ A}$ is given by

$$B = \left(\frac{\mu_0}{4 \pi} \right) \frac{2 \pi n i}{r}$$

$$\text{Here } \frac{\mu_0}{4 \pi} = 10^{-7} \text{ Wb m}^{-1} \text{ A}^{-1}, \quad r = 5 \text{ cm} = 5 \times 10^{-2} \text{ m}$$

$$B = \frac{10^{-7} \times 2 \pi \times 50 \times i}{5 \times 10^{-2}}$$

The magnetic induction due to the earth's magnetic field

$$= 0.314 \times 10^{-4} \text{ T}$$

$$\text{Therefore, } \frac{10^{-7} \times 2 \pi \times 50 \times i}{5 \times 10^{-2}} = 0.314 \times 10^{-4}$$

$$\begin{aligned}\therefore i &= \frac{0.314 \times 10^{-4} \times 5 \times 10^{-2}}{2 \pi \times 50 \times 10^{-7}} \\ &= 5 \times 10^{-2} \text{ A}\end{aligned}$$

The potential difference across the coil $= i R$

$$= 5 \times 10^{-2} \times 10$$

$$= 0.5 \text{ V}$$

The plane of the coil should be normal to the magnetic meridian.

Note: B for earth's magnetic field is not obtained by equating it to $\mu_0 H$, because H is in CGS units (namely, gauss) and not in ampere per metre. Here, in CGS units, B and H are identical and $B = 2 \pi n i / r \text{ G}$ if i is in emu and r in cm.

Example 8: Two coils each of 100 turns are held such that one lies in the vertical plane and the other in the horizontal plane with their centres coinciding. The radius of the vertical coil is 0.20 m and that of the horizontal coil is 0.30 m. How would you neutralize the magnetic field of the earth at their common centre? What is the current to be passed through each coil? Horizontal component of earth's magnetic field $= 0.35 \times 10^{-4}$ tesla and angle of dip $= 30^\circ$. (IIT 1968)

Solution: Angle of dip $\theta = \tan^{-1} \left(\frac{B_V}{B_H} \right)$ where B_V and B_H are the vertical and horizontal components of earth's field respectively.

Since $B_H = 0.35 \times 10^{-4}$ T, we have

$$\begin{aligned} B_V &= B_H \tan 30^\circ \\ &= 0.35 \times 10^{-4} \times 0.577 \\ &= 0.202 \times 10^{-4} \text{ T} \end{aligned}$$

The direction of the magnetic field will be perpendicular to the plane of either coil. If the plane of the vertical coil is perpendicular to the magnetic meridian, the field produced by it can neutralize the horizontal component of earth's field if

$$B_H = \left(\frac{\mu_0}{4\pi} \right) \frac{2\pi n i}{r} = 0.35 \times 10^{-4}$$

$$\begin{aligned} \text{or } i &= \frac{0.35 \times 10^{-4} \times r}{2\pi n} \left(\frac{4\pi}{\mu_0} \right) \\ &= \frac{0.35 \times 10^{-4} \times 0.20 \times 10^7}{2\pi \times 100} = 0.111 \text{ A} \end{aligned}$$

Similarly, the magnetic field produced by the horizontal coil will be vertical and will neutralize the vertical component B_V of the earth's field, if

$$B_V = \left(\frac{\mu_0}{4\pi} \right) \frac{2\pi n i}{r} = 0.202 \times 10^{-4}$$

$$\begin{aligned} \text{or } i &= \frac{0.202 \times 10^{-4} \times 10^7 \times 0.30}{2\pi \times 100} \\ &= 0.0964 \text{ A} \end{aligned}$$

Example 9: A particle of mass 1×10^{-26} kg and charge $+1.6 \times 10^{-19}$ C travelling with a velocity 1.28×10^6 m s⁻¹ in the positive x-direction enters a region in which a uniform electric field E and a uniform magnetic field of induction B are present such that $E_x = E_y = 0$, $E_z = -102.4$ kV m⁻¹ and $B_x = B_z = 0$, $B_y = 8 \times 10^{-2}$ T. The particle enters this region at time $t = 0$. Determine the location (x , y and z co-ordinates) of the particle at $t = 5 \times 10^{-6}$ s. If the electric field is switched off at this instant (with the magnetic field still present), what will be the position of the particle at $t = 7.45 \times 10^{-6}$ s? (IIT 1982)

Solution: It is clear from Fig. 14.14 that the force on the particle due to the electric field E is

$$\begin{aligned} F_e &= qE = 1.6 \times 10^{-19} \times 102.4 \times 10^3 \\ &= 163.8 \times 10^{-16} \text{ N} \end{aligned}$$

in the negative z -direction. Similarly, the force due to the magnetic field B_y is given by

$$F_m = q v B_y = 1.6 \times 10^{-19} \times 1.28 \times 10^6 \times 8 \times 10^{-2} \\ = 163.8 \times 10^{-16} \text{ N}$$

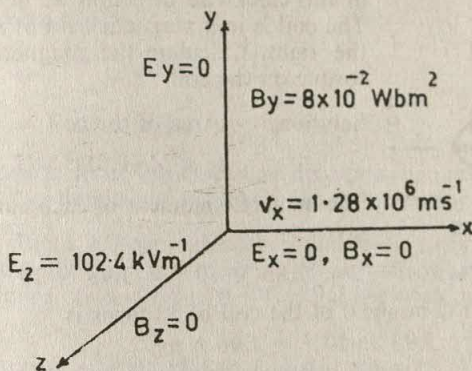


Fig. 14.14

in the positive z -direction. These two forces are equal and opposite and are collinear. Hence, the net force on the particle is zero and it, therefore, continues to travel along the x -axis. The distance travelled by it in 5×10^{-6} s is equal to

$$5 \times 10^{-6} \times 1.28 \times 10^6 = 6.4 \text{ m}$$

Hence, the co-ordinates of the particle are (6.4 m, 0, 0).

When the electric field is switched off, the particle moves in a circular path under the magnetic field B_y in the x - z plane in an anticlockwise direction as seen along the $+y$ -axis.

$$\frac{mv^2}{r} = B q v \quad \text{or} \quad r = \frac{mv}{Bq} \\ r = \frac{1 \times 10^{-26} \times 1.28 \times 10^6}{8 \times 10^{-2} \times 1.6 \times 10^{-19}} \\ = 1.0 \text{ m}$$

The time period of revolution of the particle is

$$T = \frac{2\pi r}{v} = \frac{2\pi m}{Bq} \\ = \frac{2\pi \times 10^{-26}}{8 \times 10^{-2} \times 1.6 \times 10^{-19}} \\ = 4.9 \times 10^{-6} \text{ s}$$

The time for which the magnetic field acts is equal to

$$7.45 \times 10^{-6} - 5.0 \times 10^{-6} = 2.45 \times 10^{-6} \text{ s}$$

Hence, the number of revolutions completed by the particle during this time is

$$\frac{2.45 \times 10^{-6}}{4.9 \times 10^{-6}} = \frac{1}{2} \text{ revolution.}$$

Therefore, the displacement of the particle along the z -direction is equal to 2 m. Hence, its co-ordinates will be (6.4 m, 0, 2 m).

Example 10: A circular coil of wire 0.05 m in radius, having 40 turns lies in a horizontal plane as shown in Fig. 14.15. It carries a current of 5 A, in anti-clockwise direction as seen from above. The coil is in a magnetic field of 1.2 T directed to the right. Calculate the magnetic moment and torque on the coil.

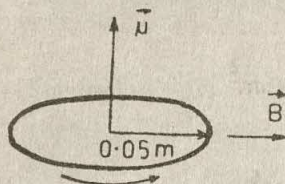


Fig. 14.15

Solution: Area of the coil $= \pi r^2$
 $= \pi \times (0.05)^2 = 7.85 \times 10^{-3} \text{ m}^2$

The magnetic moment of each turn of the coil is given by

$$M = i A = 5 \times 7.85 \times 10^{-3} = 3.93 \times 10^{-2} \text{ A m}^2$$

The total magnetic moment of the coil of 50 turns is

$$50 \times 3.93 \times 10^{-2} = 1.96 \text{ A m}^2$$

The torque on the coil is given by

$$\tau = M B \sin \alpha$$

Since the angle α between the direction of B and the normal to the plane of the coil is 90° , we have

$$\tau = M B = 1.96 \times 1.2 = 2.36 \text{ N m}$$

Alternatively,

$$\begin{aligned} \tau &= (i B A \sin \alpha) \times 50 \\ &= 5 \times 1.2 \times 7.85 \times 10^{-3} \times 1 \times 50 \\ &= 2.36 \text{ N m} \end{aligned}$$

Example 11: A 50 cm long bar AB is moved with a speed of 4 m s^{-1} in a magnetic field $B = 0.01 \text{ Wb m}^{-2}$, as shown in Fig. 14.16. Find the emf generated and the power expended if the resistance of the circuit is 0.1Ω .

Solution: Induced emf $e = B l v$
 $= 0.01 \times 0.5 \times 4 = 0.02 \text{ V}$

Power dissipated $P = \frac{e^2}{R} = \frac{0.02 \times 0.02}{0.1}$
 $= 0.004 \text{ W}$

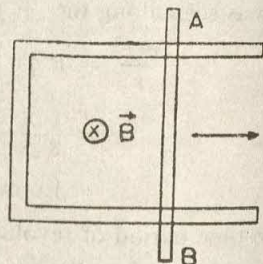


Fig. 14.16

Example 12: A square loop of wire of side 5 cm is lying on a horizontal table. An electromagnet above and to one side of the loop is turned on, causing a uniform magnetic field that is downwards at an angle of 30° to the vertical, as shown in Fig. 14.17. The magnetic induction is 0.50 T. Calculate the average induced emf in the loop, if the field increases from zero to its final value in 0.2 s.

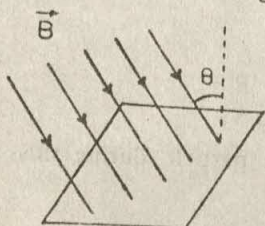


Fig. 14.17

Solution: Area of the loop, $A = (5 \times 10^{-2})^2 \text{ m}^2$
 $= 2.5 \times 10^{-3} \text{ m}^2$

Magnetic flux, $\phi = A B \cos \theta$

where $B \cos \theta$ is the component of magnetic induction perpendicular to the plane of the loop. Therefore,

$$\begin{aligned}\phi &= 2.5 \times 10^{-3} \times 0.50 \times \cos 30^\circ \\ &= 1.0825 \times 10^{-3} \text{ Wb}\end{aligned}$$

Average induced emf is

$$\begin{aligned}\bar{e} &= \frac{\Delta \phi}{\Delta t} = \frac{1.0825 \times 10^{-3}}{0.2} \\ &= 5.4 \times 10^{-3} \text{ V}\end{aligned}$$

The induced emf is in an anticlockwise direction around the loop during the time when the downward magnetic flux is increasing.

Example 13: In an AC generator, the number of turns in the coil is 2000 and the area of the coil is 0.15 m^2 . The coil is rotated at a rate of 60 cycles per second in a field of 0.1 T . Find the peak value of the emf induced.

Solution: Actual emf induced $= n A B \omega \sin \omega t$

Here $\omega = 2\pi \times 60 \text{ rad s}^{-1}$

The peak value e_0 of the emf is obtained whenever ωt is an integral multiple of $\pi/2$ and hence equals

$$\begin{aligned}e_0 &= n A B \omega \\ &= 2000 \times 0.15 \times 0.1 \times 2\pi \times 60 \\ &= 1.131 \times 10^4 \text{ V}\end{aligned}$$

Example 14: A square metal wire loop of side 10 cm and resistance 1Ω is moved with a constant velocity v_0 in a uniform magnetic field of induction $B = 2 \text{ Wb m}^{-2}$, as shown in Fig. 14.18. The magnetic field lines are perpendicular to the plane of the loop (directed into the paper). The loop is connected to network of resistors each of value 3Ω . The resistances of the lead wires OS and PQ are negligible. What should be the speed of the loop so as to have a steady current of 1 mA in the loop? Give the direction of the current in the loop. (IIT 1983)

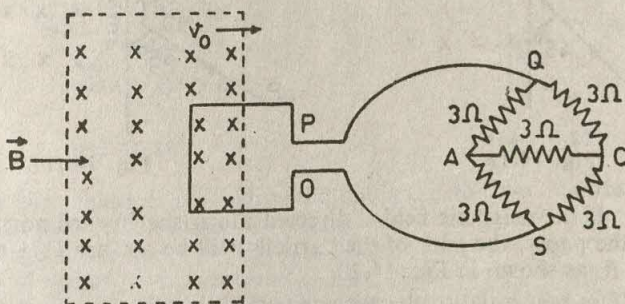


Fig. 14.19

Solution: From the symmetry of the network $AQCS$, it is clear that the points A and C are at the same potential and hence no current flows between A and C through the 3Ω resistance. The resistance R of the network

is, therefore, given by $\frac{1}{R} = \frac{1}{6} + \frac{1}{6}$ or $R = 3 \Omega$

Resistance of the loop $r = 1 \Omega$

Total resistance $= 3 + 1 = 4 \Omega$.

The emf induced by a wire of length l moving in a uniform magnetic field B with a constant velocity v is given by

$$e = B l v_0 = 2 \times 0.1 v_0 = 0.2 v_0$$

$$\text{Current in the circuit, } i = \frac{e}{R} = \frac{B l v_0}{R} = \frac{0.2 v_0}{4},$$

But $i = 1 \text{ mA} = 1 \times 10^{-3} \text{ A}$. Hence

$$v_0 = \frac{4 \times 10^{-3}}{0.2} = 2 \times 10^{-2} \text{ m s}^{-1}$$

The direction of the induced current in the loop is from O and P (i.e. clockwise).

Example 15: A particle of mass $m = 1.6 \times 10^{-27} \text{ kg}$ and charge $q = 1.6 \times 10^{-19} \text{ C}$ enters a region of uniform magnetic field of strength 1 T along the direction shown in Fig. 14.19. The speed of the particle is 10^7 m s^{-1} .

(i) The magnetic field is directed along the inward normal to the plane of the paper. The particle leaves the region of the field at the point F . Find the distance EF and the angle θ . (ii) If the direction of the magnetic field is along the outward normal to the plane of the paper, find the time spent by the particle in the region of the magnetic field after entering it at E .
(IIT 1984)

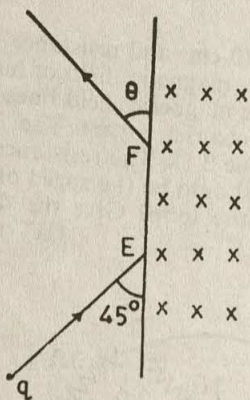


Fig. 14.19

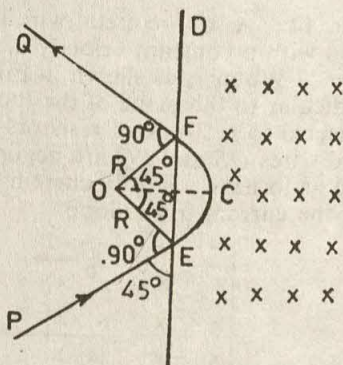


Fig. 14.20

Solution: If the magnetic field is directed along the inward normal to the plane of the paper, the path of the particle will be an arc FCE of a circle of radius R , as shown in Fig. 14.20.

Since PE is tangential to the arc and $\angle OEP$ is 90° , therefore, $\angle COE = 45^\circ$. Similarly, $\angle QFD = 45^\circ$, because QF is a tangent to the arc CF at F .

The angle between the lines DF and FQ is equal to the angle between their perpendiculars. Hence $\angle FOC = 45^\circ$. Also

$$EF = 2 R \cos 45^\circ = 2 R \times \frac{1}{\sqrt{2}} = \sqrt{2} R$$

The radius of the circular path is given by

$$R = \frac{mv}{Bq}$$

Here $m = 1.6 \times 10^{-27}$ kg, $v = 10^7$ m s⁻¹, $B = 1$ T and $q = 1.6 \times 10^{-19}$ C. Substituting these values, we get

$$R = 0.1 \text{ m}$$

Hence $EF = \sqrt{2} R = 0.1 \times \sqrt{2} = 0.1414 \text{ m}$

(ii) When the magnetic field is directed along the outward normal to the plane of the paper, the circular path EPQ of the particle will be clockwise, as shown in Fig. 14.21.

Since the radii $P'E$ and $P'Q$ are perpendicular to their respective tangents drawn at points E and Q , we have

$$\text{Length of arc } EPQ = 2\pi R - \frac{\pi R}{2} = \frac{3\pi R}{2}$$

where R is the radius of the arc. ($\pi R/2$, length of the arc in the previous case (i)).

The time spent by the particle along the arc EPQ is given by

$$t = \frac{\text{distance covered}}{\text{speed}} = \frac{3\pi R}{2v}$$

where $R = 0.1$ m and $v = 10^7$ m s⁻¹. Putting these values, we get

$$t = \frac{3\pi \times 0.1}{2 \times 10^7} = 4.7 \times 10^{-8} \text{ s}$$

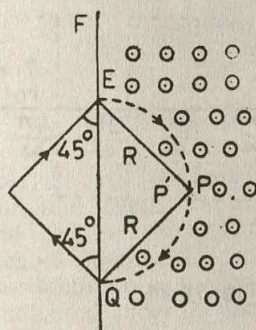


Fig. 14.21

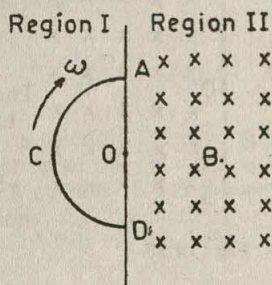


Fig. 14.22

Example 16: Space is divided by the line AD into two regions. Region I is field free and region II has a uniform magnetic field B directed into the plane of the paper as shown in Fig. 14.22. ACD is a semicircular conducting loop of radius r with centre at O , the plane of the loop being in the plane of the paper. The loop is now made to rotate with a constant angular velocity ω about an axis passing through O and perpendicular to the plane of the paper. The effective resistance of the loop is R .

- Obtain an expression for the magnitude of the induced current in the loop.
- Show the direction of the current when the loop is entering into the region II.

- (iii) Plot a graph between the induced emf and the time of rotation for two periods of rotation. (IIT 1985)

Solution: (i) Induced emf $e = -\frac{d\phi}{dt} = -B \frac{dA}{dt}$

where A is the area of loop, $\phi = BA$ because the normal to the loop and field B are along the same line.

When a loop of radius r rotates with an angular velocity $\omega = \frac{d\theta}{dt}$, the rate of change of area is given by

$$\frac{dA}{dt} = \frac{d}{dt} \left(\frac{1}{2} r^2 \theta \right)$$

where $\frac{1}{2} r^2 \theta$ = area swept out by the loop when it rotates through an angle θ . Therefore, we have

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} r^2 \omega$$

$$e = -\frac{1}{2} B r^2 \omega$$

Since R is the effective resistance of the loop, the magnitude of the induced current is

$$i = \frac{e}{R} = \frac{B r^2 \omega}{2 R}$$

- (ii) The direction of the current is anti-clockwise, i.e. opposite to the direction of rotation of the loop and is shown in Fig. 14.23.

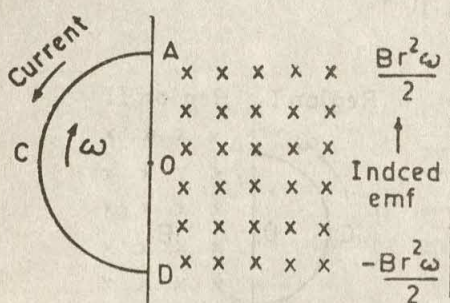


Fig. 14.23

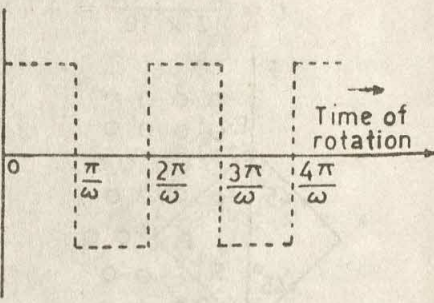


Fig 14.24

- (iii) The time period of rotation is equal to $2\pi/\omega$. The graph between the induced emf and the time of rotation is shown in Fig. 14.24.

Example 17: A beam of protons with a velocity $4 \times 10^5 \text{ m s}^{-1}$ enters a uniform magnetic field of 0.3 tesla at an angle of 60° to the magnetic field. Find the radius of the helical path taken by the proton beam. Also find the pitch of the helix (which is the distance travelled by a proton during one period of rotation). (IIT 1986)

Solution: Refer to Fig. 14.10 of Ex. 4. This problem is exactly similar except that here the angle is 60° . Following the procedure of Ex. 4, we have

$$v_x = v \cos 60^\circ = 2 \times 10^5 \text{ m s}^{-1}$$

$$v_y = v \sin 60^\circ = 3.464 \times 10^5 \text{ m s}^{-1}$$

$$\begin{aligned}
 \text{Radius of helical path } r &= \frac{m v}{q B} \\
 &= \frac{1.67 \times 10^{-27} \times 3.464 \times 10^5}{1.6 \times 10^{-19} \times 0.3} \\
 &= 1.205 \times 10^{-2} \text{ m} = 1.2 \text{ cm}
 \end{aligned}$$

This is the radius of the circular path which spirals along the x -axis. The pitch of the helix is found by computing the distance covered along the x -axis during the time taken to complete one circle, which is equal to

$$T = \frac{2 \pi r}{v_y} = \frac{2 \pi \times 1.205 \times 10^{-2}}{3.464 \times 10^5} = 2.19 \times 10^{-7} \text{ s}$$

$$\begin{aligned}
 \text{Pitch} &= v_x \times T = 2 \times 10^5 \times 2.19 \times 10^{-7} = 4.37 \times 10^{-2} \text{ m} \\
 &= 4.4 \text{ cm}
 \end{aligned}$$

Example 18: A circular coil of 1000 turns of wire has an enclosed area of 0.05 m^2 . It is kept perpendicular to a magnetic field of induction 0.2 T and rotated by 180° about a diameter perpendicular to the field in 0.05 s . Calculate the emf induced. How much charge will pass when the coil is connected to a galvanometer with a combined resistance of 50Ω ?

Solution: Induced emf $e = \frac{\text{change in flux}}{\text{time}} = \frac{\Delta \phi}{\Delta t}$

$$\begin{aligned}
 \Delta \phi &= n A B (\cos \theta_1 - \cos \theta_2) \\
 &= n A B (\cos 0^\circ - \cos 180^\circ) = 2 n A B
 \end{aligned}$$

now $n = 1000, A = 0.05 \text{ m}^2, B = 0.2 \text{ T}, \Delta t = 0.05 \text{ s}$

$$e = \frac{2 \times 1000 \times 0.05 \times 0.2}{0.05} = 400 \text{ V}$$

$$\text{Charge } q = i \Delta t = \frac{e \Delta t}{R} = \frac{400 \times 0.05}{50} = 0.4 \text{ C}$$

Example 19: An iron core is inserted into a solenoid 0.5 m long with 400 turns per unit length. The area of cross-section of the solenoid is 0.01 m^2 . (a) Find the permeability of the core when a current of 5 A flows through the solenoid winding. Under these conditions, the magnetic flux through the cross-section of the solenoid is $1.6 \times 10^{-3} \text{ Wb}$. (b) Find the inductance of the solenoid under these conditions.

Solution: The magnetic induction on the axis of the solenoid is given by

$$B = \mu \left(\frac{\mu_0}{4 \pi} \right) 4 \pi n i$$

where μ is the permeability of the medium, n the number of turns per unit length and i is the current.

(a) Magnetic flux $\phi = B A$, since the normal to the area is along the direction of the field.

$$\text{Given } \phi = 1.6 \times 10^{-3} \text{ Wb}, A = 0.001 = 10^{-3} \text{ m}^2$$

$$\text{Therefore, } B = \phi/A = \frac{1.6 \times 10^{-3}}{10^{-3}} = 1.6 \text{ Wb m}^{-2}$$

Since $n = 400$, $i = 5$ A and $\frac{\mu_0}{3\pi} = 10^{-7}$ H m⁻¹, we have

$$1.6 = \mu \times 10^{-7} \times 4\pi \times 400 \times 5$$

which gives

$$\mu = \frac{1.6 \times 10^7}{4\pi \times 5 \times 400} = 636.7 \approx 637$$

(b) Total number of turns in the solenoid is given by

$$N = n \times l = 400 \times 0.5 = 200$$

Total flux through the solenoid is

$$N\phi = 200 \times 1.6 \times 10^{-3} = 0.32 \text{ Wb}$$

$$\text{Self inductance } L = \frac{N\phi}{i} = \frac{0.32}{5} = 0.064 \text{ H} = 64 \text{ mH}$$

Alternatively,

$$\begin{aligned} L &= \frac{\mu \mu_0 N^2 A}{l} = \mu \mu_0 n^2 l A \\ &= 637 \times (4\pi \times 10^{-7}) \times 400 \times 400 \times 0.5 \times 10^{-3} \\ &= 6.4 \times 10^{-2} \text{ H} = 64 \text{ mH} \end{aligned}$$

Example 20: A power transformer is used to step up an alternating emf of 220 V to 4.4 kV to transmit 6.6 kW of power. If the primary coil has 1000 turns, find (a) the number of turns in the secondary and (b) the current rating of the secondary. Assume a 100% efficiency for the transformer.

Solution: (a) Using Eq. (14.29), the number of turns in the secondary is given by

$$\begin{aligned} N_s &= \left(\frac{e_s}{e_p} \right) N_p \\ &= \frac{4.4 \times 1000 \times 1000}{220} = 20,000 \end{aligned}$$

(b) Current in the primary is given by

$$i_p e_p = 6.6 \times 10^3 \text{ W}$$

$$\text{or } i_p = \frac{6.6 \times 10^3}{220} = 30 \text{ A}$$

$$\text{But } \frac{i_s}{i_p} = \frac{N_p}{N_s} = \frac{1000}{20,000} = \frac{1}{20}$$

$$i_s = \frac{1}{20} \times i_p = \frac{30}{20} = 1.5 \text{ A}$$

Thus the current rating of the secondary is 1.5 A

Example 21: A 5 Ω coil of 100 turns and diameter 5 cm, is placed between the poles of a magnet so that the flux is maximum through its area. When the coil is suddenly removed from the field of the magnet, a charge of 10^{-4}

C flows through a 495Ω galvanometer connected to the coil. Compute the magnetic induction B between the poles of the magnet.

Solution: As the coil is removed, the change in flux is given by

$$\Delta \phi = B A$$

where A is the area of the coil. Therefore,

$$|e| = N \frac{\Delta \phi}{\Delta t} = \frac{N B A}{\Delta t}$$

But from Ohm's law

$$\frac{|e|}{R} = i = \frac{\Delta q}{\Delta t}$$

$$\text{or } |e| = R \frac{\Delta q}{\Delta t}$$

$$\therefore R \frac{\Delta q}{\Delta t} = \frac{N B A}{\Delta t}$$

$$\text{or } B = \frac{R \Delta q}{N A}$$

$$\text{Here } R = 495 + 5 = 500 \Omega, A = \pi r^2 = \pi (2.5 \times 10^{-2})^2$$

$$B = \frac{500 \times 10^{-4}}{100 \times \pi \times 6.24 \times 10^{-4}} = 0.255 \text{ T}$$

Example 22: The two rails of a railway track, insulated from each other, and the ground, are connected to a millivoltmeter. What is the reading of the voltmeter when a train travels at a speed of 180 km h^{-1} along the track? Given vertical component of earth's magnetic field $= 0.2 \times 10^{-4} \text{ T}$ and the separation between the rails $= 1 \text{ m}$.

Solution: The induced emf generated is given by

$$e = - \frac{d\phi}{dt} = - \frac{d}{dt} (BA) = - B \frac{dA}{dt}$$

where A is the area and B , the magnetic field. If l is the distance between the rails and v , the speed of the train, then

$$\frac{dA}{dt} = l v$$

$$e = - B l v$$

Here $l = 1 \text{ m}$ and $B = 0.2 \times 10^{-4} \text{ T}$. Thus we have

$$v = \frac{180 \times 1000}{3600} = 50 \text{ m s}^{-1}$$

$$|e| = 0.2 \times 10^{-4} \times 1 \times 50 = 1 \times 10^{-3} \text{ V} = 1 \text{ mV}$$

Hence, the millivoltmeter will read 1 mV .

Example 23i A beam of electrons accelerated by a potential difference of 300 V , is directed into a uniform magnetic field of flux density $1.46 \times 10^{-3} \text{ Wb m}^{-2}$, which is perpendicular to the plane of the paper, and outwards

as shown in Fig. 14.25. The field acts over a width $l = 2.5$ cm. In the absence of the magnetic field, the electron beam produces a spot at point F on a fluorescent screen AA' at a distance of $l_1 = 5$ cm from the edge of the poles of the magnet. When the magnetic field is switched on, the spot moves to a point B . Find the displacement $x = FB$ of the electron beam. Electron charge $= -1.6 \times 10^{-19}$ C and electron mass $(m) = 9.1 \times 10^{-31}$ kg.

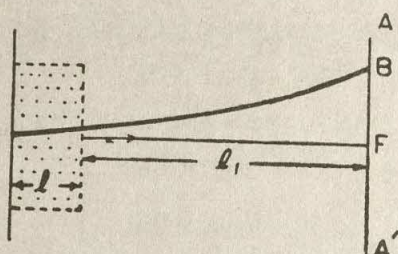


Fig. 14.25

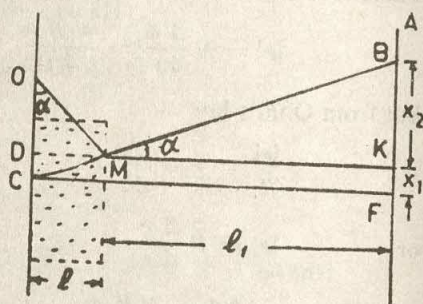


Fig. 14.26

Solution: The total displacement of the electron beam is

$$x = x_1 + x_2$$

where x_1 is the displacement of the electron in the magnetic field, as shown in Fig. 14.26. The electron moves along a circular path whose radius of curvature is

$$R = \frac{mv}{eB}$$

The displacement x_1 can be found from the relation

$$x_1 = DC = OC - OD$$

But $OC = R$ and $OD = \sqrt{(OM)^2 - (DM)^2} = \sqrt{R^2 - l^2}$

Thus $x_1 = R - \sqrt{R^2 - l^2}$

The displacement x_2 can be found from the proportionality

$$\frac{x_2}{l_1} = \frac{DM}{OD} \quad \text{or} \quad x_2 = \frac{l_1 l}{\sqrt{R^2 - l^2}}$$

Hence $x = R - \sqrt{R^2 - l^2} + \frac{l_1 l}{\sqrt{R^2 - l^2}}$

Velocity v of the electron is given by

$$eV = \frac{1}{2} m v^2$$

where V is the applied potential difference. Therefore,

$$R = \frac{1}{B} \sqrt{\frac{2Vm}{e}}$$

Now $V = 300$ volts and $B = 1.46 \times 10^{-3}$ tesla

$$R = \frac{1}{1.46 \times 10^{-3}} \left(\frac{2 \times 300 \times 9.1 \times 10^{-31}}{1.6 \times 10^{-19}} \right)^{1/2}$$

$$\therefore 4.0 \times 10^{-2} \text{ m} = 4.0 \text{ cm}$$

Now $l = 2.5$ cm and $l_1 = 5$ cm. Thus

$$x = 4.0 - \sqrt{16 - 6.25} + \frac{5 \times 2.5}{\sqrt{16 - 6.25}}$$

$$= 4.0 - 3.12 + 4.00$$

$$= 4.88 \text{ cm}$$

14.5 TRUE-FALSE STATEMENTS WITH REASONS

We give below some statements. We have to decide whether they are true or false giving, in brief, reasons for our answers.

1. In Fig. 14.27 the current increases in magnitude from A to B . The direction of the induced current in the loop is clockwise.

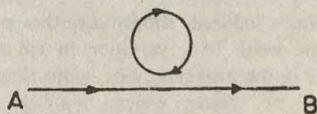


Fig. 14.27

2. An emf can be induced between the ends of a straight metal wire by moving it through a uniform magnetic field.
3. No net force acts on a rectangular coil carrying a steady current when suspended freely in a uniform magnetic field.
4. The self induced emf produced by a changing current in a coil always tends to decrease the current.
5. The presence of a large magnetic flux through a coil maintains a current in the coil if the circuit is continuous.
6. Inserting an iron core in a coil increases its coefficient of self induction.
7. There is no change in the energy of a charged particle moving in a magnetic field, although a magnetic force is acting on it.
8. A charged particle enters a region of uniform magnetic field at an angle of 85° to the magnetic lines of force. The path of the particle is a circle.
9. A coil of metal wire is kept stationary in a non-uniform magnetic field. An emf is induced in the coil.

ANSWERS

1. *True* According to Lenz's law, the induced current always acts in a direction so as to oppose the change which is causing it.
2. *True* An electron moving across a magnetic field experiences an induced emf. As the wire moves across the magnetic field, an emf is generated. The magnetic force comes into play which drives the electrons along the wire. If the wire is not connected to a closed circuit, the electrons will continue to collect at one end of the wire and thus the other end will be at a higher potential.

3. *True* The forces on the opposite sides of the coil are equal and opposite.
4. *False* If the change in current is positive (increase), the self induced emf will tend to decrease the current. If the current is decreasing (negative change in current), the self induced emf will tend to increase the current, in accordance with Lenz's law.
5. *False* Only a change in magnetic flux (by the motion of the lines of induction or the motion of the coil) will maintain an induced current in the coil. If the magnetic flux through the coil is not changing with time, there is no induced emf.
6. *True* The magnetic permeability of iron is greater than that of free space. Therefore, the magnetic field and hence the magnetic flux increases μ times, where μ is the permeability of iron. Consequently, the coefficient of self induction ($L = \text{flux/current}$) increases.
7. *True* A charged particle moves in a magnetic field when the magnetic force is perpendicular to the path of the charged particle. Thus, there is no change in the velocity or energy of the charged particle moving in a magnetic field, although a magnetic force is acting on it.
8. *False* The velocity of the charged particle, due to the application of a uniform magnetic field at an angle of 85° to the field lines, can be resolved into two components which are unequal in magnitude and are parallel and perpendicular to the lines of force. The resultant path of the particle is, therefore, not a circle.
9. *False* An emf cannot be induced simply by the presence of a non-uniform magnetic field. There must be a variation in the magnetic field with time so that there is a change in the magnetic flux with time which is responsible for the induced emf.

14.6 MULTIPLE CHOICE QUESTIONS

Select the correct answer from the given alternatives.

1. A magnetic needle is kept in a non-uniform magnetic field. It experiences
 - (a) a force as well as a torque
 - (b) a force but no torque
 - (c) a torque but no force
 - (d) neither a force nor a torque
2. A conducting circular loop of radius r carries a constant i . It is placed in a uniform magnetic field B such that B is perpendicular to the plane of the loop. The magnetic force acting on the loop is given by
 - (a) $i r B$
 - (b) $2\pi i r B$
 - (c) zero
 - (d) $\pi i r B$
3. A proton moving with a constant velocity passes through a region of space without any change in its velocity. If E and B represent the electric and magnetic fields respectively, this region of space may have
 - (a) $E = 0, B = 0$
 - (b) $E = 0, B \neq 0$
 - (c) $E \neq 0, B = 0$
 - (d) $E \neq 0, B \neq 0$
4. A rectangular loop carrying a current i is situated near a long straight wire such that the wire is parallel to one of the sides of the loop. If a steady current I is established in the wire, as shown in Fig. 14.28, the loop will
 - (a) rotate about an axis parallel to the wire
 - (b) move away from the wire
 - (c) move towards the wire
 - (d) remain stationary.

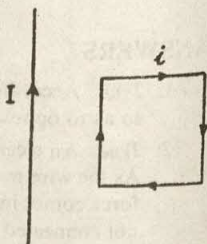


Fig. 14.28

5. When there is a current in the coils shown in Fig. 14.29, they attract each other. The electrons must be flowing in the same direction in

(a) A and B

(b) A and C

(c) C and D

(d) A and D

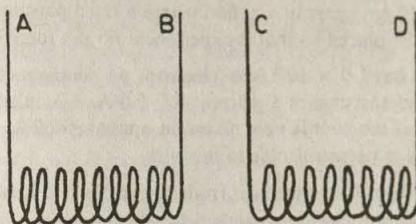


Fig. 14.29

6. At magnetic poles of the earth, the total magnetic field intensity is
 (a) greater than the vertical component
 (b) equal to the vertical component
 (c) less than the vertical component
 (d) independent of the vertical component.
7. As one journeys along the magnetic equator, the angle of dip will
 (a) increase
 (b) decrease
 (c) remain unchanged
 (d) increase or decrease depending on the direction of the journey
8. A magnetic field
 (a) always exerts a force on a charged particle
 (b) never exerts a force on a charged particle
 (c) exerts a force if the charged particle is moving across the magnetic field lines
 (d) exerts a force if the charged particle is moving along the magnetic field lines.
9. A ship with a vertical steel mast navigates the Indian Ocean in the latitude of the magnetic equator. To induce the greatest emf in the mast, the ship should proceed
 (a) northward (b) southward (c) eastward (d) westward
10. An electrostatic field E and a magnetic field B act over the same region. An electron enters the region (see Fig. 14.30). Which one of the combination shown in the figure can cause the electrons to pass undeflected?

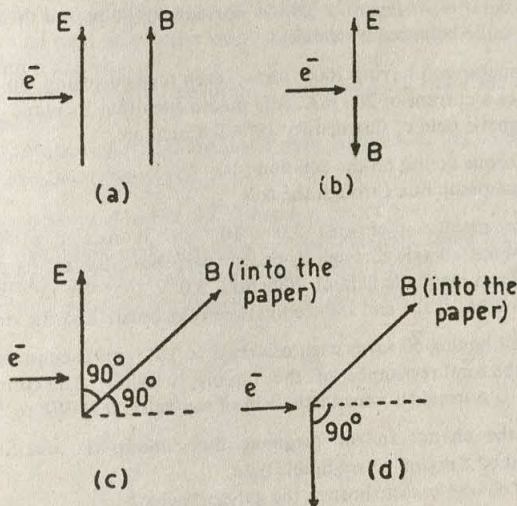


Fig. 14.30

14.7 NUMERICAL EXERCISES

- Two long parallel wires carry currents of 12 A and 8 A in the same direction. If the wires are 10 cm apart in air, find where a third parallel wire, also carrying a current, must be placed so that it experiences no net force.
- A copper wire has 1.0×10^{29} free electrons per cubic metre, a cross-sectional area of 6.0 mm^2 and carries a current of 5.0 A. Calculate the force acting on each electron, if the wire is now placed in a magnetic field of flux density 0.15 Wb m^{-2} which is perpendicular to the wire.
- A beam of protons is accelerated from rest by a potential difference of 2000 V and then enters a uniform magnetic field which is perpendicular to the direction of the proton beam. If the magnetic flux density is 0.2 Wb m^{-2} , calculate the radius of the path which the beam describes. Mass of a proton $= 1.7 \times 10^{-27} \text{ kg}$, electron charge $= 1.6 \times 10^{-19} \text{ C}$.
- A charge q enters a region where a uniform electric field $E = 10^5 \text{ V m}^{-1}$ is directed downwards. A magnetic field of flux density 0.4 T and perpendicular to E is directed into the page. What should be the speed of the particle so that it remains undeflected by these crossed electric and magnetic fields?
- A charge of $1.6 \times 10^{-19} \text{ C}$ is moving at 1 km s^{-1} through a magnetic field in such a direction that the magnetic force on it is maximum. What is the magnetic induction, if the maximum force on the charge is $8 \times 10^{-8} \text{ N}$?
- During a space walk, two astronauts are separated by a connecting wire 10 m long that is moving horizontally, broadside to its length, at an orbital speed of 8.0 km s^{-1} over the north magnetic pole of the earth. The earth's magnetic field at this point is $6 \times 10^{-5} \text{ T}$. Calculate the induced emf, which is also the potential difference between the two astronauts.
- A flat circular wire loop of radius 5 cm and resistance 4Ω is in a region of space where the magnetic field strength is 0.05 T , perpendicular to the plane of the loop. Calculate the current in the loop, if the field decreases at a steady rate, becoming zero after $200 \mu\text{s}$.
- A straight conductor of length 20 cm and mass 2 g kept horizontal and a uniform magnetic induction of 10^{-2} T is applied in a horizontal direction and perpendicular to its length. Find the current to be passed through it so that its weight can be balanced by the field.
- A rectangular coil having 1000 turns, each turn enclosing an area of $3 \times 10^{-2} \text{ m}^2$ carries a current of 200 mA. It is placed such that its plane is inclined at 60° to a magnetic field of flux density 10^{-3} T . Calculate
 - the torque acting on the coil and
 - the magnetic flux through the coil
- A circular metal disc of area $3.0 \times 10^{-3} \text{ m}^2$ is rotated at 50 revolutions per second about an axle through its centre perpendicular to its plane. The disc is in a uniform magnetic field of induction $5.0 \times 10^{-3} \text{ Wb m}^{-2}$ in the direction of the axle. What is the emf induced between the centre and the rim of the disc?
- A flat coil having 50 turns each of area $2 \times 10^{-4} \text{ m}^2$ is connected to a galvanometer; the total resistance of the circuit is 100Ω . The coil is placed so that its plane is normal to a magnetic field of induction 0.25 Wb m^{-2} .
 - Find the change in the magnetic flux linking the circuit when the coil is moved to a region of negligible field.
 - What charge passes through the galvanometer?

12. A long solenoid of cross-sectional area $8 \times 10^{-4} \text{ m}^2$ and length 0.5 m, has 2000 turns wound uniformly on a hollow wooden cylinder. Closely wound over the middle region of this solenoid is a short secondary coil of 600 turns. Calculate
- the flux density within the solenoid when a steady current of 5A flows in it
 - the flux linked with the solenoid when it carries a steady current of 5A
 - the self inductance of the solenoid
 - the mutual inductance between the solenoid and the secondary coil
 - the magnitude of the emf induced in the secondary coil at the instant when the primary current is increasing at a rate of 12 amperes per second.
13. Considering the magnetic field along the axis of a circular loop of radius r , at what distance from the centre of the loop is the field 1/10 of its value at the centre?
14. A rectangular loop is pivoted about the y -axis and carries a current 10A in the direction shown in Fig. 14.31.

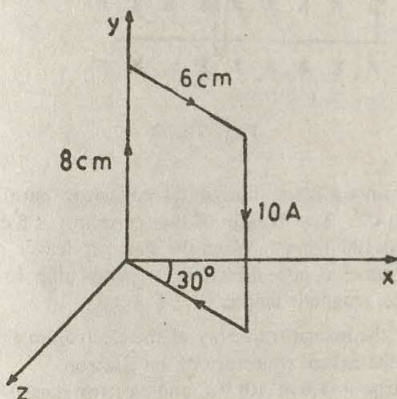


Fig. 14.31

- if the loop is in a uniform magnetic field of 0.2 T, parallel to the x -axis, find the force on each side of the loop and the torque required to hold the loop in the position shown.
 - determine the force and the torque, as in case (a), when the field is parallel to the z -axis.
 - find the torque required if the loop were pivoted about an axis through its centre, parallel to the y -axis.
15. An electron at point P in Fig. 14.32 has a speed of $v = 10^7 \text{ m s}^{-1}$.
- find the magnitude and direction of the magnetic field that will cause the electron to follow the semi-circular path from P to Q .
 - find the time required for the electron to move from P to Q .

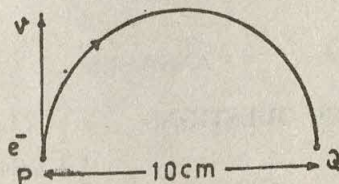


Fig. 14.32

16. An aircraft with a wing span of 40 m flies with a ground speed of 1000 km h^{-1} in the eastward direction at a constant altitude in a region of the northern hemisphere where the horizontal component of the earth's magnetic field is $1.6 \times 10^{-5} \text{ T}$ and the angle of dip is 71.6° . Find the potential difference that exists between the tips of the wings of the aircraft.
17. A 0.50 m long metal rod PQ completes the circuit shown in Fig. 14.33. The area of the circuit is perpendicular to a magnetic field of flux density 0.15 T . If the resistance of the total circuit is 3Ω , calculate the force needed to move the rod as indicated with a constant speed of 2 m s^{-1} .

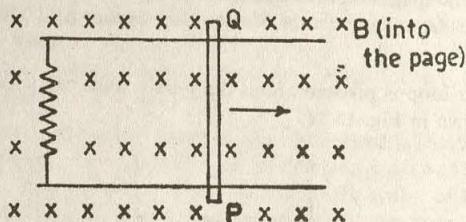


Fig. 14.33

18. An electron flies into a plane horizontal capacitor parallel to its plates with a velocity of 10^7 m s^{-1} . The length of the capacitor is 5 cm and the intensity of its electric field is 100 V cm^{-1} . When the electron leaves the capacitor, it enters a magnetic field which is in a direction perpendicular to the electric field. The flux density of the magnetic field is 10^{-2} T . Find
- the radius of the helical trajectory of the electron in the magnetic field.
 - the pitch of the helical trajectory of the electron.
- Electron charge = $1.6 \times 10^{-19} \text{ C}$ and electron mass = $9.1 \times 10^{-31} \text{ kg}$.
19. Two coils have mutual inductance of $5 \times 10^{-8} \text{ H}$. The current in one coil changes according to the relation $I = I_0 \sin \omega t$, where $I_0 = 10 \text{ A}$, $\omega = 2\pi/T$ and $T = 0.02 \text{ s}$. Find
- the time dependence of the induced emf in the other coil
 - the maximum value of the induced emf.
20. A square frame made of copper wire with a cross-section of 1 mm^2 is placed in a magnetic field whose induction changes according to the relation $B = B_0 \sin \omega t$ where $B_0 = 0.01 \text{ T}$, $\omega = 2\pi/T$ and $T = 0.02 \text{ s}$. The area of the frame is 25 cm^2 . The plane of the frame is perpendicular to the direction of the magnetic field. The resistivity of copper is $1.7 \times 10^{-8} \Omega \text{ m}$. Find the dependence on time and the maximum value of
- the magnetic flux piercing the frame,
 - the emf induced in the frame,
 - the current flowing in the frame if the total length of the copper wire is 200 m .

ANSWERS

MULTIPLE CHOICE QUESTIONS

- | | | | |
|----------------|---------|---------------------|--------|
| 1. (a) and (b) | 2. (c) | 3. (a), (b) and (d) | 4. (c) |
| 5. (b) | 6. (b) | 7. (c) | 8. (c) |
| 9. (c) and (d) | 10. (c) | | |

NUMERICAL EXERCISES

1. 6 cm from the 12 A wire
2. 3.75×10^{-24} N
3. 3.26 cm
4. 2×10^5 m s⁻⁴
5. 0.5 T
6. 4.8 V
7. 0.491 A
8. 9.8 A
9. 3×10^{-3} N m, 0.026 Wb
10. 7.5×10^{-4} V
11. (a) 2.5×10^{-3} Wb (b) 2.5×10^{-5} C
12. (a) 0.025 T (b) 4.02×10^{-2} Wb
(c) 8.04 mH (d) 2.41 mH,
(e) 28.9 mV
13. 1.91 r
14. (a) 0.16 N, 0.06 N, 8.3×10^{-3} Nm (b) 0.16 N, 0.104 N, 4.8×10^{-3} N m
(c) same torque
15. (a) 1.14×10^{-3} T, into the page (b) 1.57×10^{-8} s
16. 0.53 V
17. 3.75×10^{-3} N
18. (a) $r = 5$ mm, (b) 3.6 cm
19. (a) $e = -M dI/dt = -M \omega \cos \omega t = -45.7 \cos 100 \pi t$ volt
(b) $e_{\max} = 15.7$ V
20. (a) $\phi = B_0 A \sin \omega t = 2.5 \times 10^{-5} \sin 100 \pi t$ Wb, $\phi_{\max} = 2.5 \times 10^{-5}$ Wb
(b) $e = -7.85 \times 10^{-3} \cos 100 \pi t$ V, $e_{\max} = 7.85 \times 10^{-3}$ V
(c) $i = -2.3 \cos 100 \pi t$ A, $i_{\max} = 2.3$ A

Reflection of Light

15.1 REVIEW OF BASIC CONCEPTS

15.1.1 Reflection of Light When light strikes the surface of an object, some part of the light is sent back into the medium from which it is coming. This is the reflected light. The surface which reflects light is called a *mirror*. Mirrors can be plane or curved.

15.1.2 Laws of Reflection When a ray of light strikes a surface, plane or curved, it is reflected obeying the following two laws.

- (i) The angle of incidence and the angle of reflection, at the point of incidence are always equal.
- (ii) The incident ray, the reflected ray and the normal to the surface at the point of incidence, lie in one and the same plane. This is called the plane of incidence which coincides with the plane of reflection.

15.1.3 Regular and Irregular (or Diffused) Reflection If the reflecting surface is plane, the normals at different points on the surface are parallel to each other. Thus, a parallel beam of light is reflected from a plane surface as a parallel beam. Such surfaces show regular reflection. If the reflecting surface has irregularities, a parallel beam will be reflected in various different directions. These surfaces give irregular or diffused reflection.

15.1.4 Real and Virtual Objects The object for a mirror can be real or virtual (see Fig. 15.1). If the rays from a point on an object actually diverge from it and fall on the mirror, the object is said to be real. But if the rays incident on the mirror appear to converge to a point, then that point is the virtual point object for the mirror.

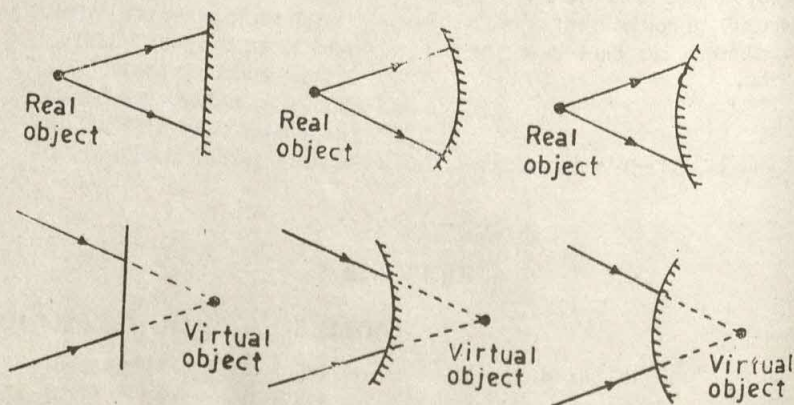


Fig. 15.1

15.1.5 Real and Virtual Images If the rays from a point O on an object, after reflection (or refraction) actually meet at or appear to diverge from a point I , then I is said to be the image of O . Images can be real or virtual. If the reflected or refracted rays actually meet at the point I , then I is the real image of O . But, if the reflected or refracted rays do not actually meet but only appear to diverge from the point I , then I is said to be the virtual image of O .

15.1.6 Characteristics of Reflection from Plane Mirrors

- (1) The image of a real object is always virtual, erect and laterally inverted with respect to the object.
- (2) The image is as far behind the mirror as the object is in front of it.
- (3) The size of the image is equal to the size of the object, irrespective of the distance of the object from the mirror.
- (4) For a given incident ray, if the mirror is rotated through a certain angle, the reflected ray rotates through twice that angle.
- (5) If two mirrors are inclined to each other at an angle θ and an object is placed between them, we get many images due to multiple reflections. The image formed by one mirror acts as a virtual object for the second and so on. The number n of the images seen is given by

$n = \text{the least value of the integer which satisfies the condition}$

$$(n + 1) \theta \geq 360^\circ$$

15.1.7 Spherical Mirrors A spherical mirror is a part of a hollow sphere or a spherical surface. Spherical mirrors can be concave or convex. Some terms used in relation to spherical mirrors are defined as follows.

- (1) *Pole or Vertex* The centre of the spherical surface of the mirror is called the pole or vertex of the mirror. In Fig. 15.2 the point O is the pole.

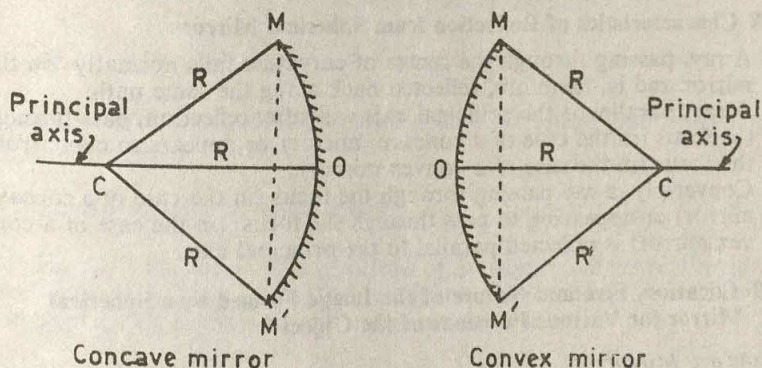


Fig. 15.2

- (2) *Centre Of Curvature* The centre C of the sphere of which the mirror is a part is the centre of curvature of the mirror.

- (3) *Radius of Curvature* The radius of curvature R is the radius of the sphere of which the mirror is a part. The distance of any point on the surface of the mirror from the centre of curvature equals R .

- (4) *Principal Axis* The line OC (produced on both sides) joining the pole and the centre of curvature of the mirror is called the principal axis of the mirror.

- (5) **Linear Aperture** The distance MM' between the points of extremities of the mirror is called its linear aperture.
- (6) **Angular Aperture** The angle MCM' subtended by the periphery of the mirror at the centre of curvature is called its angular aperture.
- (7) **Focus and Focal Length** If a parallel beam of rays, parallel and close to the principal axis, is incident on a spherical mirror, the reflected rays converge to a point F (in the case of concave mirror) or appear to diverge from a point F (in the case of a convex mirror). This point F is called the principal focus or simply focus of the mirror (see Fig. 15.3). The distance between the pole and the focus of a spherical mirror is called its focal length and is usually denoted by the letter f .
- (8) **Focal Plane** The plane passing through the focus and perpendicular to the principal axis is called the focal plane.

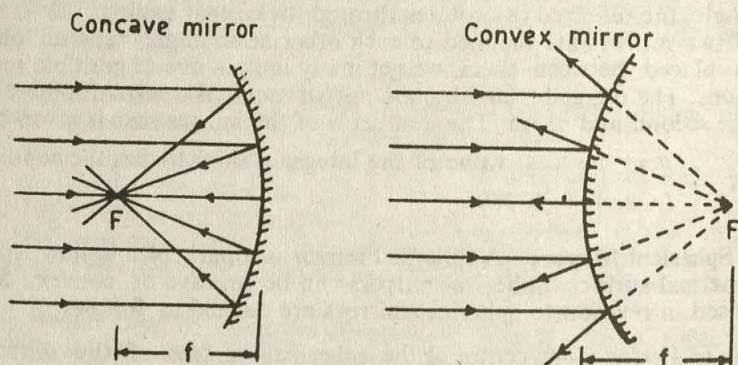


Fig. 15.3

15.1.8 Characteristics of Reflection from Spherical Mirrors

- (1) A ray, passing through the centre of curvature falls normally on the mirror and is, therefore, reflected back along the same path.
- (2) A ray, parallel to the principal axis will, after reflection, pass through the focus (in the case of a concave mirror) or appears to come from the focus (in the case of a convex mirror).
- (3) Conversely, a ray passing through the focus (in the case of a concave mirror) or appearing to pass through the focus (in the case of a convex mirror) is reflected parallel to the principal axis.

15.1.9 Location, Size and Nature of the Image Formed by a Spherical Mirror for Various Positions of the Object

(1) Concave Mirror

Table 15.1

Position of the object	Position of the image	Size of the image	Nature of the image
Between O and F	Behind the mirror	Magnified	Virtual, erect
At F	At infinity	Highly magnified	Virtual, erect
Between F and C	Beyond C	Magnified	Real, inverted
At C	At C	Same size	Real, inverted
Beyond C	Between F and C	Diminished	Real, inverted
At infinity	At F	Highly diminished	Real, inverted

(2) *Convex Mirror* A convex mirror forms only virtual images for all positions of the object. The image is always virtual, erect, smaller than the object and is located between the pole and the focus. The image becomes smaller and moves closer to the focus as the object is moved away from the mirror.

15.2 SIGN CONVENTIONS

- (1) Distances of the object and the image are measured from the pole of the mirror.
- (2) Distances measured in the direction of the incident rays are positive, while those measured in a direction opposite to the incident rays are negative.
- (3) Heights measured upwards and perpendicular to the principal axis are positive, while those measured downwards are negative.

15.3 FORMULAE

1. The spherical mirror formula is

$$\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$$

where u = object distance and v = image distance, both measured from the pole.

2.
$$f = \frac{R}{2}$$

3. Linear magnification = $\frac{\text{height or size of the image}}{\text{height or size of the object}}$

$$\text{or } m = \frac{I}{O} = -\frac{v}{u} = \frac{f-v}{f} = \frac{f}{f-u}$$

Linear magnification m is positive for an erect image and negative for an inverted image.

4. Superficial or area magnification = $\frac{v^2}{u^2}$

5. *Newton's Formula* The positions of an object and its real image are interchangeable. They are called conjugate positions. Newton's formula for a pair of real object and real image positions is

$$xy = f^2$$

where x = distance, along the principal axis, of the real object from the focus of the concave mirror

y = distance, along the principal axis, of the real image from the focus.

This formula is applicable only to concave mirrors because convex mirrors do not form real images.

An Important Note Students often experience difficulties in the use of the sign convention while solving numerical problems. To solve any problem, remember the following procedure.

1. Use the mirror formulae $\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$ and $m = -\frac{v}{u}$ for convex as well as concave mirrors.
2. Substitute the numerical values of the given quantities with the proper positive or negative sign.
3. Do not give any sign to the quantity to be determined. The unknown quantity will be obtained with its proper sign.

15.4 NUMERICAL EXAMPLES

Example 1: Figure 15.4 (a) shows an arrow placed in front of a plane mirror. Draw the rays to locate the position of an observer's eye so that he can see the image of the arrow in the mirror.

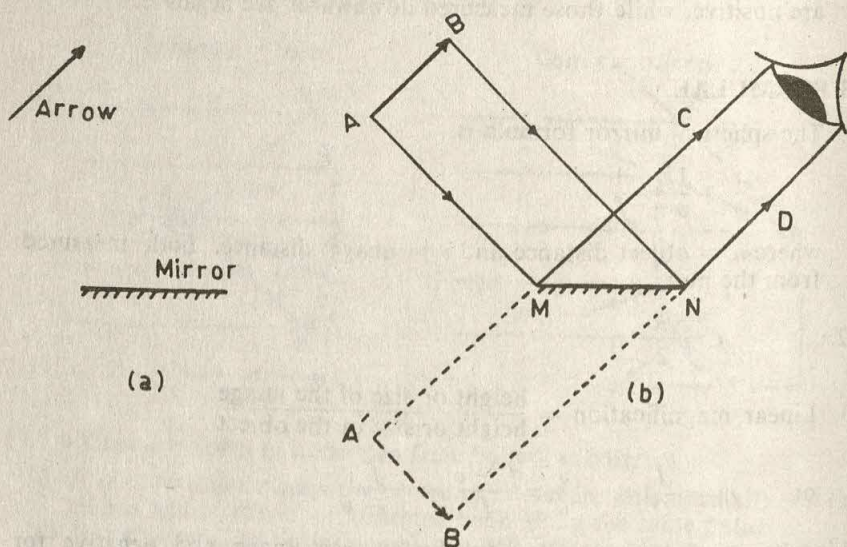


Fig. 15.4

Solution: Figure 15.4(b) shows the ray diagram to locate the image $A'B'$ of the arrow AB as seen in the mirror MN by reflection. The eye must be placed at the indicated position so that the extreme reflected rays MC and ND reach the eye to enable the observer to see the entire image.

Example 2: Two plane mirrors are placed at right angles to each other. A ray strikes one mirror at an angle of incidence θ such that it is also reflected from the second mirror. Show that, for any value of θ , the ray reflected from the second mirror is parallel to that incident on the first mirror, but its direction is reversed.

Solution: The incident ray AB strikes the first mirror M_1 at an angle θ and is reflected along BC , as shown in Fig. 15.5 (a). The ray BC is incident on mirror M_2 at an angle $\alpha = 90^\circ - \theta$ and is reflected along CD making an angle $\alpha = 90^\circ - \theta$ with the mirror M_2 . It is clear from Fig. 15.5 (b) that the ray CD is parallel to the ray AB but its direction is reversed.

Note Two reflecting surfaces are often combined in optical devices to alter the path of the incident beam.

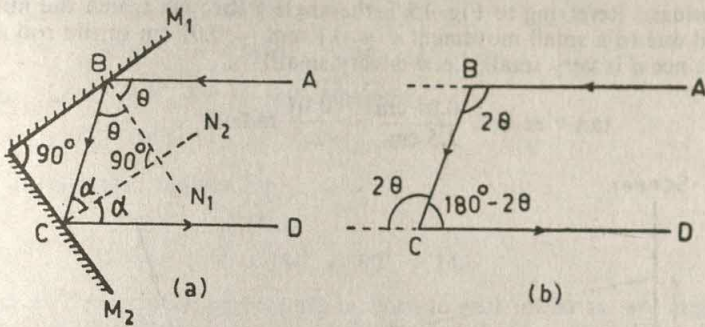


Fig. 15.5

Example 3: Two plane mirrors are inclined to each other at an angle of 70° . A ray is incident on one mirror at an angle θ . The ray reflected from this mirror falls on the second mirror from where it is reflected parallel to the first mirror. Determine value of θ .

Solution: A ray AB is incident at an angle θ on mirror M_1 and is reflected along BC as shown in Fig. 15.6. The ray makes an angle γ with the normal CN_2 and is reflected along CD making an equal angle γ with the normal. Since CD is parallel to mirror OM_1 ,

$$\angle M_2CD = \angle M_2OM_1 = 70^\circ$$

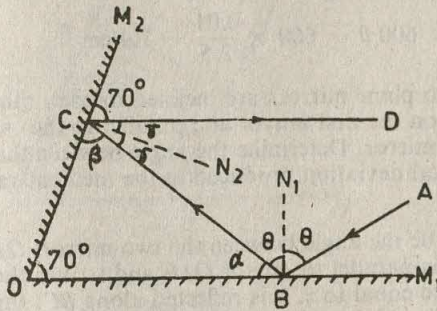


Fig. 15.6

It is clear from the diagram that

$$\alpha = 90^\circ - \theta \quad \text{and} \quad \beta = 20^\circ + \theta$$

Now $\angle N_2CO = \gamma + \beta = 90^\circ$. Hence $\gamma = 70^\circ - \theta$. But

$\angle M_2CN_2 = 90^\circ$ giving $\gamma = 20^\circ$. Thus

$$\theta = 50^\circ$$

Example 4: One end of a strip of a plane mirror 2.5 cm in length, is fixed. The other end rests on the top of a small vertical rod. A beam of light is reflected from the mirror and forms a spot on a screen 3 m away from the mirror. If the top of the rod is moved upwards 0.1 mm, what will be the movement of the spot?

Solution: Referring to Fig. 15.7, the angle θ through which the mirror is tilted due to a small movement $d = 0.1 \text{ mm} = 0.01 \text{ cm}$ of the rod is given by (since d is very small, i.e. θ is very small)

$$\tan \theta \approx \theta = \frac{0.01 \text{ cm}}{2.5 \text{ cm}} = \frac{0.01}{2.5} \text{ radian}$$

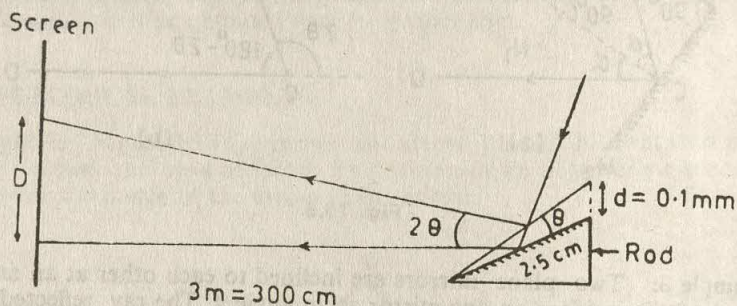


Fig. 15.7

Now, if the mirror rotates through an angle θ , the reflected ray rotates through 2θ . It is clear from the diagram that

$$\tan 2\theta \approx 2\theta = \frac{D}{300}$$

where D = movement of the spot in cm. Thus

$$D = 600\theta = 600 \times \frac{0.01}{2.5} = 2.4 \text{ cm}$$

Example 5: Two plane mirrors are inclined to each other such that a ray of light incident on the first mirror and parallel to the second is reflected from the second mirror. Determine the angle between the two mirrors. Also determine the total deviation produced in the incident ray due to the two reflections.

Solution: Let θ be the angle between the two mirrors OM_1 and OM_2 . The incident ray AB is parallel to mirror OM_2 and strikes the mirror OM_1 at an angle of incidence equal to α . It is reflected along BC ; the angle of reflection being α . From Fig. 15.8 we have

$$\angle M_1BA = \angle OBC = \angle M_1OM_2 = \theta$$

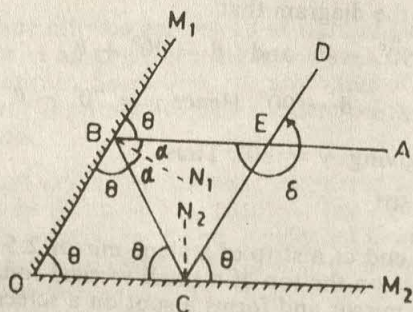


Fig. 15.8

Similarly for reflection at mirror OM_2 , we have

$$\angle M_2CD = \angle BCO = \angle M_2OM_1 = \theta$$

Now in triangle OBC ; $3\theta = 180^\circ$. Therefore

$$\theta = 60^\circ$$

The total deviation is given by

$$\begin{aligned}\delta &= 180^\circ + \angle AED \\ &= 180^\circ + \theta = 180^\circ + 60^\circ = 240^\circ\end{aligned}$$

Example 6: Two plane mirrors are inclined to each other at an angle θ . A ray of light is reflected first at one mirror and then at the other. Find the total deviation of the ray and show that it is independent of the angle of incidence at the first mirror.

Solution: A ray AB is incident on mirror OM_1 at an angle α and is reflected along BC suffering a deviation

$$\delta_1 = \angle FBC$$

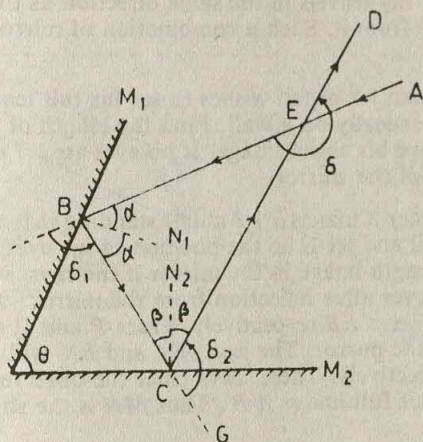


Fig. 15.9

The ray BC falls on mirror OM_2 at an angle of incidence β and is reflected along CD suffering another deviation

$$\delta_2 = \angle GCD$$

The total deviation is $\delta = \delta_1 + \delta_2$ (see Fig. 15.9).

It is clear from the diagram that

$$\delta_1 = 180^\circ - 2\alpha \quad \text{and} \quad \delta_2 = 180^\circ - 2\beta$$

$$\therefore \delta = \delta_1 + \delta_2 = 360^\circ - 2(\alpha + \beta)$$

Now, in triangle OBC , $\angle OBC + \angle BCO + \angle BOC = 180^\circ$

$$\text{or} \quad (90^\circ - \alpha) + (90^\circ - \beta) + \theta = 180^\circ$$

$$\text{or} \quad \alpha + \beta = \theta$$

Hence $\delta = 360^\circ - 2\theta$

which is independent of the angle of incidence α at the first mirror.

Alternative Method To find the total deviation δ , we rotate the incident ray AB in the counter clockwise sense until it coincides with the final reflected ray CD . It is obvious from Fig. 15.9 that

$$\delta = \angle BEC + \angle CEA + \angle AED$$

Now $\angle BEC = \angle AED = 180^\circ - 2(\alpha + \beta) = 180^\circ - 2\theta$

$$[\because \theta = 2(\alpha + \beta)]$$

And $\angle CEA = 2\alpha + 2\beta = 2(\alpha + \beta) = 2\theta$

Hence $\delta = (180^\circ - 2\theta) + 2\theta + (180^\circ - 2\theta)$
 $= 360^\circ - 2\theta$

Note If the angle θ between the two mirrors is 90° (as in Ex. 2) the total deviation $\delta = 360^\circ - 2 \times 90^\circ = 180^\circ$; i.e. the final reflected ray is parallel to the first incident ray but its direction is reversed. If the two mirrors are parallel to each other, then $\theta = 0^\circ$ giving $\delta = 360^\circ$. Thus, in this case, the final reflected ray travels in the same direction as the first incident ray; but it is displaced from it. Such a combination of mirrors is used in a periscope.

Example 7: A man 1.8 m tall wishes to see his full length image in a plane mirror hanging vertically on a wall. Find the length of the shortest mirror in which he can see his entire image. If his eyes are 1.7 m above the ground, find the position of the mirror.

Solution: Consider a man AB 1.8 m tall standing in front of a plane mirror hanging on a wall and let E be the position of his eyes (see Fig. 15.10). He can see his full length image in the mirror if the rays which start from A and B reach his eyes after reflection from the mirror. Let P and Q be the midpoints of AE and EB respectively. From P and Q draw perpendiculars PM and QN on the mirror. The rays AM and BN will be reflected along ME and NE respectively. These rays enter the observer's eyes, thus enabling him to see his full image $A'B'$. Thus MN is the shortest length of the mirror. Now

$$MN = PQ = PE + EQ$$

$$= \frac{1}{2} AE + \frac{1}{2} EB = \frac{1}{2} (AE + EB) = \frac{1}{2} AB$$

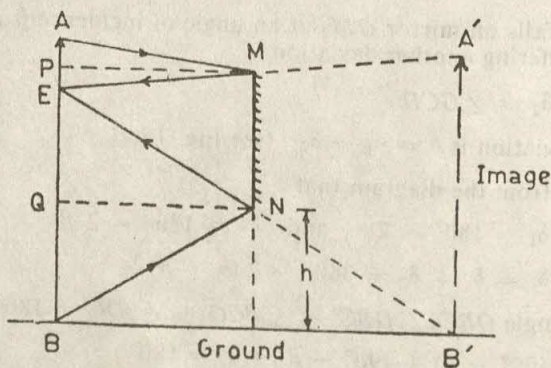


Fig. 15.10

Hence the shortest length of the mirror = half the height of the observer = $\frac{1}{2} \times 1.8 = 0.9$ m

This mirror must be placed with its lower edge at a height h above the ground where

$$\begin{aligned} h &= QB = EB - EQ \\ &= EB - \frac{1}{2} EB = \frac{1}{2} EB = \frac{1}{2} \times 1.7 = 0.85 \text{ m} \end{aligned}$$

Example 8: Find the smallest size of a looking glass a person with a face 24 cm by 16 cm should purchase which will enable him to see his whole face at a time. The distance between his eyes is 8 cm.

Solution: As in Example 7, the smallest length of the mirror = half the length of the face = $\frac{1}{2} \times 24 = 12$ cm

To find the smallest breadth of the mirror, refer to Fig. 15.11 where PQ is the breadth of the face and E_1 and E_2 are the two eyes.

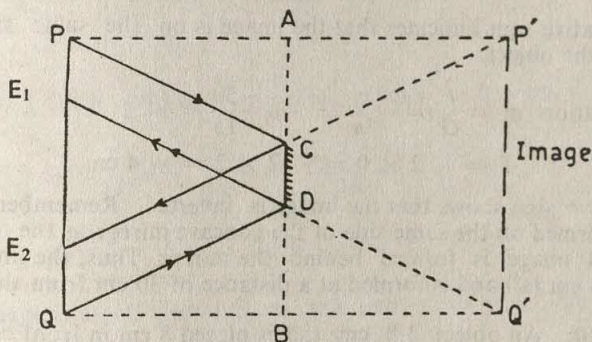


Fig. 15.11

A ray PC , after reflection, will travel along CE_2 and will appear to come from P' . Similarly, a ray QD , after reflection, will travel along DE_1 and will appear to come from Q' . Thus $P'Q'$ is the image of PQ and CD is the smallest breadth of the looking glass the person requires to see the entire breadth of his face in the mirror.

In triangle $PP'E_2$, A and C are the midpoints of sides PP' and $P'E_2$ respectively. Hence

$$AC = \frac{1}{2} (PE_2) = \frac{1}{2} (PE_1 + E_1E_2)$$

Similarly, in triangle $QQ'E_1$, we have

$$BD = \frac{1}{2} (QE_1) = \frac{1}{2} (QE_2 + E_1E_2)$$

$$\begin{aligned} \therefore AC + BD &= \frac{1}{2} (PE_1 + QE_2) + E_1E_2 \\ &= PE_1 + E_1E_2 \quad (\because PE_1 = QE_2) \\ &= PE_2 \end{aligned}$$

$$\begin{aligned}
 \text{Now } CD &= AB - (AC + BD) = PQ - PE_2 = QE_2 \\
 &= \frac{1}{2} PQ - \frac{1}{2} E_1 E_2 \\
 &= \frac{1}{2} (\text{breadth of face} - \text{distance between the eyes}) \\
 &= \frac{1}{2} (16 - 8) = 4 \text{ cm}
 \end{aligned}$$

Hence the shortest size of the mirror is 12 cm by 4 cm.

Example 9: A 2.0 cm tall object is placed 15 cm from a concave mirror of focal length 10 cm. Find the position, size and nature of the image.

$$\begin{aligned}
 \text{Solution: } u &= -15 \text{ cm} \\
 f &= -10 \text{ cm (concave mirror)} \\
 O &= +2.0 \text{ cm}
 \end{aligned}$$

Using $\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$ and substituting for u and f we get

$$\frac{1}{v} + \frac{1}{-15} = \frac{1}{-10}$$

$$\text{or } v = -30 \text{ cm}$$

The negative sign indicates that the image is on the same side of the mirror as the object.

$$\text{Magnification } m = \frac{I}{O} = \frac{v}{u} = \frac{-30}{-15} = +2$$

$$\text{or } I = +2 \times O = +2 \times 2 = +4 \text{ cm}$$

The negative sign shows that the image is inverted. Remember, the real image is formed on the same side of the concave mirror as the object, and the virtual image is formed behind the mirror. Thus, the image is real, inverted, 4 cm tall and is formed at a distance of 30 cm from the mirror.

Example 10: An object 1.0 cm tall is placed 8 cm in front of a concave mirror of radius of curvature 24 cm. Find the position, size and nature of the image.

$$\text{Solution: } u = -8 \text{ cm, } R = -24 \text{ cm, } f = \frac{R}{2} = -12 \text{ cm and}$$

$$O = +1.0 \text{ cm}$$

$$\text{Using } \frac{1}{v} + \frac{1}{u} = \frac{1}{f}$$

$$\text{we have } \frac{1}{v} + \frac{1}{-8} = \frac{1}{-12} \text{ or } v = +24 \text{ cm}$$

The positive sign shows that the image is virtual.

$$m = \frac{I}{O} = \frac{v}{u} = \frac{+24}{-8} = +3$$

$$\text{or } I = +3 \times O = +3 \times 1.0 = +3.0 \text{ cm}$$

The positive sign shows that the image is erect. Thus the image is erect, virtual, 3.0 cm tall and is formed 24 cm behind the mirror.

Example 11: A concave mirror forms a real image four times the size of the object at a distance of 10 cm from the mirror. Find the radius of curvature of the mirror.

Solution: Since the image is erect, the magnification is negative. Hence

$$m = -4$$

Also $u = -10$ cm

Now $m = -\frac{v}{u}$ or $-4 = -\frac{v}{-10}$ or $v = -40$ cm

Substituting for u and v in the formula $\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$ we have

$$f = -8 \text{ cm. Hence } R = 2f = -16 \text{ cm}$$

The negative sign is expected since the mirror is concave.

Example 12: An object 5 cm tall is placed 10 cm from a convex mirror of focal length 15 cm. Find the position, size and nature of the image.

Solution: $u = -10$ cm

$$f = +15 \text{ cm (convex mirror)}$$

$$O = +5 \text{ cm}$$

Substituting for u and f in $\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$ gives $v = +6$ cm.

Now $m = \frac{I}{O} = -\frac{v}{u} = -\frac{+6}{-10} = +\frac{3}{5}$

or $I = \frac{3}{5} \times 5 = +3$ cm

The image is erect, virtual, 3 cm tall and is formed 6 cm behind the mirror.

Example 13: A convex mirror of radius of curvature 20 cm forms an image which is half the size of the object. Locate the position of the object.

Solution: Since a convex mirror can form only virtual image, its magnification will be positive. Thus

$$m = +\frac{1}{2}$$

But $m = -\frac{v}{u}$ or $v = -\frac{u}{2}$

Using this value of v in $\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$ we have ($\because f = +10$ cm)

or $u = -10$ cm

Thus the object is placed at a distance of 10 cm from the mirror.

Example 14: How far should an object be held from a concave mirror of focal length 40 cm so as to obtain an image twice the size of the object?

Solution: $f = -40$ cm

There are two possibilities : (a) the image is real and (b) the image is virtual.

(a) If the image is real then

$$m = -\frac{v}{u} = -2 \quad \text{or} \quad v = +2u$$

Putting $v = 2u$ in the mirror formula gives $u = -60$ cm

(b) If the image is virtual, then

$$m = -\frac{v}{u} = +2 \quad \text{or} \quad v = -2u$$

Putting $v = -2u$ in the mirror formula gives $u = -20$ cm.

Hence the possible object distance is 20 cm or 60 cm depending on whether the image is virtual or real.

Example 15: A concave mirror produces a real image twice the size of the object when placed at a distance of 22.5 cm from it. Where should the object be placed so that the size of the image becomes three times that of the object?

Solution: When $u_1 = -22.5$ cm, $m_1 = -2$ (real image). Now

$$m_1 = \frac{f}{f - u_1}$$

or
$$-2 = \frac{f}{f - (-22.5)} \text{ which gives } f = -15 \text{ cm.}$$

The negative sign is expected since the mirror is concave.

To obtain a magnification of $m_2 = -3$, the object distance u_2 is given by

$$m_2 = \frac{f}{f - u_2}$$

or
$$-3 = \frac{-15}{-15 - u_2}$$

which gives $u_2 = -20$ cm.

Example 16: Two objects A and B when placed in turn in front of a concave mirror of focal length 7.5 cm, give images of equal size. If A is three times the size of B and is placed 30 cm from the mirror, find the distance of B from the mirror.

Solution: $m_A = \frac{I_A}{O_A}$, $m_B = \frac{I_B}{O_B}$

where $I_A = O_A$ and $O_A = 3O_B$

$$\therefore \frac{m_A}{m_B} = \frac{I_A}{O_A} \times \frac{O_B}{I_B} = \frac{I_A}{I_B} \times \frac{O_B}{O_A} = \frac{1}{3}$$

But
$$m_A = \frac{f}{f - u_A} \quad \text{and} \quad m_B = \frac{f}{f - u_B}$$

$$\therefore \frac{m_A}{m_B} = \frac{f - u_B}{f - u_A}$$

It is given that $f = -7.5$ cm and $u_A = -30$ cm. Thus

$$\frac{1}{3} = \frac{-7.5 - u_B}{-7.5 - (-30)}$$

which gives $u_A = -15$ cm

Example 17: The sun's disc subtends an angle of $\frac{1}{2}^\circ$ at the earth. What must be the radius of curvature of the mirror which will produce on a screen an image of the sun 2 cm in diameter?

Solution: Since the image is formed on a screen, it must be real. Hence the mirror used is concave. Since the distance of the sun is very large (effectively infinite), the image of the sun is formed at the focal plane of the mirror, i.e. the image is at a distance equal to the focal length (f) from the mirror (see Fig. 15.12).

AB = diameter of the disc of the sun

$A'B'$ = diameter of the image = 2 cm

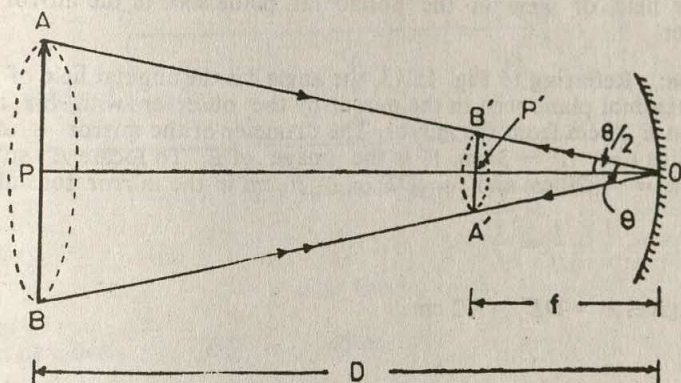


Fig. 15.12

Angle subtended by the disc of the sun $= \theta = \frac{1}{2}^\circ$

$$= \frac{1}{2} \times \frac{\pi}{180} \text{ rad}$$

$$= \frac{1}{2} \times \frac{3.142}{180} \text{ rad}$$

In triangle APO we have

$$\tan \frac{\theta}{2} = \frac{AP}{OP}$$

Since $OP \gg AP$, angle θ is very small. Hence $\tan \frac{\theta}{2} = \frac{\theta}{2}$ where θ is expressed in radians. Thus

$$\frac{\theta}{2} = \frac{AP}{OP} = \frac{AB/2}{OP}$$

or
$$\frac{AB}{OP} = \theta$$

Now triangles APO and $A'P'O$ are similar. Therefore,

$$\frac{AP}{A'P'} = \frac{OP}{OP'}$$

or
$$\frac{AB}{A'B'} = \frac{OP}{OP'}$$

or
$$OP' = A'B' \times \frac{OP}{AB}$$

or
$$f = \frac{2 \text{ cm}}{\theta} = \frac{2 \text{ cm} \times 180 \times 2}{3.142} = 229 \text{ cm}$$

Hence
$$R = 2f = 2 \times 229 = 458 \text{ cm} = 4.58 \text{ m}$$

Example 18: A circular convex mirror of diameter 6 cm and focal length 30 cm is fixed with its principal axis horizontal. An observer is sitting with his eyes on the axis of the mirror and 20 cm away from it. Calculate the angular field of view in the horizontal plane seen in the mirror by the observer.

Solution: Referring to Fig. 15.13, the angle θ is the angular field of view in the horizontal plane seen in the mirror by the observer with his eyes at position E 20 cm from the mirror. The diameter of the mirror is AB and its radius $r = AP = 3 \text{ cm}$. E' is the image of E . To locate E' , substitute $u = OE = -20 \text{ cm}$ and $f = OF = +30 \text{ cm}$ in the mirror formula

$$\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$$

which gives $v = OE' = 12 \text{ cm}$.

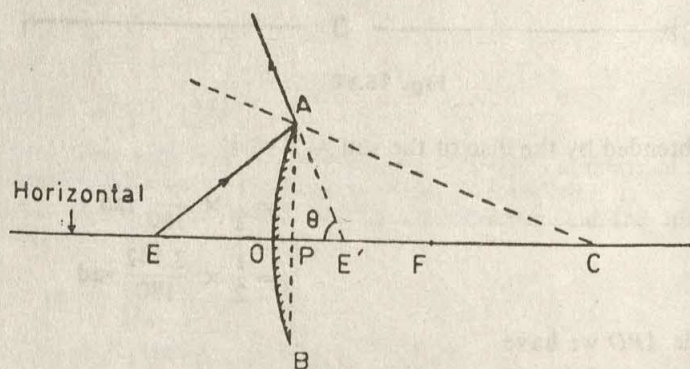


Fig. 15.13

Let $OP = x$. In triangle APC

$$(AC)^2 = (PC)^2 + (AP)^2$$

or
$$R^2 = (R - x)^2 + r^2$$

or
$$R - x = (R^2 - r^2)^{1/2}$$

Putting $R = 60 \text{ cm}$ and $r = 3 \text{ cm}$ we have $x = 0.075 \text{ cm}$

Now in triangle APE' , we have

$$\tan \theta = \frac{AP}{PE'} = \frac{AP}{OE' - OP} = \frac{3}{12 - 0.075} = 0.2516$$

or

$$\theta = 14.1^\circ$$

Example 19. A small strip of plane mirror A is set with its plane normal to the principal axis of a convex mirror B and placed 15 cm in front of B which it partly covers. An object is placed 30 cm from A and the two virtual images formed by reflection in A and B coincide without parallax. Find the radius of curvature of B .

Solution: $P'Q'$ and $P''Q'$ are virtual images of object PQ due to reflections in A and B respectively. Since the image $P'Q'$ of object PQ due to reflection from plane mirror A is as far behind the mirror as the object is in front of it, the image distance for the convex mirror B is (see Fig. 15.14).

$$v = + OQ' = + 15 \text{ cm}$$

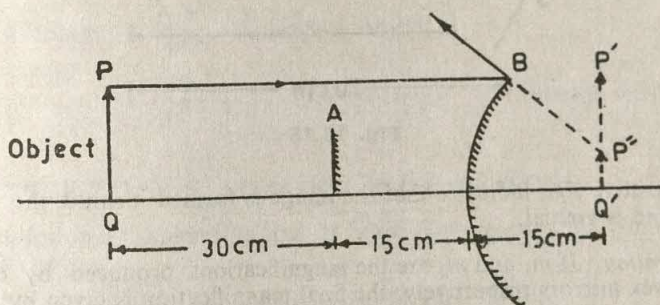


Fig. 15.14

Object distance $u = - OQ = - 45 \text{ cm}$

$$\text{Now } \frac{1}{f} = \frac{1}{v} + \frac{1}{u} = \frac{1}{15} + \frac{1}{-45}$$

which gives $f = 22.5 \text{ cm}$

Hence $R = 2f = 45 \text{ cm}$

Example 20: A concave mirror (focal length 15 cm) and a convex mirror (focal length 10 cm) are placed co-axially 70 cm apart facing each other. A 2 cm tall object is placed perpendicular to the common axis 20 cm from the concave mirror. Find the position, size and nature of the final image formed by two reflections, first at concave mirror and then at convex mirror.

Solution: The concave mirror forms a real image $A'B'$ of the object AB . The image $A'B'$ acts as a virtual object for the convex mirror forming the final image at $A''B''$ (see Fig. 15.15).

For reflection at concave mirror

$$f_1 = - 15 \text{ cm}, \quad u_1 = - 20 \text{ cm}$$

$$\frac{1}{v_1} = \frac{1}{f_1} - \frac{1}{u_1} = -\frac{1}{15} + \frac{1}{20} = -\frac{1}{60}$$

or

$$v_1 = - 60 \text{ cm}$$

For reflection at convex mirror

$$u_2 = -10 \text{ cm}, f_2 = +10 \text{ cm}$$

$$\frac{1}{v_2} = \frac{1}{f_2} - \frac{1}{u_2} = \frac{1}{10} - \frac{1}{-10} = \frac{1}{5}$$

or

$$v_2 = +5 \text{ cm}$$

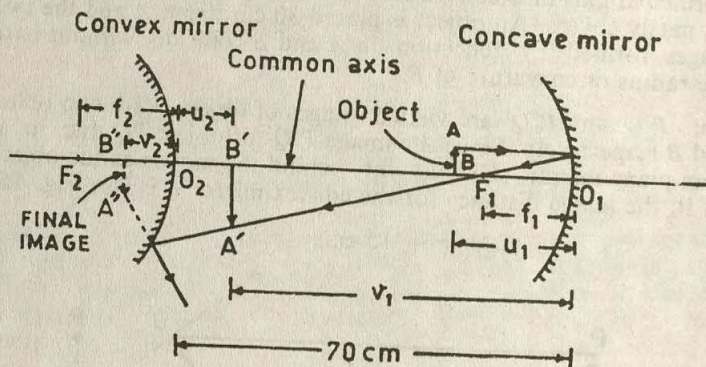


Fig. 15.15

The positive sign indicates that the image is formed behind the convex mirror and is virtual.

Magnification If m_1 and m_2 are the magnifications produced by concave and convex mirrors respectively, the final magnification is given by

$$m = |m_1 \times m_2|$$

$$= \frac{v_1}{u_1} \times \frac{v_2}{u_2} = \frac{60}{20} \times \frac{5}{10} = \frac{3}{2}$$

$$\text{Size of the image} = \frac{3}{2} \times \text{size of the object}$$

$$= \frac{3}{2} \times 2 = 3 \text{ cm}$$

Example 21: A motor car is fitted with a convex driving mirror of focal length 20 cm. A second motor car 2 m broad and 1.6 m high is 6 m away from the first car. (a) Calculate the position and size of the image of the second car seen in the mirror of the first. (b) If the second car is overtaking at a relative speed of 15 m s^{-1} , how fast will the image be moving and in what direction?

Solution: (a) $f = +20 \text{ cm}$, $u = -600 \text{ cm}$

Using $\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$ we get $v = +19.4 \text{ cm}$

$$\text{Also } m = \frac{I}{O} = -\frac{v}{u} = +\frac{19.4}{600}$$

i.e. $\text{Size of image} = \frac{19.4}{600} \times (\text{size of object})$

$$\therefore \text{Breadth} = 200 \times \frac{19.4}{600} = 6.4 \text{ cm}$$

$$\text{and Height} = 160 \times \frac{19.4}{600} = 5.2 \text{ cm}$$

(b) We know that

$$\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$$

Differentiating with respect to time t we have

$$-\frac{1}{v^2} \frac{dv}{dt} - \frac{1}{u^2} \frac{du}{dt} = 0$$

$$\text{or } \frac{dv}{dt} = -\frac{v^2}{u^2} \frac{du}{dt}$$

$$\begin{aligned} \text{Speed of image} &= \frac{-v^2}{u^2} \times (\text{speed of object}) \\ &= \left(\frac{19.4}{600}\right)^2 \times 15 = 0.016 \text{ ms}^{-1} \text{ towards the mirror} \end{aligned}$$

15.5 TRUE-FALSE STATEMENTS WITH REASONS

Are the following statements true or false? Give reasons for your answers in each case.

- (1) The laws of reflection of light hold only for plane surfaces.
- (2) The size of a virtual image can be measured by receiving it on a screen.
- (3) A plane mirror lying on the ground is not visible from all directions but a piece of paper can be seen from any direction.
- (4) A dentist uses a convex mirror to examine a small cavity.
- (5) The focal length of a spherical mirror is half the radius of curvature for all rays.

ANSWERS

- (1) *False* The laws of reflection are also applicable in the case of a spherical mirror. The reason is that a spherical mirror can be regarded as an arrangement of a large number of tiny plane mirrors, all equally inclined to each other so as to form a spherical mirror. Thus, when a ray of light falls on any point, it is reflected from a small plane mirror at that point, in accordance with the usual laws of reflection.
- (2) *False* The reflected rays do not actually meet at any point, but only appear to diverge from a point where the virtual image is seen. Hence a virtual image can not be received on a screen.
- (3) *True* When a parallel beam of sunlight falls on a plane mirror, it is reflected in one direction. Therefore, the surface of the mirror is seen only if we are looking in that direction. But a piece of paper, due to its surface irregularities, shows diffused reflection. Thus, when a parallel beam of light falls on it, the angle of incidence is different for different rays and, therefore, they are reflected in various different directions, enabling us to see the paper from every direction.
- (4) *False* A convex mirror forms only diminished virtual images. A dentist uses a concave mirror which can form a magnified virtual image of an object.

- (5) *False* The relation $f = R/2$ does not hold for all rays; it holds only for paraxial rays, i.e. those rays which are close to the principal axis of the mirror and subtend extremely small angle with the axis.

15.6 MULTIPLE CHOICE QUESTIONS

Choose the correct answer from the given choices.

- An observer moves towards a stationary plane mirror at a speed of 4 m s^{-1} . With what speed will his image move towards him?
 (a) 2 m s^{-1} (b) 4 m s^{-1}
 (c) 8 m s^{-1} (d) the image will stay at rest
- A plane mirror is moved towards a stationary observer with a speed of 4 m s^{-1} . The speed with which his image will move towards him will be
 (a) 2 m s^{-1} (b) 4 m s^{-1}
 (c) 8 m s^{-1} (d) zero
- Which mirror should be used to obtain a parallel beam of light from a small lamp?
 (a) Plane mirror (b) Convex mirror
 (c) Concave mirror (d) Any one of the above
- Choose the only wrong statement from the following.
 (a) A convex mirror forms virtual images for all positions of the object.
 (b) A concave mirror forms real images for all positions of the object.
 (c) A concave mirror, if suitably placed in front of an object, can form a virtual magnified image.
 (d) The magnification produced by a convex mirror is always less than unity.

Questions 5 to 10 are based on the following passage.

'An object is placed at a distance of $x \text{ cm}$ from a concave mirror and its real image is received on a screen placed at a distance of $y \text{ cm}$ from the mirror. The focal length of the mirror is $z \text{ cm}$.'

- What is the relation between x , y and z ?
 (a) $z = \frac{xy}{x+y}$ (b) $z = \frac{xy}{y-x}$
 (c) $xyz = 1$ (d) $x+y = z$
- The values of x are changed and the corresponding values of y are measured. Which one of the graphs shown in Fig. 15.16 represents the variation of y with x ?

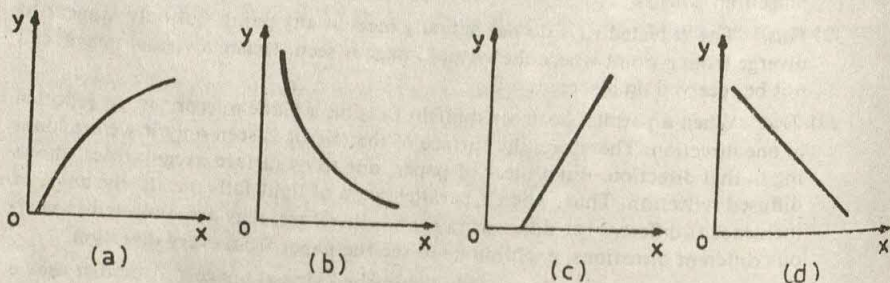


Fig. 15.16

7. Which of the following graphs approximately represents the variation of $1/y$ against $1/x$?

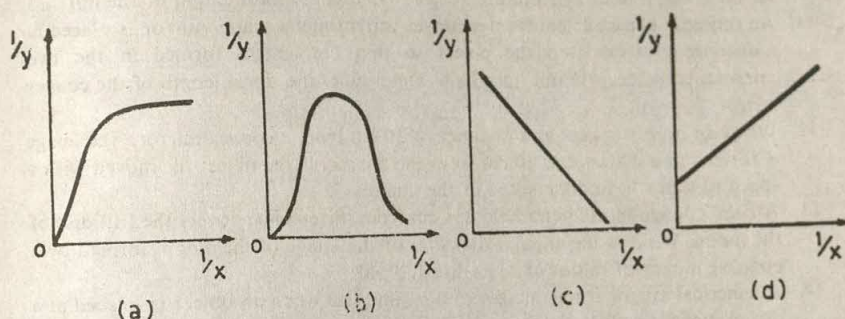


Fig. 15.17

8. What is the slope of the correct graph in Fig. 15.17?

- (a) + 1
(b) zero
(c) - 1
(d) - 2

9. What is the value of $1/y$ when $1/x$ is zero?

- (a) z
(b) $\sqrt{z}$
(c) z^2
(d) $1/z$

10. The magnification m is given by

- (a) $m = \frac{y - z}{z}$
(b) $m = \frac{y + z}{y}$
(c) $m = \frac{y - z}{y}$
(d) $m = \frac{y + z}{z}$

15.7 NUMERICAL EXERCISES

- Two plane mirrors are inclined to each other at an angle θ . A ray incident on one mirror at 50° after reflection from it falls on the second mirror where it is reflected parallel to the first mirror. Determine the value of θ .
- Two plane mirrors are placed parallel to each other 20 cm apart. A luminous point object is placed between them 5 cm from one mirror. Determine the distance from each mirror of the three nearest images in each mirror.
- Two plane mirrors are inclined to each other at 45° . A ray of light is reflected first at one mirror and then at the other. Find the total deviation of the ray.
- A man is running away from a plane mirror at a speed of 5 m s^{-1} . With what speed is he receding from his own image?
- By drawing ray diagrams, show that a person with two eyes requires a mirror less broad than that required by a one-eyed person to see his whole face at a time; but the length of the mirror remains the same.
- A narrow beam of light after reflection from a plane mirror falls on a scale 100 cm from the mirror. When the mirror is rotated a little, the spot moves through 20 cm. Determine the angle through which the mirror is rotated.
- A lamp is 4 m from a wall. Find the focal length of a concave mirror which will form a five times magnified image of the lamp on the wall. How far from the wall must the mirror be placed?
- A dentist's concave mirror has a radius of curvature of 30 cm. How far must it be placed from a small cavity in order to give a virtual image magnified 5 times?
- When an object is moved 12 cm towards a concave mirror, the real image moves out from 25 cm to 75 cm. Find the radius of curvature of the mirror.

10. The diameter of the curved surface of a convex mirror is 6 cm. A parallel beam of light falls on it and the reflected beam outlines a patch 50 cm in diameter on a wall 2 m away from the mirror. What is the focal length of the mirror?
11. An object is placed 25 cm from a convex mirror and a plane mirror is placed at a distance of 20 cm from the object so that the images formed in the two mirrors coincide without parallax. Determine the focal length of the convex mirror.
12. When an object is kept at a distance of 30 cm from a convex mirror, the image is formed at a distance of 10 cm from the mirror. If the object is moved with a speed of 9 m s^{-1} , find the speed of the image.
13. A coin 2.54 cm in diameter held 254 cm from the eye just covers the full disc of the moon. What is the angular diameter of the image of the moon formed by a concave mirror of radius of curvature 1.27 m?
14. A spherical mirror forms images of the same size when an object is placed at a distance of 8 cm or 16 cm from the mirror. Determine the radius of curvature of the mirror. Is it concave or convex?
15. A convex mirror produces a magnification of $1/2$ when the object is at a position *A* and a magnification of $1/3$ when the object is at a position *B*. If positions *A* and *B* are 60 cm apart, find the focal length of the mirror.
16. Two plane mirrors are held parallel and facing each other at a distance x cm apart. A small luminous object is placed y cm from one of them. Find the distance from the object of an image produced by four reflections. Deduce the corresponding distance for an image formed by $2n$ reflections.

ANSWERS

MULTIPLE CHOICE QUESTIONS

- | | | | | |
|--------|--------|--------|--------|---------|
| 1. (c) | 2. (b) | 3. (c) | 4. (b) | 5. (a) |
| 6. (b) | 7. (c) | 8. (c) | 9. (d) | 10. (a) |

NUMERICAL EXERCISES

1. 70°
2. 5, 35, 45; 15, 25, 55 cm
3. 270°
4. 10 m s^{-1}
6. 5.7°
7. 82.3 cm, 5 m
8. 1.2 cm
9. 26.4 cm
10. 27.25 cm
11. 37.5 cm
12. 1 m s^{-1}
13. 114°
14. 24 cm, concave
15. 60 cm
16. $4x \text{ cm}$, $2nx \text{ cm}$

Refraction of Light

16.1 REVIEW OF BASIC CONCEPTS

16.1.1 Refraction of Light When a ray of light travels from one medium into another, the speed of light undergoes a change. This phenomenon is known as *refraction*. Refraction is accompanied by a change in the direction of light, except in the case of normal incidence.

16.1.2 Laws of Refraction and Refractive Index When a ray of light suffers refraction at any surface (plane or curved) between two media, it obeys the following two laws of refraction.

(i) The incident ray, the normal to the surface at the point of incidence, and the refracted ray, all lie in the same plane. This common plane is called the plane of incidence.

(2) The ratio of the sine of the angle of incidence (in the first medium) to the sine of the angle of refraction (in the second medium) is a constant for all angles of incidence. This constant ratio is denoted by ${}^1\mu_2$ and is called the refractive index of the second medium with respect to the first. Thus

$${}^1\mu_2 = \frac{\sin i}{\sin r}$$

If the first medium is air (or vacuum), the refractive index is simply represented by the symbol μ_m and is called the refractive index of the medium.

16.1.3 Refraction and Speed of Light The phenomenon of refraction occurs because the speed of light is different in different media. When light passes from one medium into another, the amount of refraction or bending (for a given angle of incidence) depends upon the speed of light in each medium.

The refractive index ${}^1\mu_2$ of medium 2 with respect to medium 1 is also defined as

$${}^1\mu_2 = \frac{\text{speed of light in medium 1}}{\text{speed of light in medium 2}} = \frac{c_1}{c_2}$$

The refractive index μ_m of a medium is defined as

$$\mu_m = \frac{\text{speed of light in air (or vacuum)}}{\text{speed of light in the medium}} = \frac{c}{c_m}$$

16.1.4 Total Internal Reflection If a ray of light travelling from a denser medium into a rarer medium, is incident at an angle greater than the critical angle for the two media, the ray is totally reflected back into the same medium, obeying the laws of reflection. The critical angle is the angle of incidence (in the denser medium) for which the angle of refraction is 90° .

16.1.5 Dispersion of Light Refraction of light occurs because the velocity of light changes as it travels from one medium into another. The velocity of light in a given medium depends upon its wavelength (or frequency). Light of a single wavelength or frequency is called monochromatic. White light (sunlight) is not monochromatic; it consists of many wavelengths ranging from violet ($\sim 4500 \text{ \AA}$) to red ($\sim 7800 \text{ \AA}$).

When white light (or any composite light) enters a refracting medium, the different constituent colours are refracted unequally. The red is refracted the most and the violet the least. From Snell's law

$$\mu = \frac{\sin i}{\sin r}$$

it follows that the refractive index for red is less than for violet, i.e. the speed of red light is greater than that of violet light. Thus, a medium does not have one definite refractive index; it has a range of refractive indices corresponding to the range of colours or wavelengths of the composite light. Since each colour has its own characteristic wavelength (or frequency), the refractive index of a medium will be different for different wavelengths (or frequencies). The variation of refractive index of a medium (and hence of the velocity of light in the medium) with the wavelength or frequency is referred to as *dispersion*. The prism disperses the colours of white light and produces its spectrum.

16.2 FORMULAE

1. Refractive index ${}^1\mu_2$ of a medium 2 with respect to medium 1 is given by

$${}^1\mu_2 = \frac{c_1}{c_2} = \frac{\sin i}{\sin r}$$

where c_1 = velocity of light in medium 1

c_2 = velocity of light in medium 2

i = angle of incidence in medium 1

r = angle of refraction in medium 2

$$2. \quad {}^1\mu_2 = \frac{1}{{}^2\mu_1}$$

$$3. \quad {}^2\mu_3 = \frac{{}^1\mu_3}{{}^1\mu_2}$$

4. *Refraction Through a Slab with Parallel Sides* When a ray of light AB falls obliquely at an angle i on a parallel-sided slab of a refracting material, such as glass, having a thickness t and refractive index μ , it suffers refractions at B and C and the emergent ray CD is parallel to the incident ray AB but shifted due to refraction (see Fig. 16.1).

The lateral shift (x) is given by

$$x = \frac{t}{\cos r} \sin(i - r) = t \sec r \sin(i - r)$$

where the angle of refraction (r) is given by

$$\mu = \frac{\sin i}{\sin r}$$

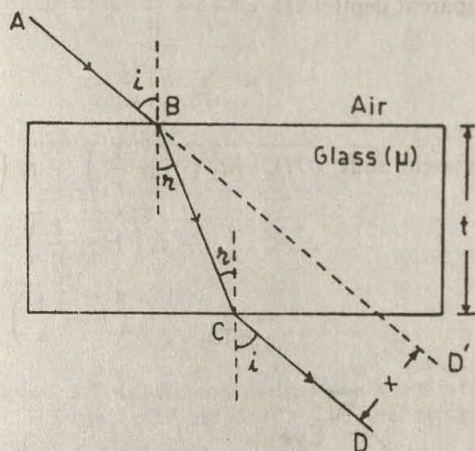


Fig. 16.1

Apparent Depth and Apparent Shift

(a) *Object in the Denser Medium and Observer in the Rarer Medium* When an object O in the denser medium of thickness t and refractive index μ is viewed from the rarer medium (air), the image is seen at I , as shown in Fig. 16.2. The real depth of the object is AO and the apparent depth is AI ; the apparent shift being OI .

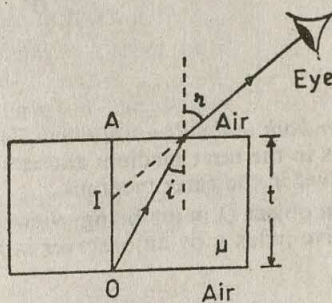


Fig. 16.2

Now
$$\mu = \frac{\text{real depth}}{\text{apparent depth}} = \frac{AO}{AI} = \frac{t}{AI}$$

$\therefore$ Apparent depth (AI) = $\frac{t}{\mu}$

Apparent shift (OI) = $OA - AI = t \left(1 - \frac{1}{\mu} \right)$

If the image is seen after successive refractions through a number of slabs of thicknesses $t_1, t_2, t_3, \dots$ and refractive indices $\mu_1, \mu_2, \mu_3, \dots$, then (see Fig. 16.3)

$$\begin{aligned}\text{Apparent depth (AI)} &= \frac{t_1}{\mu_1} + \frac{t_2}{\mu_2} + \frac{t_3}{\mu_3} + \dots \\ &= \sum_{n=1,2,\dots} \frac{t_n}{\mu_n}\end{aligned}$$

$$\begin{aligned}\text{Apparent shift (OI)} &= t_1 \left(1 - \frac{1}{\mu_1}\right) + t_2 \left(1 - \frac{1}{\mu_2}\right) \\ &\quad + t_3 \left(1 - \frac{1}{\mu_3}\right) + \dots \\ &= \sum_{n=1,2,\dots} t_n \left(1 - \frac{1}{\mu_n}\right)\end{aligned}$$

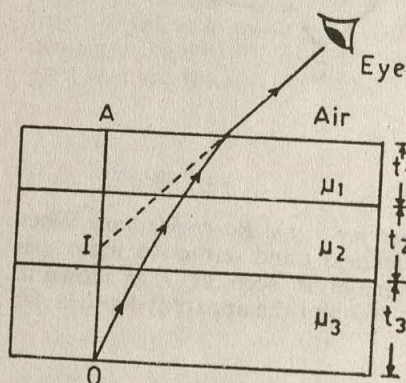


Fig. 16.3

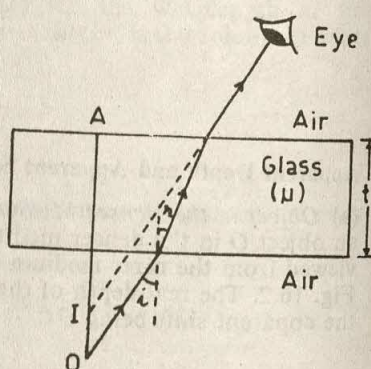


Fig. 16.4

(b) *Object and Observer both in the Rarer Medium* The above relations hold even when the object is in the rarer medium and is seen through a parallel-sided slab by an observer in the rarer medium.

Figure 16.4 shows an object O in air being viewed through a slab of thickness t and refractive index μ by an observer in air. The image is seen at O .

$$\text{Apparent shift (OI)} = t \left(1 - \frac{1}{\mu}\right)$$

(c) *Object in the Rarer Medium and the Observer in the Denser Medium* When the object O is in the rarer medium (air), the image I as seen by an observer in the denser medium (water), appears to be raised up as shown in Fig. 16.5.

The real and apparent depths are now related as

$$\frac{AO}{AI} = \mu_w = \frac{1}{\mu_a}$$

where μ is the refractive index of water.

$$\text{Apparent shift } (IO) = AI - AO = (\mu - 1) AO$$

where AO is the distance of the object from the surface of water.

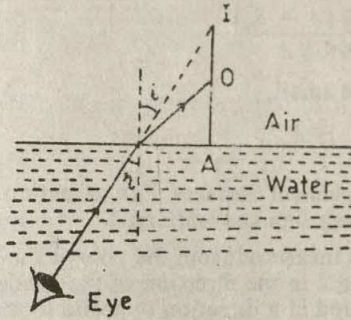


Fig. 16.5

16.1.5 Critical Angle for Total Internal Reflection When a ray of light travels from a denser to a rarer medium ($\mu_2 > \mu_1$), the critical angle i_c for total internal reflection is given by

$$\sin i_c = \frac{\mu_1}{\mu_2} = \frac{1}{\mu_2} = \mu_1$$

The critical angle i_c is the value of angle of incidence i when the corresponding value of the angle of refraction $r = 90^\circ$ (see Fig. 16.6).

If the rarer medium is air, $\mu_1 = 1$, then

$$\sin i_c = \frac{1}{\mu}$$

where μ is the refractive index of the denser medium.

Total internal reflection will occur if

- the ray travels from a denser to a rarer medium and
- the angle of incidence in the denser medium is greater than the critical angle for the two media.

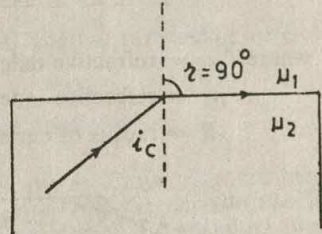


Fig. 16.6

16.1.6 Refraction through a Prism Referring to Fig. 16.7, when a ray of monochromatic light is refracted by a prism, the deviation δ produced by the prism is given by

$$\delta = i + e - A$$

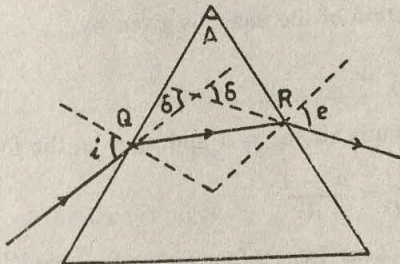


Fig. 16.7

where A is the refracting angle of the prism.

$$A = r_1 + r_2$$

For minimum deviation δ_m , we have

$$\mu = \frac{\sin \frac{1}{2}(A + \delta_m)}{\sin \frac{1}{2}A}$$

If angles i and A are small,

$$\delta = (\mu - 1)A$$

16.1.7 Refraction at Spherical Surfaces For refraction at spherical surfaces we will use the following sign conventions.

- (1) All distances are measured from the pole of the refracting surface.
- (2) Distances measured in the direction of the incident rays are positive and those measured in a direction opposite to the direction of the incident rays are negative.
- (3) Heights measured upwards and perpendicular to the principal axis are positive and those measured downwards are negative.

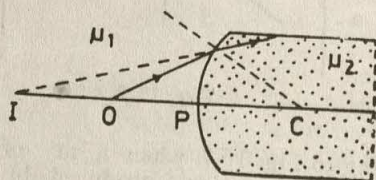
(a) *Refraction at a Single Spherical Surface* A single refracting surface (convex or concave) forms images of objects by virtue of refraction. If the rays from an object travelling in medium 1 are refracted at the surface of medium 2 (see Fig. 16.8), the object and image distances are related by the formula (valid only for paraxial rays and for both convex and concave surfaces)

$$\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}$$

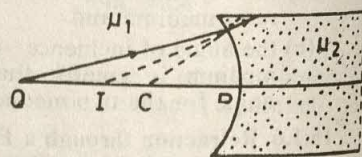
where μ_1 = refractive index of medium 1

μ_2 = refractive index of medium 2

R = radius of curvature of the surface



Convex surface



Concave surface

Fig. 16.8

Linear magnification of the image is given by

$$m = \frac{I}{O} = \frac{v}{u} \frac{\mu_1}{\mu_2}$$

If the first medium is air, $\mu_1 = 1$ and $\mu_2 = \mu$, the formula becomes

$$\frac{\mu}{v} - \frac{1}{u} = \frac{\mu - 1}{R}$$

(b) *Refraction at a Lens* Lenses (convex and concave) also form images of objects. The formulae given below hold for thin lenses and for paraxial

rays. A lens is said to be thin if its thickness is negligible compared to the distances of the object or its image.

(i) The relation between the object distance u , the image distance v and the focal length f of a lens (convex or concave) is

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

The linear magnification of the image is given by

$$m = \frac{I}{O} = \frac{v}{u} = \frac{f-v}{f} = \frac{f}{f+u}$$

(ii) The focal length f of a lens of refractive index μ_2 surrounded by a medium of refractive index μ_1 is given by

$$\frac{1}{f} = \left(\frac{\mu_2 - \mu_1}{\mu_1} \right) \left(\frac{1}{R_1} - \frac{1}{R_2} \right)$$

where R_1 = radius of curvature of the surface of the lens on which the rays are incident

R_2 = radius of curvature of the surface of the lens from which the rays emerge after refraction through the lens.

If the lens is surrounded by air, $\mu_1 = 1$ and $\mu_2 = \mu$, then

$$\frac{1}{f} = (\mu - 1) \left(\frac{1}{R_1} - \frac{1}{R_2} \right)$$

This is known as the 'lens maker's formula'.

(iii) *Power of lens* The reciprocal of the focal length (expressed in metres) is known as the power of a lens.

Power $p = \frac{1}{f}$ where f is expressed in metres

(iv) *Displacement method* For a convex lens, the minimum distance between the object and its real image is $4f$. If a convex lens is placed between an object O and a screen S such that the distance $OS \geq 4f$, there are two positions L_1 and L_2 of the lens which give a sharp image on the screen (see Fig. 16.9).

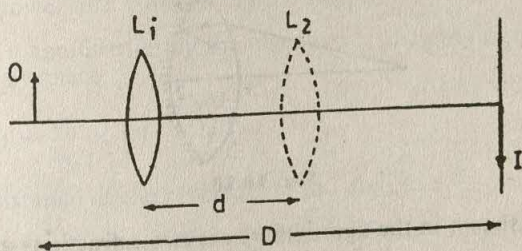


Fig. 16.9

The focal length of the lens is given by

$$f = \frac{D^2 - d^2}{4D}$$

where D = distance between the screen and the object

d = distance between the two positions of the lens.

(v) *Co-axial Combination of Lenses* When a number of thin lenses of focal lengths $f_1, f_2, \dots$ etc. are placed in contact coaxially (i.e. with their principal axes coinciding), the equivalent focal length F of the combination is given by

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2} + \dots$$

(vi) *Chromatic Aberration and Achromatism* The focal length of a lens depends on the refractive index of its material. But the refractive index of a lens is different for different colours of light. Hence the focal length of a lens is different for different colours. The focal length of the lens for red light (f_r) is greater than that for violet light (f_v).

Axial chromatic aberration = $f_r - f_v = \omega f$

where $\omega = \frac{\mu_v - \mu_r}{\mu - 1}$, $\mu = \frac{1}{2}(\mu_v + \mu_r)$ and $f = \frac{1}{2}(f_r + f_v)$

Chromatic aberration is eliminated if two lenses (one concave and the other convex) of different materials are placed in contact and satisfy the following condition of achromatism.

$$\frac{f_1}{f_2} = - \frac{\omega_1}{\omega_2}$$

16.1.8 Effect of Silvering one of the Refracting Surfaces of a Lens If one of the surfaces of a lens is silvered, as shown in Fig. 16.10, the rays are refracted at surface 1, reflected at surface 2 and again refracted at surface 1. The focal length F of the effective lens is given by

$$\frac{1}{F} = \sum_{n=1,2,\dots} \frac{1}{f_n}$$

where f_n = focal length of the lens or mirror repeated as many times as there are refractions and reflections.

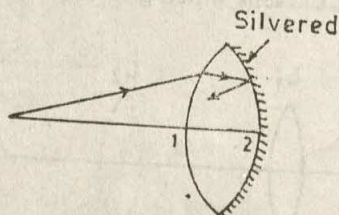


Fig. 16.10

In the case shown in the figure, there are two refractions and one reflection. Hence

$$\frac{1}{F} = \frac{1}{f_l} + \frac{1}{f_l} + \frac{1}{f_m} = \frac{2}{f_l} + \frac{1}{f_m}$$

where f_l = focal length of the lens

f_m = focal length of the spherical mirror of radius of curvature R_2 .

If the second surface is plane, as shown in Fig. 16.11 (a), $f_m = \infty$, therefore

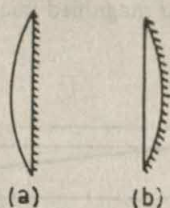
$$\frac{1}{F} = \frac{2}{f_l}$$

where $\frac{1}{f_l} = (\mu - 1) \frac{1}{R}$

If the convex surface of a plano-convex lens is silvered, as shown in Fig. 16.11 (b), we have

$$\frac{1}{F} = \frac{2}{f_l} + \frac{1}{f_m} = 2(\mu - 1) \frac{1}{R} + \frac{2}{R} = \frac{2\mu}{R} \quad \text{Fig. 16.11 (a) and (b)}$$

$$(\because R = 2f_m)$$



16.1.9 Microscope In its simplest form, a compound microscope consists of two convergent lenses, one of very short focal length called the *objective* and the other of a longer focal length called the *eye-piece*. The lenses are mounted coaxially and the separation between them can be varied. The object AB (to be examined) is placed just outside the focus F of the objective which forms its real image at $A'B'$ which serves as the object for the eyepiece. Image $A'B'$ lies within the focus F_e of the eyepiece which forms the final magnified virtual image at $A''B''$, as shown in Fig. 16.12.

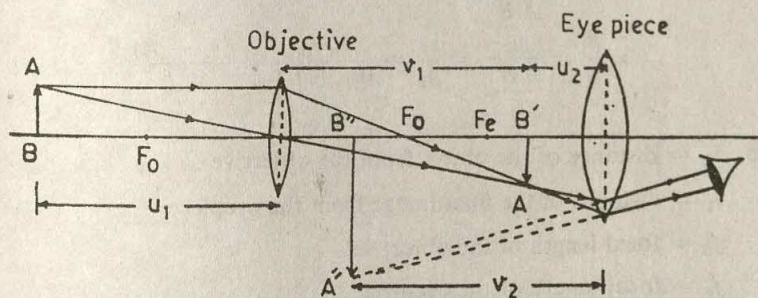


Fig. 16.12

Magnifying power of a compound microscope

= magnification by the objective $\times$ magnifying power of the eyepiece

i.e. $mp = \frac{v_1}{u_1} \times \left(1 - \frac{v_2}{f_c}\right)$

where u_1 = distance of the object from the objective

v_1 = distance (from the objective) of the image formed by the objective

f_c = focal length of the eyepiece

v_2 = distance of the final image from the eyepiece

16.1.10 Telescope A telescope consists of two convergent lenses called the objective and the eyepiece. The objective has a much longer focal length

than the eyepiece. The objective forms an image $A'B'$ of a distant object AB . The image $A'B'$ serves as the object for the eyepiece which forms the final magnified image $A''B''$ (see Fig. 16.13).

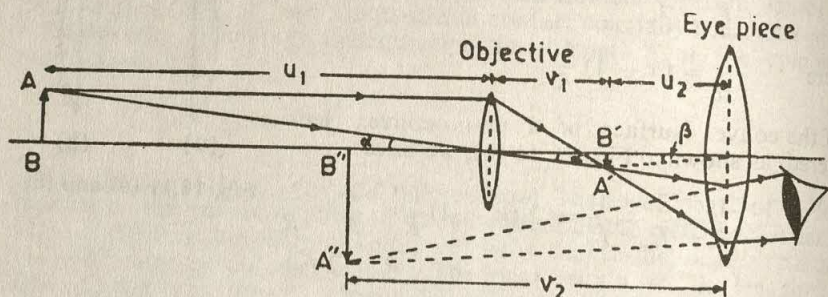


Fig. 16.13

If u_1 is large and the eye is held close to the eyepiece, the magnifying power of a telescope is given by

$$\begin{aligned} \text{mp} &= \frac{\text{angle subtended by the image at the eye}}{\text{angle subtended by the object at the eye}} \\ &= \frac{\beta}{\alpha} = \frac{A'B'}{A'B''} = \frac{v_1}{u_2} = \frac{u_1 f_0}{(u_1 + f_0)} \times \frac{(v_2 + f_e)}{v_2 f_e} \end{aligned}$$

where u_1 = distance of the object from the objective

v_2 = distance of the final image from the eyepiece

f_0 = focal length of the objective

f_e = focal length of the eyepiece

Special Cases (1) If the final image is formed at the least distance of distinct vision, $v_2 = D$, then

$$\text{mp} = \frac{u_1 f_0}{(u_1 + f_0)} \frac{(D + f_e)}{D f_e}$$

(2) If the final image is formed at infinity, $v_2 \rightarrow \infty$, then

$$\begin{aligned} \text{mp} &= \frac{u_1 f_0}{(u_1 + f_0)} \times \frac{\left(1 + \frac{f_e}{v}\right)}{f_e} = \frac{u_1}{(u_1 + f_0)} \frac{f_0}{f_e} \\ &\quad \left(\because \frac{f_e}{v} \rightarrow 0\right) \end{aligned}$$

(3) If the object is at infinity ($u_1 \rightarrow \infty$) and the image is at the distance of distinct vision ($v_2 = D$), we have

$$\text{mp} = \frac{f_0 (D + f_e)}{f_e D}$$

(4) When both the object and the image are at infinity ($u_1 \rightarrow \infty, v_2 \rightarrow \infty$) the telescope is said to be in normal adjustment. In this case,

$$mp = \frac{f_0}{\left(1 + \frac{f_0}{u_1}\right)} \times \frac{\left(1 + \frac{f_e}{v_2}\right)}{f_e} = \frac{f_0}{f_e}$$

16.3 NUMERICAL EXAMPLES

Example 1: A ray of light is incident at an angle i on a parallel-sided plate of thickness t and refractive index μ . Show that the lateral displacement of the emergent ray is given by

$$x = t \sin i \left\{ 1 - \frac{\cos i}{(\mu^2 - \sin^2 i)^{\frac{1}{2}}} \right\}$$

Can the displacement exceed the thickness of the plate?

Solution: In triangle BCD (see Fig. 16.14)

$$\frac{x}{BC} = \sin(i - r)$$

$$x = BC \sin(i - r)$$

Incident ray

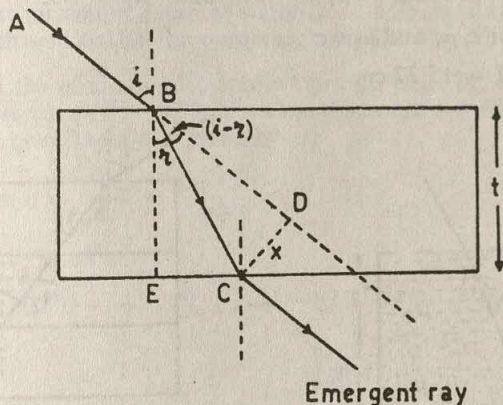


Fig. 16.14

In triangle BCE , we have

$$BC = \frac{BE}{\cos r} = \frac{t}{\cos r}$$

Hence

$$x = \frac{t}{\cos r} \sin(i - r)$$

or

$$x = \frac{t}{\cos r} (\sin i \cos r - \cos i \sin r)$$

$$= t \left(\sin i - \frac{\cos i \sin r}{\cos r} \right) \quad (i)$$

Now $\mu = \frac{\sin i}{\sin r}$ or $\sin r = \frac{\sin i}{\mu}$ (ii)

and $\cos r = (1 - \sin^2 r)^{\frac{1}{2}} = \left(1 - \frac{\sin^2 i}{\mu^2}\right)^{\frac{1}{2}}$ (iii)

Using Eqs (ii) and (iii) in Eq. (i) we get

$$x = t \left\{ \sin i - \frac{\cos i \sin i}{\mu \left(1 - \frac{\sin^2 i}{\mu^2}\right)^{\frac{1}{2}}} \right\}$$

or $x = t \sin i \left\{ 1 - \frac{\cos i}{(\mu^2 - \sin^2 i)^{\frac{1}{2}}} \right\}$

It is clear that the displacement x tends to a maximum value equal to t if $i \rightarrow 90^\circ$ so that $\sin i \rightarrow 1$ and $\cos i \rightarrow 0$. Hence x cannot exceed t .

Example 2: A vessel 20 cm deep is half-filled with oil of refractive index 1.37 and the other half is filled with water of refractive index 1.33. Find the apparent depth of the vessel when viewed from above.

Solution: Referring to Fig. 16.15, the apparent depth of the vessel is

$$AI = \frac{t_1}{\mu_1} + \frac{t_2}{\mu_2} = \frac{t}{2} \left(\frac{1}{\mu_1} + \frac{1}{\mu_2} \right) = \frac{t(\mu_1 + \mu_2)}{2\mu_1\mu_2}$$

where $t = 20$ cm, $\mu_1 = 1.37$ and $\mu_2 = 1.33$

Substituting for t , μ_1 and μ_2 we get

$$AI = 15.37 \text{ cm}$$

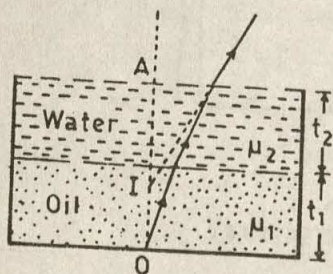


Fig. 16.15

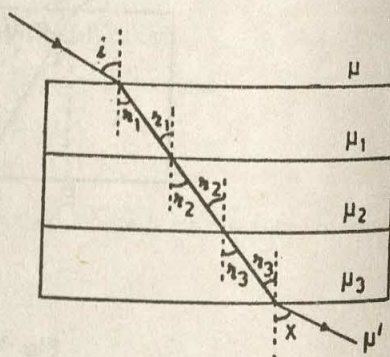


Fig. 16.16

Example 3: A ray of light, travelling in a medium of refractive index μ , is incident at an angle i on a composite transparent plate consisting of three plates of refractive indices μ_1 , μ_2 and μ_3 . The ray emerges from the composite plate into a medium of refractive index μ' at an angle x . Show that angle x is independent of μ_1 , μ_2 and μ_3 . Also show that $x = i$ if $\mu = \mu'$.

Solution: From Snell's law, we have for successive refractions (see Fig. 16.16)

$$\frac{\sin i}{\sin r_1} = \frac{\mu}{\mu_1}$$

$$\frac{\sin r_1}{\sin r_2} = \frac{\mu_2}{\mu_1}$$

$$\frac{\sin r_2}{\sin r_3} = \frac{\mu_3}{\mu_2}$$

$$\frac{\sin r_3}{\sin x} = \frac{\mu'}{\mu_3}$$

Multiplying these equations we get

$$\frac{\sin i}{\sin x} = \frac{\mu'}{\mu}$$

or $\sin x = \frac{\mu}{\mu'} \sin i$

This shows that angle x at which the ray leaves the composite plate depends only on the angle of incidence on the plate and the refractive indices of the media on both sides of the plate and is independent of μ_1 , μ_2 and μ_3 . If $\mu = \mu'$,

$$\sin x = \sin i \quad \text{or} \quad x = i$$

Example 4: A cubical vessel with opaque walls, is so placed that the eye of an observer cannot see its bottom but can see the entire wall CD (see Fig. 16.17 a). A small object is placed at O at a distance $b = 10$ cm from corner D . What minimum depth of water ($\mu = 4/3$) should be poured into the vessel which will enable the observer to see the object?

Solution: Let the minimum depth of water be x [see Fig. 16.17 (b)]. Since the vessel is cubical ($AB = AD$), it is clear that angle $i = 45^\circ$. From Snell's law, the angle of refraction is given by

$$\sin r = \frac{\sin i}{\mu} = \frac{\sin 45^\circ}{4/3} = 0.5303$$

$$\therefore r = 32.03^\circ$$

Since $\angle QPD = 45^\circ$, $PQ (= x) = QD$. Hence $QO = x - b$

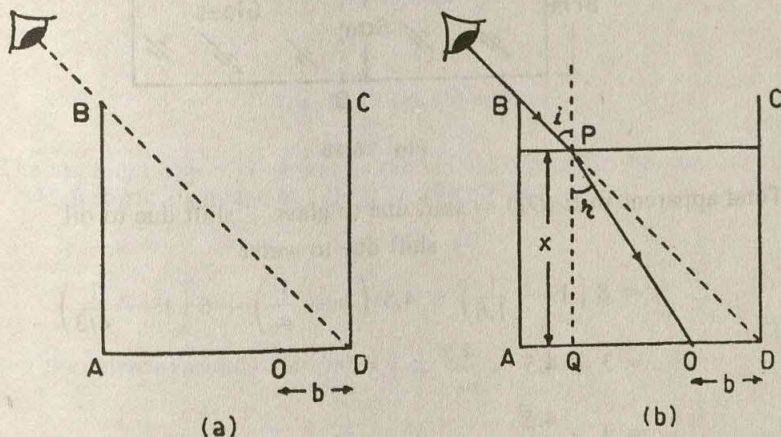


Fig. 16.17

Now in triangle PQO ,

$$QO = x \tan r$$

or $x - b = x \tan r$

or $x = \frac{b}{1 - \tan r}$

Substituting for b and r we get

$$x = 26.7 \text{ cm}$$

Example 5: A tank contains a slab of glass 8 cm thick and of refractive index 1.6. Above this is a depth of 4.5 cm of an oil and upon this floats a layer of water 6 cm thick and of refractive index $4/3$. To an observer looking down from above, a mark at the bottom of the glass slab appears to be raised up to a position 6 cm from the bottom of the slab. Calculate the refractive index of oil.

Solution: O is the mark at the bottom of the glass slab (Fig. 16.18). Since the boundaries of the media are parallel, the total shift of the mark O is equal to the shifts due to each of the three media. Let μ_0 be the refractive index of the oil.

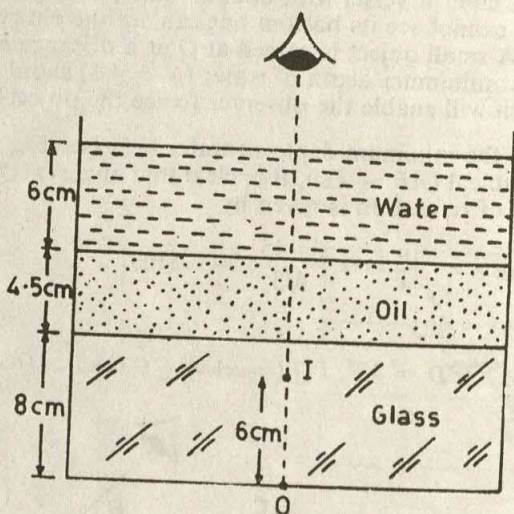


Fig. 16.18

Total apparent shift (OI) = shift due to glass + shift due to oil + shift due to water

$$= 8 \left(1 - \frac{1}{1.6} \right) + 4.5 \left(1 - \frac{1}{\mu_0} \right) + 6 \left(1 - \frac{1}{4/3} \right)$$

$$= 3 + 4.5 - \frac{4.5}{\mu_0} + 1.5$$

$$= 9 - \frac{4.5}{\mu_0}$$

But $OI = 6$ cm (given). Hence

$$6 = 9 - \frac{4.5}{\mu_0}$$

or $\mu_0 = 1.5$

Example 6: An object is placed 21 cm in front of a concave mirror of radius of curvature 20 cm. A glass slab of thickness 3 cm and refractive index 1.5 is placed close to the mirror in the space between the object and the mirror. Find the position of the final image formed. The distance of the nearer surface of the slab from the mirror is 1.0 cm. (IIT 1980)

Solution: The rays starting from the object O will first be refracted by the glass slab and appear to come from point O' [Fig. 16.19 (a)].

$$\text{The apparent shift } (OO') = t \left(1 - \frac{1}{\mu} \right) = 3 \left(1 - \frac{1}{1.5} \right) = 1.0 \text{ cm}$$

$$\text{Therefore } O'P = 21 - 1 = 20 \text{ cm}$$

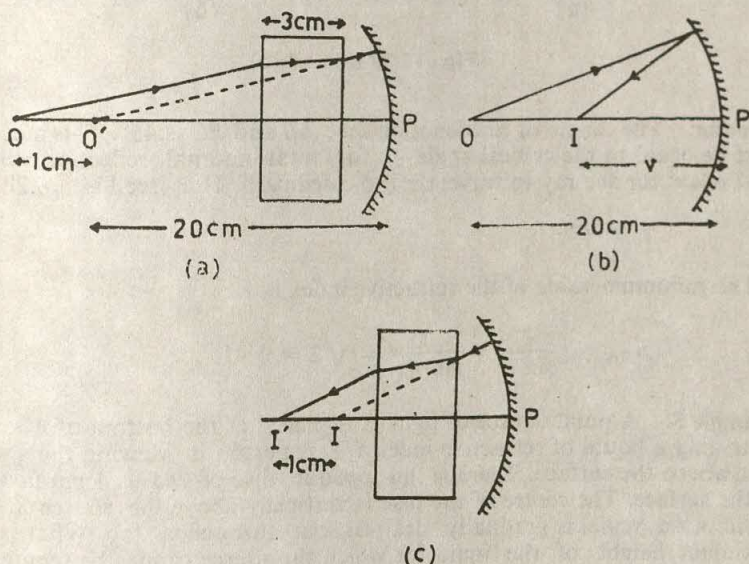


Fig. 16.19 (a), (b) and (c)

The apparent object O' serves as the virtual object for the concave mirror which forms its image at I [Fig. 16.19 (b)]. Now

$$u = -20 \text{ cm}$$

$$f = \frac{R}{2} = -10 \text{ cm}$$

From the mirror formula we have

$$\frac{1}{v} = \frac{1}{f} + \frac{1}{u} = -\frac{1}{10} - \frac{1}{20} = -\frac{3}{20}$$

or

$$v = -6.67 \text{ cm}$$

The reflected rays will again pass through the slab, suffering a further shift of 1 cm, producing the final image at I' [Fig. 16.19 (c)]. Thus the final image I' will be at a distance of $PI' = 6.67 + 1 = 7.67$ cm from the mirror.

Example 7: What must be the minimum value of the refractive index of an isosceles prism ABC (with $AB = BC$) so that a ray falling normally on face AC can travel as shown in Fig. 16.20 (a)?

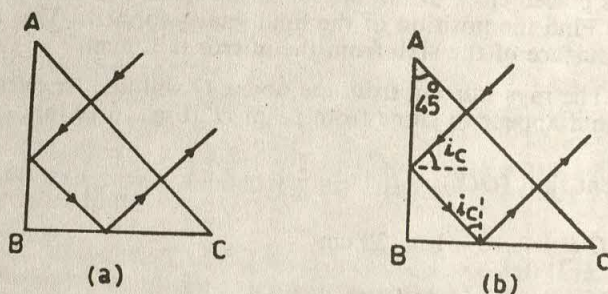


Fig. 16.20 (a) and (b)

Solution: The angle of incidence on face AB and BC is 45° . This must at least be equal to the critical angle i_c for total internal reflection which must occur for the ray to travel the indicated path. Thus [see Fig. 16.20 (b)]

$$i_c = 45^\circ$$

The minimum value of the refractive index is

$$\mu_{\min} = \frac{1}{\sin i_c} = \frac{1}{\sin 45^\circ} = \sqrt{2} = 1.41$$

Example 8: A point source of light S is placed at the bottom of a vessel containing a liquid of refractive index $5/3$. A person is viewing the source from above the surface. There is an opaque disc of radius 1 cm floating on the surface. The centre of the disc is vertically above the source S . The liquid in the vessel is gradually drained out through a tap. What is the maximum height of the liquid for which the source cannot be seen at all from above?
(IIT 1970)

Solution: Referring to Fig. 16.21, the source S cannot be seen at all from above when the water level attains a critical maximum height x so that

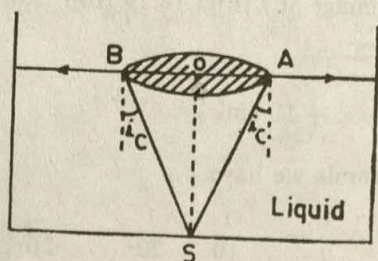


Fig. 16.21

rays such as SA and SB suffer internal reflection. The critical angle i_c is given by

$$\sin i_c = \frac{1}{\mu} = \frac{1}{5/3} = 0.6$$

or $i_c = 36.87^\circ$

In triangle OAS , we have

$$\frac{r}{x} = \tan i_c$$

or $x = \frac{r}{\tan i_c} = \frac{1}{\tan 36.87^\circ} = 1.33 \text{ cm}$

Example 9: The principal section of a glass prism is an isosceles triangle ABC with $AB = AC$. The face AC is silvered. A ray is incident normally on face AB and after two reflections, it emerges from the base BC , perpendicular to it. Find the $\angle BAC$ of the prism.

Solution: The path of the ray is shown in Fig. 16.22. It is clear that

$$\alpha + 2\beta = 180^\circ$$

and $\beta = 2\alpha$

Hence $\alpha = 36^\circ$

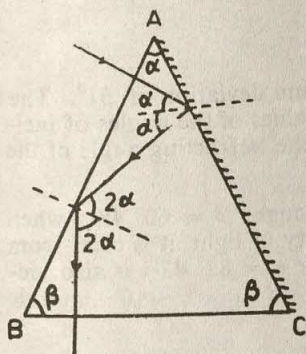


Fig. 16.22

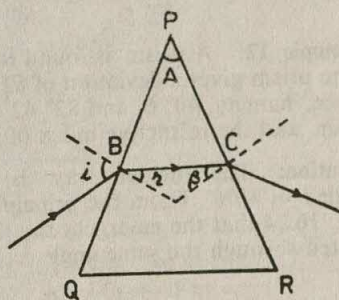


Fig. 16.23

Example 10: A ray of light incident on the face of a prism is refracted and escapes through an adjacent face. Find the maximum permissible refracting angle of the prism if it is made of glass of refractive index 1.5.

Solution: The path of the ray through the prism is shown in Fig. 16.23. The ray is incident on face PQ and emerges out of the face PR if the refracted ray BC does not suffer total internal reflection at face PR . The necessary condition is that

$$\sin \beta \leq \frac{1}{\mu} \quad \text{or} \quad \beta_{\max} = \sin^{-1} \left(\frac{1}{1.5} \right) = 41.8^\circ$$

It is clear from the diagram that

$$\beta = A - r$$

where A is the refracting angle of the prism. Hence the greater the value of r , the higher is the permissible value of A . Angle r will be maximum if $i = 90^\circ$, i.e.

$$\sin r_{\max} = \frac{\sin 90^\circ}{\mu} = \frac{1}{1.5} = 0.6667$$

or $r_{\max} = 41.8^\circ$

The maximum value of A is

$$A_{\max} = \beta_{\max} + r_{\max} = 41.8^\circ + 41.8^\circ = 83.6^\circ$$

Example 11: Show that, when a ray passes symmetrically through a prism, the refractive index of the prism is given by

$$\mu = \frac{\sin \frac{1}{2}(A + \delta_m)}{\sin \frac{1}{2}A}$$

where A = refracting angle of the prism and δ_m = angle of minimum deviation.

Solution: Refer to Fig. 16.7. When the ray passes symmetrically through a prism, i.e. when $i = e$, then the deviation δ is minimum equal to δ_m .

$$\delta_m = i + i - A = 2i - A$$

or $i = \frac{1}{2}(A + \delta_m)$

Also $r_1 = r_2$, therefore $A = 2r_1$ or $r_1 = A/2$.

Hence
$$\mu = \frac{\sin i}{\sin r_1} = \frac{\sin \frac{1}{2}(A + \delta_m)}{\sin \frac{1}{2}A}$$

Example 12: A prism is found to give a minimum deviation of 51° . The same prism gives a deviation of $62^\circ 48'$ for two values of the angles of incidence, namely, $40^\circ 6'$ and $82^\circ 42'$. Determine the refracting angle of the prism and the refractive index of its material.

Solution: The incident ray is deviated through $\delta = 60^\circ 48'$ when angle $i = 40^\circ 6'$. From the principle of reversibility of light, it is clear from Fig. 16.24 that the emergent ray (for which angle $e = 82^\circ 42'$) is also deviated through the same angle δ . Now

$$\delta = (i + e) - A$$

or $A = (i + e) - \delta$

$$= 40^\circ 6' + 82^\circ 42' - 62^\circ 48'$$

or $A = 60^\circ$

which is the refracting angle of the prism.

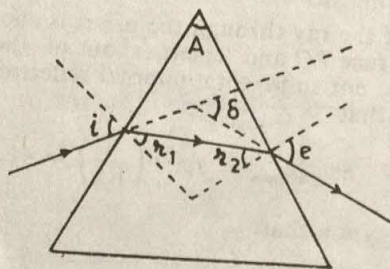


Fig. 16.24

For minimum deviation, $i = e$. Hence

$$\delta_{\min} = 2i - A$$

$$\begin{aligned} \text{or } i &= \frac{1}{2}(\delta_{\min} + A) \\ &= \frac{1}{2}(51^\circ + 60^\circ) = 55^\circ 30' \end{aligned}$$

which is the angle of incidence at minimum deviation. The refractive index of the material of the prism is given by

$$\begin{aligned} \mu &= \frac{\sin \frac{1}{2}(\delta_{\min} + A)}{\sin \frac{1}{2}A} \\ &= \frac{\sin \frac{1}{2}(51^\circ + 60^\circ)}{\sin \frac{1}{2}(60^\circ)} \\ &= 1.648 \end{aligned}$$

Example 13: A ray of light PQ is incident at an angle i on face ML of a prism and is refracted along QR (Fig. 16.25). This ray, after refraction at face MN , travels along RN at grazing emergence. If μ is the refractive index and A refracting angle of the prism, show that

$$\mu = \left\{ 1 + \left(\frac{\sin i + \cos A}{\sin A} \right)^2 \right\}^{1/2}$$

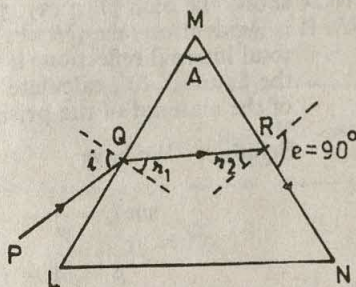


Fig. 16.25

Solution: Referring to Fig. 16.25 we have, for refraction at face ML

$$\mu = \frac{\sin i}{\sin r_1}$$

$$\text{Also } r_1 + r_2 = A \quad \text{or} \quad r_1 = A - r_2$$

$$\therefore \sin r_1 = \sin (A - r_2)$$

$$\therefore \mu = \frac{\sin i}{\sin (A - r_2)} = \frac{\sin i}{(\sin A \cos r_2 - \cos A \sin r_2)} \quad (i)$$

For refraction at face MN , we have

$$\mu = \frac{\sin e}{\sin r_2} = \frac{\sin 90^\circ}{\sin r_2} = \frac{1}{\sin r_2}$$

$$\text{or } \sin r_2 = \frac{1}{\mu} \quad (ii)$$

$$\therefore \cos r_2 = \frac{1}{\mu} (\mu^2 - 1)^{1/2} \quad (iii)$$

Example 16: A mark is made on the surface of a glass sphere. It is viewed through the glass from a portion directly opposite. If the diameter of the sphere is 10 cm and the refractive index of glass is 1.5, find the position of the image.

Solution: Let P be the position of the mark. Q is the position of its image. Since the incident ray PA lies in a medium of refractive index μ_2 , and is refracted into a medium of refractive index μ_1 (see Fig. 16.27), our formula becomes

$$\frac{\mu_1}{v} - \frac{\mu_2}{u} = \frac{\mu_1 - \mu_2}{R}$$

where $u = -2R = -10$ cm, $R = -5$ cm, $\mu_2 = 1.5$ and $\mu_1 = 1$

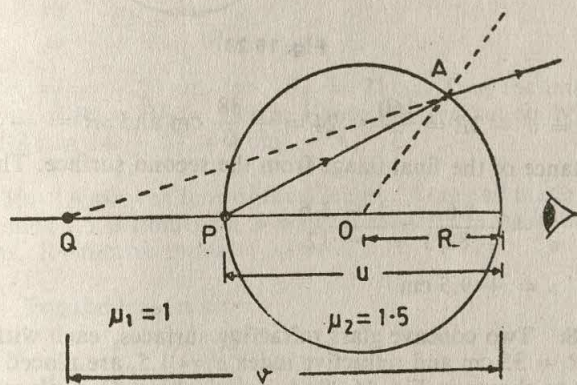


Fig. 16.27

Putting these values in the above formula, we have

$$\frac{1}{v} - \frac{1.5}{-10} = \frac{1 - 1.5}{-5}$$

or $v = -20$ cm

i.e. the image is formed at a distance of 10 cm from point P .

Example 17: An object is placed 50 cm from the surface of a glass sphere of radius 12 cm along a diameter. Where will the final image be formed by refraction at both surfaces? Refractive index of glass = 1.6.

Solution: Let v' be the distance of the image from the first surface, due to refraction at this surface (see Fig. 16.28). Then

$$\frac{\mu_2}{v'} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}$$

where $\mu_1 = 1$, $\mu_2 = 1.6$, $u = -50$ cm and $R = +12$ cm.

Substituting these numerical values in the above formula we get

$$v' = +\frac{160}{3} \text{ cm}$$

This image serves as the virtual object for the second surface so that $(v' - 2R)$ is the object distance for this surface. Since the incident rays travel from glass into air, our formula for this refraction becomes

$$\frac{\mu_1}{v} - \frac{\mu_2}{u} = \frac{\mu_1 - \mu_2}{R}$$

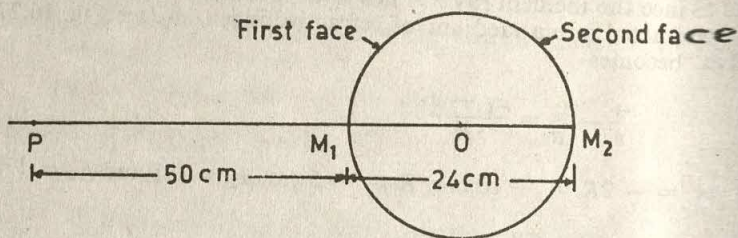


Fig. 16.28

where $u = v' - 2R = \frac{160}{3} - 24 = +\frac{88}{3}$ cm and $R = -12$ cm; v being the distance of the final image from the second surface. Thus we have

$$\frac{1}{v} - \frac{1.6}{+88/3} = \frac{1 - 1.6}{-12}$$

or $v = +9.5$ cm

Example 18: Two concave glass refracting surfaces, each with radius of curvature $R = 35$ cm and refractive index $\mu = 1.5$, are placed facing each other in air as shown in Fig. 16.29. A point object O is placed at a distance of $R/2$ from one of the surfaces as shown. Find the separation between the images of O formed by each refracting surface.

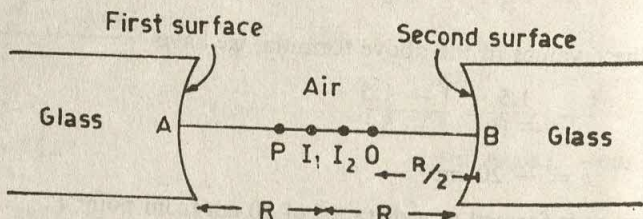


Fig. 16.29

Solution: For refraction at the first surface, we have

$$u = AO = -\frac{3R}{2} = -\frac{105}{2} \text{ cm}$$

$$\mu_2 = 1.5$$

$$\mu_1 = 1$$

$$R = -35 \text{ cm}$$

The distance v_1 of the image I_1 from A is given by the relation

$$\frac{\mu_2}{v_1} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}$$

$$\text{or} \quad \frac{1.5}{v_1} - \frac{1}{-105/2} = \frac{1.5 - 1}{-35}$$

$$\text{or} \quad v_1 = -45 \text{ cm} \quad \text{or} \quad AI_1 = 45 \text{ cm.}$$

Since the radius of curvature is 35 cm, this image is 10 cm to the right of P , i.e. $PI_1 = 10$ cm.

For refraction at the second surface we have

$$u = BO = -\frac{R}{2} = -\frac{35}{2} \text{ cm}$$

The distance v_2 of the image I_2 from the second surface is given by

$$\frac{\mu_2}{v_2} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}$$

which gives $v_2 = -21$ cm or $BI_2 = 21$ cm. Thus the image is 14 cm to the right of P i.e. $PI_2 = 14$ cm. Hence the separation between the two images $(I_1I_2) = 14 - 10 = 4$ cm

Example 19: A convex lens of focal length 20 cm and made of glass of refractive index 1.5 is immersed in water. Find the change in the focal length of the lens. Refractive index of water is 1.33.

Solution: For the lens in air

$$\frac{1}{f_a} = (\mu - 1) \left(\frac{1}{R_1} - \frac{1}{R_2} \right)$$

where $f_a = +20$ cm and $\mu = 1.5$. Thus we have

$$\frac{1}{20} = (1.5 - 1) \left(\frac{1}{R_1} - \frac{1}{R_2} \right)$$

$$\text{or} \quad \left(\frac{1}{R_1} - \frac{1}{R_2} \right) = \frac{1}{10}$$

For the lens in water

$$\begin{aligned} \frac{1}{f_w} &= \left(\frac{\mu_g - \mu_w}{\mu_w} \right) \left(\frac{1}{R_1} - \frac{1}{R_2} \right) \\ &= \left(\frac{1.5 - 1.33}{1.33} \right) \times \frac{1}{10} \end{aligned}$$

$$\text{or} \quad f_w = 78.2 \text{ cm}$$

$$\text{Change in focal length} = f_w - f_a = 78.2 - 20 = 58.2 \text{ cm}$$

Note that the focal length of a lens increases when it is immersed in water.

Example 20: A convex lens of crown glass ($\mu_g = 1.5$) has a focal length of 15 cm. The lens is placed in (a) water ($\mu_w = 1.33$) and (b) carbon bisulphide ($\mu_c = 1.65$). Determine in each case, whether the lens behaves as a converging or diverging lens and determine its focal length.

Solution: For lens in air

$$\begin{aligned}\frac{1}{15} &= (\mu - 1) \left(\frac{1}{R_1} - \frac{1}{R_2} \right) = (1.5 - 1) \left(\frac{1}{R_1} - \frac{1}{R_2} \right) \\ &= 0.5 \left(\frac{1}{R_1} - \frac{1}{R_2} \right)\end{aligned}\quad (i)$$

For lens in water

$$\begin{aligned}\frac{1}{f_w} &= \left(\frac{\mu_g - \mu_w}{\mu_w} \right) \left(\frac{1}{R_1} - \frac{1}{R_2} \right) = \left(\frac{1.5 - 1.33}{1.33} \right) \left(\frac{1}{R_1} - \frac{1}{R_2} \right) \\ &= \frac{0.17}{1.33} \left(\frac{1}{R_1} - \frac{1}{R_2} \right)\end{aligned}\quad (ii)$$

For lens in carbon bisulphide

$$\begin{aligned}\frac{1}{f_c} &= \left(\frac{\mu_g - \mu_c}{\mu_c} \right) \left(\frac{1}{R_1} - \frac{1}{R_2} \right) = \left(\frac{1.5 - 1.65}{1.65} \right) \left(\frac{1}{R_1} - \frac{1}{R_2} \right) \\ &= -\frac{0.15}{1.65} \left(\frac{1}{R_1} - \frac{1}{R_2} \right)\end{aligned}\quad (iii)$$

Dividing (i) by (ii) we get

$$f_w = + 58.7 \text{ cm}$$

The positive sign indicates that the lens is converging.

Dividing (i) by (iii) we get

$$f_c = - 82.5 \text{ cm}$$

The negative sign indicates that the lens behaves as a diverging lens when it is immersed in carbon bisulphide.

Example 21: A luminous object and a screen are mounted on an optical bench and a converging lens is placed between them so that a sharp image is received on the screen; the linear magnification of the image is found to be 2.5. The lens is now moved 30 cm nearer the screen and a sharp image is again formed. Find the focal length of the lens.

Solution: If O and I are the object and the screen respectively and L_1 and L_2 are the two conjugate positions of the lens (Fig. 16.30), then

$$OL_1 = IL_2$$

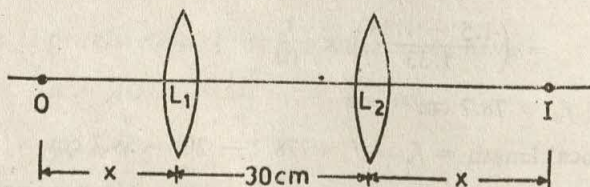


Fig. 16.30

For lens at L_1 we have $u = -x$ and $v = 30 + x$

Linear magnification $m = + \frac{v}{u} = - 2.5$

$$\text{Thus} \quad + \frac{30 + x}{-x} = -2.5$$

$$\text{or} \quad x = 20 \text{ cm}$$

$$\text{Hence} \quad u = -20 \text{ cm} \quad \text{and} \quad v = 30 + 20 = 50 \text{ cm.}$$

$$\text{Substituting in } \frac{1}{v} - \frac{1}{u} = \frac{1}{f} \text{ we have}$$

$$\frac{1}{50} - \frac{1}{-20} = \frac{1}{f}$$

$$\text{which gives } f = 14.3 \text{ cm}$$

Example 22: A lamp and a screen are set up 100 cm apart and a convex lens is placed between them. The two positions of the lens forming real images on the screen are 40 cm apart. Find the focal length of the lens and linear magnifications in the two positions.

Solution: Refer again to Fig. 16.30 where now

$$L_1 L_2 = 40 \text{ cm and } OI = 100 \text{ cm}$$

$$\text{Therefore } x + 40 + x = 100 \quad \text{or} \quad x = 30 \text{ cm}$$

For lens at L_1 , we have

$$u = -30 \text{ cm} \quad \text{and} \quad v = +70 \text{ cm. Thus}$$

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

$$\text{or} \quad \frac{1}{f} = \frac{1}{70} - \frac{1}{-30}$$

$$\text{or} \quad f = +21 \text{ cm}$$

Linear magnification at this position of lens is

$$m_1 = \frac{v}{u} = \frac{70}{-30} = -\frac{7}{3}$$

For lens at L_2 we have $u = -70 \text{ cm}$ and $v = +30 \text{ cm}$. Linear magnification at this position of the lens is

$$m_2 = \frac{v}{u} = \frac{30}{-70} = -\frac{3}{7}$$

The negative sign indicates that the image is inverted.

Example 23: A convex lens forms a real image of an object on a screen; the magnification of the image being $3/2$. The object and the screen are kept fixed and the lens is moved through a distance of 16 cm when a sharp image is again formed on the screen; the magnification now being $2/3$. Find the focal length of the lens.

Solution: Refer again to Fig. 16.30. The separation $L_1 L_2$ now is 16 cm. For lens at L_1 , we have

$$u = -x, \quad v = 16 + x \quad \text{and} \quad m_1 = \frac{3}{2}$$

Now $m_1 = \frac{v}{u} = -\frac{3}{2}$

or $\frac{16+x}{-x} = -\frac{3}{2}$

or $x = 32 \text{ cm}$

Therefore $u = -32 \text{ cm}$ and $v = +48 \text{ cm}$. Substituting these values in the lens formula we get

$$f = 19.2 \text{ cm}$$

Example 24: An object and its real image are located at distances 25 cm and 40 cm respectively from the two principal foci of a lens. Find its focal length and the linear magnification of the image.

Solution: In Fig. 16.31 the two principal foci are F and F' . Let f be the focal length of the lens. Then

$$u = -(f + 25) \text{ cm}$$

$$v = +(f + 40) \text{ cm}$$

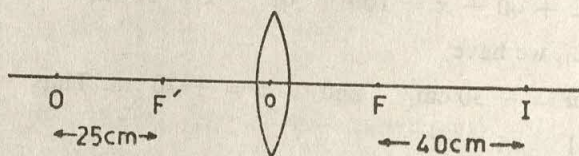


Fig. 16.31

Substituting in the lens formula $\frac{1}{f} = \frac{1}{v} - \frac{1}{u}$ we have

$$\frac{1}{f} = \frac{1}{f+40} + \frac{1}{f+25}$$

or $(f+40)(f+25) = f(f+25) + f(f+40)$

or $f = \sqrt{40 \times 25}$
 $= 31.6 \text{ cm}$

Alternatively, we could use Newton's formula $f^2 = xy$ where $x = 25 \text{ cm}$ and $y = 40 \text{ cm}$.

Linear magnification

$$m = \frac{v}{u} = -\frac{(f+40)}{(f+25)} = -\frac{(31.6+40)}{(31.6+25)} \\ = -1.27$$

The negative sign shows that the image is inverted.

Example 25: Show that the least possible distance between an object and its real image in a convex lens is $4f$ where f is the focal length of the lens.

Solution: Suppose I is the real image of an object O (see Fig. 16.32). Let d be the distance between them. If the image distance is x , the object distance will be $(d - x)$. Thus

$$u = -(d - x) \quad \text{and} \quad v = +x$$

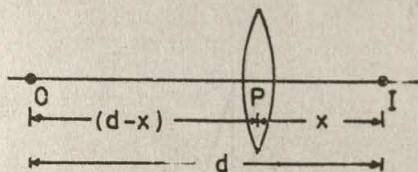


Fig. 16.32

Substituting in the lens formula we have

$$\frac{1}{x} - \frac{1}{-(d-x)} = \frac{1}{f}$$

$$\text{or} \quad \frac{1}{x} + \frac{1}{(d-x)} = \frac{1}{f}$$

$$\text{or} \quad x^2 - xd - fd = 0$$

For a real image, the value of x must be real, i.e. the roots of the above equation must be real. This is possible if

$$d^2 \geq 4fd$$

$$\text{or} \quad d \geq 4f$$

Hence $4f$ is the minimum distance between the object and its real image formed by a convex lens.

Example 26: An equi-convex lens of refractive index $\mu_g = 1.5$ and focal length 10 cm is placed on the surface of water ($\mu_w = 4/3$) such that its lower surface is immersed in water but its upper surface is in contact with air outside. (a) At what distance from the lens will a beam parallel to its principal axis come to focus? (b) How is the position of the focus altered if the lens is wholly immersed in water?

Solution: Let the radius of curvature of each face of the lens be x cm. Then $R_1 = +x$ cm and $R_2 = -x$ cm. Also $\mu_g = 1.5$ and $f = +10$ cm. Substituting in the formula

$$\frac{1}{f} = (\mu_g - 1) \left(\frac{1}{R_1} - \frac{1}{R_2} \right)$$

we have

$$\frac{1}{10} = (1.5 - 1) \left(\frac{1}{x} - \frac{1}{-x} \right)$$

which gives $x = 10$ cm

Thus $R_1 = +10$ cm and $R_2 = -10$ cm

(a) Referring to Fig. 16.33, I is the image of O due to refraction at the upper face. Since the incident ray is in air of refractive index μ_a and the

refracted ray in glass of refractive index μ_g , we have for refraction at this face (whose radius of curvature is R_1)

$$\frac{\mu_g}{v'} - \frac{\mu_a}{u} = \frac{\mu_g - \mu_a}{R_1} \quad (i)$$

where $v' = PI'$

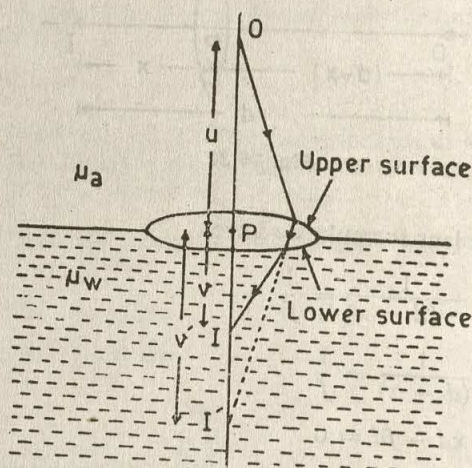


Fig. 16.33

The image I' serves as the virtual object for refraction at the lower surface. For refraction at this surface, the incident ray is in glass and the refracted ray in water. I is the final image. Thus, for refraction at the lower surface (whose radius of curvature is R_2) we have

$$\frac{\mu_w}{v} - \frac{\mu_g}{v'} = \frac{\mu_w - \mu_g}{R_2} \quad (ii)$$

Adding (i) and (ii) we get

$$\frac{\mu_w}{v} - \frac{\mu_a}{u} = \frac{\mu_g - \mu_a}{R_1} + \frac{\mu_w - \mu_g}{R_2}$$

Putting $u = -\infty$; $R_1 = +10$ cm, $R_2 = -10$ cm, $\mu_w = 4/3$, $\mu_g = 1.5$ and $\mu_a = 1$, we get

$$v = 20 \text{ cm}$$

(b) If the lens is wholly immersed in water, the formula is

$$\frac{1}{v} - \frac{1}{u} = \left(\frac{\mu_g - \mu_w}{\mu_w} \right) \left(\frac{1}{R_1} - \frac{1}{R_2} \right)$$

which (for $u = -\infty$) gives $v = 40$ cm

Example 27: An object O is placed 15 cm from a convex lens A of focal length 10 cm and its image I_1 is formed on a screen on the other side of the lens. A concave lens B is now placed midway between A and I_1 and the screen is moved back 10 cm to receive a clear image I_2 . Find (a) the focal length of lens B and (b) the linear magnification of the final image I_2 .

Solution: (a) For lens A , $u = PO = -15$ cm, $f = +10$ cm and $v = PI_1$? (see Fig. 16.34)

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

$$\frac{1}{v} = \frac{1}{f} + \frac{1}{u} = \frac{1}{10} - \frac{1}{15}$$

which gives $v = +30$ cm = PI_1

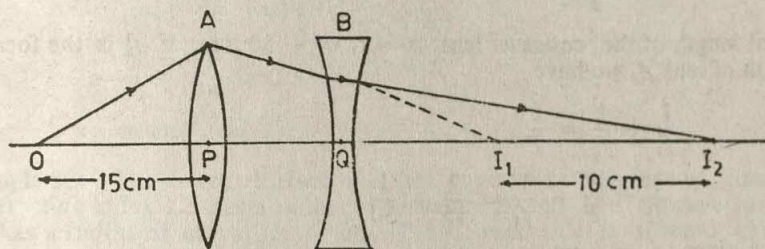


Fig. 16.34

The image I_1 serves as the virtual object for lens B which is placed midway between P and I_1 . Thus for lens B ,

$$u = QI_1 = \frac{1}{2}(PI_1) = +15$$
 cm

$$v = QI_2 = QI_1 + I_1I_2 = 15 + 10 = 25$$
 cm

The focal length f of lens B is obtained from the formula

$$\begin{aligned} \frac{1}{f} &= \frac{1}{v} - \frac{1}{u} \\ &= \frac{1}{25} - \frac{1}{15} \end{aligned}$$

which gives $f = -37.5$ cm

The negative sign is expected since lens B is concave.

(b) Linear magnification of $I_2 = m_A \times m_B$, where m_A and m_B are magnifications due to A and B respectively.

$$\text{For lens } A, \quad m_A = \frac{v}{u} = \frac{30}{-15} = -2$$

$$\text{For lens } B, \quad m_B = \frac{v}{u} = \frac{25}{15} = \frac{5}{3}$$

$$\text{Total magnification} = -2 \times \frac{5}{3} = -10/3$$

The negative sign here indicates that the final image is inverted.

Example 28: An object is placed at a distance of 10 cm from a co-axial combination of two lenses A and B . The combination forms a real image three times the size of the object. If lens B is concave with a focal length of 30 cm, find the nature and focal length of lens A .

Solution: Given $m = \frac{v}{u} = -3$ ($\because$ the image is inverted)

or $v = -3u$

Now $u = -10$ cm, therefore $v = +30$ cm.

If F is the focal length of the combination, we have

$$\frac{1}{F} = \frac{1}{v} - \frac{1}{u} = \frac{1}{30} + \frac{1}{10}$$

or $F = \frac{15}{2}$ cm

Focal length of the concave lens $B = f_1 = -30$ cm. If f_2 is the focal length of lens A , we have

$$\frac{1}{f_1} + \frac{1}{f_2} = \frac{1}{F}$$

or $\frac{1}{f_2} = \frac{1}{F} - \frac{1}{f_1} = \frac{2}{15} + \frac{1}{30}$

which gives $f_2 = 6$ cm

Example 29: A thin equi-convex lens (radius of curvature of either face being 33 cm) is placed on a horizontal plane mirror and a pin held 20 cm vertically above the lens coincides in position with its own image. The space between the lower surface of the lens and the mirror is filled with a liquid and then, to coincide with the image as before, the pin has to be raised to a distance of 25 cm from the lens. Find the refractive index of the liquid.

Solution: In the first case, the image will coincide with the pin if the rays from the pin, after refraction through the lens, fall normally on the mirror and retrace their path, as shown in Fig. 16.35(a). This means that the focal length of the convex lens is 20 cm.

$$f = 20 \text{ cm}$$

In the second case, the focal length F of the combination of the convex lens and the plano-concave liquid lens is 25 cm (see Fig. 16.35 (b)) i.e.

$$F = 25 \text{ cm}$$

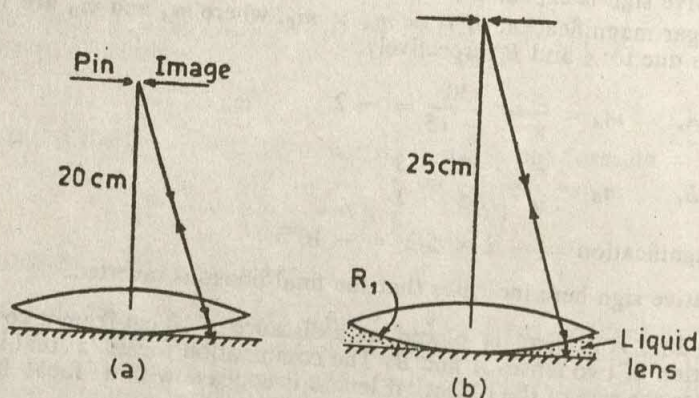


Fig. 16.35 (a) and (b)

Let f_2 be the focal length of the liquid lens, then

$$\frac{1}{f_2} = \frac{1}{F} - \frac{1}{f_1} = \frac{1}{25} - \frac{1}{20}$$

or $f_2 = -100 \text{ cm}$

For the liquid lens, $R_1 = -33 \text{ cm}$, the radius of curvature of the common surface and $R_2 = \infty$. If μ is refractive index of the liquid,

$$\frac{1}{f_2} = (\mu - 1) \left(\frac{1}{R_1} - \frac{1}{R_2} \right)$$

or $-\frac{1}{100} = (\mu - 1) \left(-\frac{1}{33} + \frac{1}{\infty} \right)$

or $\mu - 1 = \frac{33}{100} = 0.33$

or $\mu = 1.33$

Example 30: The convex surface of a thin concavo-convex lens of glass of refractive index 1.5 has a radius of curvature 20 cm. The concave surface has a radius of curvature 60 cm. The convex side is silvered and placed on a horizontal surface (see Fig. 16.36).

(a) Where should a pin be placed on the axis such that its image is formed at the same place?

(b) If the concave part is filled with water ($\mu_w = 4/3$), find the distance through which the pin should be moved so that the image of the pin again coincides with the pin. (IIT, 1981)

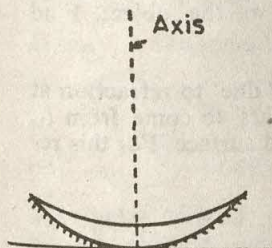


Fig. 16.36

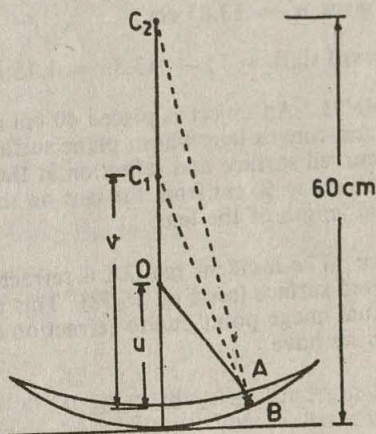


Fig. 16.37

Solution: (a) Let O be the position of the pin. The incident ray OA is refracted along AB in the lens. The image of O will coincide with O if the ray AB falls normally on the lower (silvered) face of the lens (see Fig. 16.37) and retraces its path. Let BA , when produced, meet the axis at C_2 . This is the centre of curvature of the lower face which is the position of the virtual image due to refraction at the upper face of radius of curvature R_1 . Thus

$$v = R_1 = -20 \text{ cm}$$

$$R_2 = -60 \text{ cm}$$

The object distance u is obtained from the formula

$$\frac{\mu_g}{v} - \frac{\mu_a}{u} = \frac{\mu_g - \mu_a}{R_2}$$

or
$$\frac{1.5}{-20} - \frac{1}{u} = \frac{1.5 - 1}{-60}$$

which gives
$$u = -15 \text{ cm}$$

(b) Let the object distance, in this case, be u' . When water is poured over the concave surface, apparent object distance becomes $\mu_w u' = \frac{4}{3} u'$.

Since now the ray enters the lens from water into glass, we use the formula

$$\frac{\mu_g}{v} - \frac{\mu_w}{u} = \frac{\mu_g - \mu_w}{R}$$

where $v = -20 \text{ cm}$, $u = \frac{4}{3} u'$, $R = -60 \text{ cm}$, $\mu_w = \frac{4}{3}$ and

$\mu_g = 1.5$. Substituting these values we have

$$\frac{1.5}{-20} - \frac{4/3}{\frac{4}{3} u'} = \frac{1.5 - 4/3}{-60}$$

which gives $u' = 13.85 \text{ cm}$

Downward shift $= 15 - 13.85 = 1.15 \text{ cm}$

Example 31: An object is placed 40 cm in front of the curved surface of a thin plano-convex lens whose plane surface is silvered. Due to refraction at the curved surface and reflection at the silvered surface, the real image of the object is 60 cm from the lens on the same side as the object. Find the focal length of the lens.

Solution: The incident ray OA is refracted along AB due to refraction at the curved surface (see Fig. 16.38). This ray AB appears to come from I_1 , the virtual image point due to refraction at the curved surface. For this refraction we have

$$\frac{\mu_g}{v'} - \frac{\mu_a}{u} = \frac{\mu_g - \mu_a}{R}$$

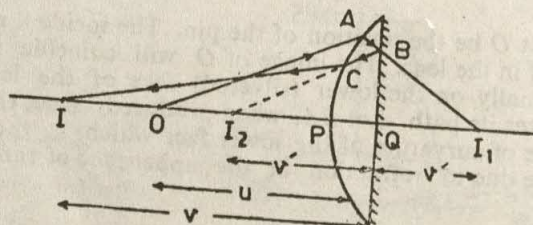


Fig. 16.38

Since the incident rays are from left to right, $v' = +v'$, $u = -u$ and $R = +R$. Therefore, we have

$$\frac{\mu_g}{v'} + \frac{\mu_a}{u} = \frac{\mu_g - \mu_a}{R} \quad (i)$$

where $u = +40$ cm (given).

The ray AB is reflected (by the plane surface) along BC . The reflected ray appears to come from I_2 such that $QI_1 = QI_2 = v'$.

The ray BC suffers refraction at the curved surface and the final image is formed at I . Thus I is the image of the virtual object I_2 due to refraction at the curved surface. For this refraction, since the incident ray travels from glass to air, we have

$$\frac{\mu_a}{v} - \frac{\mu_g}{v'} = \frac{\mu_a - \mu_g}{R}$$

From our sign convention, $v = -v$, $v' = QI = PI_2$ (since lens is thin) $= -v'$ and $R = +R$. Therefore, we have

$$-\frac{\mu_a}{v} + \frac{\mu_g}{v'} = \frac{\mu_a - \mu_g}{R} \quad (ii)$$

where $v = +60$ cm (given).

Subtracting (ii) from (i) we get

$$\frac{\mu_a}{u} + \frac{\mu_a}{v} = \frac{2}{R} (\mu_g - \mu_a)$$

$$\text{or} \quad \frac{1}{u} + \frac{1}{v} = \frac{2}{\mu_a} \frac{1}{R} (\mu_g - \mu_a)$$

$$\text{But} \quad \left(\frac{\mu_g - \mu_a}{\mu_a} \right) \frac{1}{R} = \frac{1}{f_l}$$

where f_l is the focal length of the lens. Hence

$$\frac{2}{f_l} = \frac{1}{u} + \frac{1}{v} = \frac{1}{40} + \frac{1}{60} = \frac{1}{24}$$

$$\text{or} \quad f_l = 48 \text{ cm}$$

Alternative Method Since the object (at a distance of 40 cm from the lens) has its virtual image (at a distance of 60 cm from the lens) on the same side as the object, it follows that a convex lens with the silvered face behaves like a concave mirror. The focal length F of the effective mirror is obtained from the mirror formula

$$\frac{1}{F} = \frac{1}{v} + \frac{1}{u} = \frac{1}{-60} + \frac{1}{-40}$$

which gives $F = -24$ cm

The negative sign shows that the mirror is concave.

Since there are two refractions and one reflection, we have

$$\left| \frac{1}{F} \right| = \frac{1}{f_l} + \frac{1}{f_m} + \frac{1}{f_l}$$

where f_l = focal length of the lens

and f_m = focal length of the plane mirror which is infinity. Thus we

$$\text{have } \frac{1}{24} = \frac{1}{f_l} + \frac{1}{\infty} + \frac{1}{f_l} = \frac{2}{f_l}$$

$$\text{or } f_l = 2 \times 24 = 48 \text{ cm}$$

Example 32: A plano-convex lens acts like a concave mirror of 28 cm focal length when its plane surface is silvered and like a concave mirror of 10 cm focal length when its curved surface is silvered. What is the refractive index of the material of the lens?

Solution: In the first case we have ($\because f_m = \infty$)

$$\frac{1}{28} = \frac{2}{f_l} + \frac{1}{f_m} = \frac{2}{f_l} \quad (i)$$

In the second case we have

$$\frac{1}{10} = \frac{2}{f_l} + \frac{1}{f_m} \quad (ii)$$

where f_m is the focal length of the curved silvered surface. Hence $f_m = R/2$ where R is the radius of curvature of the curved surface.

Subtracting (i) from (ii) we get

$$\frac{1}{f_m} = \frac{1}{10} - \frac{1}{28}$$

$$\text{or } f_m = \frac{140}{9} \text{ cm. Therefore } R = 2f_m = \frac{280}{9} \text{ cm}$$

From (i) we have $f_l = 28 \times 2 = 56 \text{ cm}$. Now

$$\frac{1}{f_l} = (\mu - 1) \frac{1}{R}$$

$$\text{or } \mu - 1 = \frac{R}{f_l} = \frac{280}{9 \times 56} = 0.55$$

$$\text{or } \mu = 1.55$$

Example 33: A pin is placed 10 cm in front of a convex lens of focal length 20 cm, and made of a material of refractive index 1.5. The surface of the lens farther away from the pin is silvered and has a radius of curvature of 22 cm. Determine the position of the final image. Is the image real or virtual? (IIT 1978)

Solution: The value of the effective focal length F is given by

$$\left| \frac{1}{F} \right| = \frac{1}{f_l} + \frac{1}{f_m} + \frac{1}{f_l}$$

$$= \frac{2}{f_l} + \frac{1}{f_m}$$

$$= \frac{2}{20} + \frac{2}{22}$$

or $|F| = \frac{110}{21} \text{ cm}$

Since the convex lens with a silvered surface behaves as a concave mirror of effective focal length F , we have

$$F = -\frac{110}{21} \text{ cm} \quad \text{and} \quad u = -10 \text{ cm}$$

Substituting these values in the mirror formula

$$\frac{1}{v} + \frac{1}{u} = \frac{1}{F}$$

we have $\frac{1}{v} = -\frac{21}{110} + \frac{1}{10} \quad \text{or} \quad v = -11 \text{ cm}$

The negative sign shows that the image is in front of the effective mirror and hence is real.

Example 34: The radius of curvature of the convex face of a plano-convex lens is 12 cm and its $\mu = 1.5$.

(a) Find the focal length of the lens.

The plane face of the lens is now silvered.

(b) At what distance from the lens will parallel rays incident on the convex surface converge?

(c) Sketch the ray diagram to locate the image, when a point object is placed on the axis 20 cm from the lens.

(d) Calculate the image distance when the object is placed as in (c). (IIT 1979)

Solution:

(a) $\frac{1}{f_l} = (\mu - 1) \frac{1}{R} = (1.5 - 1) \frac{1}{12} = \frac{1}{24}$

or $f_l = 24 \text{ cm}$

(b) When a face of the lens is silvered, it behaves as a concave mirror of effective focal length F given by

$$\left| \frac{1}{F} \right| = \frac{2}{f_l} + \frac{1}{f_m} = \frac{2}{f_l} + \frac{1}{\infty} = \frac{2}{f_l}$$

or $F = \frac{f_l}{2} = \frac{24}{2} = 12 \text{ cm}$

This means that the parallel rays will converge at a distance of 12 cm from the silvered surface.

(c) Figure 16.39 illustrates the formation of the image.

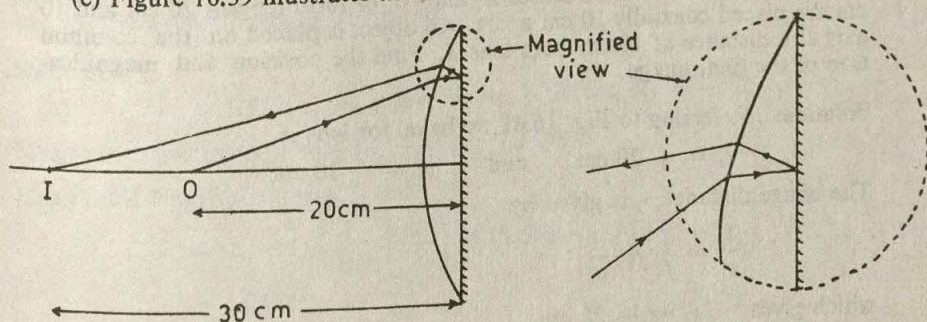


Fig. 16.39

$$(d) \quad F = -12 \text{ cm}, \quad u = -20 \text{ cm}$$

$$\frac{1}{v} = \frac{1}{F} - \frac{1}{u} = \frac{1}{-12} + \frac{1}{20}$$

$$\text{or} \quad v = -30 \text{ cm}$$

The negative sign shows that the image is real.

Example 35: A point object is placed at a distance of 12 cm on the axis of a convex lens of focal length 10 cm. On the other side of the lens, a convex mirror is placed at a distance of 10 cm from the lens such that the image formed by the combination coincides with the object itself. What is the focal length of the convex mirror?

Solution: The incident ray OA is refracted along BD by the lens. Since the image coincides with the object, the refracted ray BD must retrace its path after reflection from the mirror, i.e. it falls normally on the mirror. Let C be the point on the axis where BD produced meets the axis. It is clear that C is the centre of curvature of the mirror (see Fig. 16.40). Also C is the image of the object O formed by the lens alone. Thus

$$f = +10 \text{ cm} \quad \text{and} \quad u = -12 \text{ cm}$$

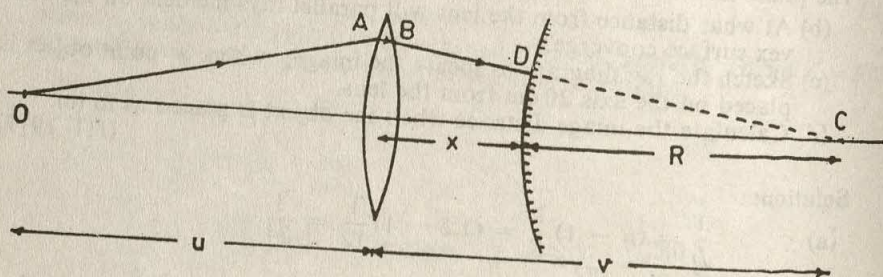


Fig. 16.40

The distance v of the image is obtained from the lens formula

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

$$\text{which gives} \quad v = -60 \text{ cm}$$

Since $x = 10 \text{ cm}$, $R = v - x = 60 - 10 = 50 \text{ cm}$. Hence the focal length of the mirror $= R/2 = 25 \text{ cm}$.

Example 36: Two convex lenses A and B of focal lengths 20 cm and 10 cm are placed coaxially 10 cm apart. An object is placed on the common axis at a distance of 10 cm from lens A . Find the position and magnification of the final image.

Solution: Referring to Fig. 16.41 we have, for lens A ,

$$f_1 = +20 \text{ cm} \quad \text{and} \quad u_1 = -10 \text{ cm}$$

The image distance v_1 is given by

$$\frac{1}{v_1} = \frac{1}{f_1} + \frac{1}{u_1}$$

$$\text{which gives} \quad v_1 = -20 \text{ cm}$$

Thus a virtual image is formed at I_1 at a distance of 20 cm from lens A , if the lens B were absent. This image acts as a virtual object for lens B which forms the final image at I_2 at a distance v_2 from lens B .

For lens B we have, since $x = 10$ cm,

$$u_2 = -(20 + 10) = -30 \text{ cm}, \quad f_2 = +10 \text{ cm}$$

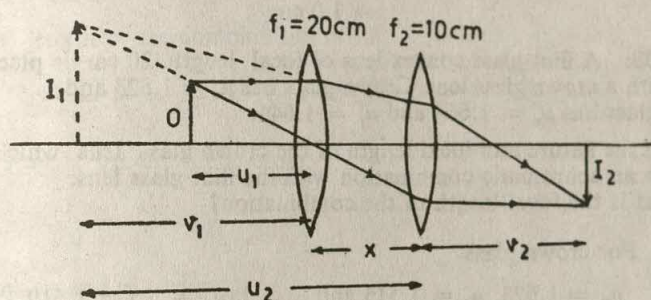


Fig. 16.41

The image distance v_2 is given by

$$\frac{1}{v_2} = \frac{1}{f_2} + \frac{1}{u_2} = \frac{1}{10} - \frac{1}{30} = \frac{1}{15}$$

which gives $v_2 = +15$ cm

Thus a real image I_2 is formed at a distance of 15 cm from lens B .

$$\text{Magnification due to } A (m_1) = \frac{v_1}{u_1} = \frac{-20}{-10} = +2$$

$$\text{Magnification due to } B (m_2) = \frac{v_2}{u_2} = \frac{15}{-30} = -\frac{1}{2}$$

Magnification of the final image is

$$m = m_1 \times m_2 = +2 \times \left(-\frac{1}{2}\right) = -1$$

This shows that the final image is inverted and is of the same size as the object.

Example 37: A convex lens of radii of curvature 20 cm and 25 cm is made of crown glass whose refractive indices for violet and red lights are 1.528 and 1.504 respectively. Determine (a) the mean focal length of the lens and (b) the axial chromatic aberration.

Solution:

$$(a) \quad \mu_v = 1.528, \quad \mu_r = 1.504, \quad R_1 = +20 \text{ cm}, \quad R_2 = -25 \text{ cm}$$

$$\text{Mean refractive index } \mu = \frac{1}{2} (\mu_v + \mu_r) = 1.516$$

Mean focal length f is given by

$$\begin{aligned} \frac{1}{f} &= (\mu - 1) \left(\frac{1}{R_1} - \frac{1}{R_2} \right) = (1.516 - 1) \left(\frac{1}{20} + \frac{1}{25} \right) \\ &= 0.0464 \end{aligned}$$

$$\therefore f = 21.5 \text{ cm}$$

(b) Axial chromatic aberration $= \omega f$

where ω is the mean dispersive power which is given by

$$\omega = \frac{\mu_v - \mu_r}{\mu - 1} = \frac{1.528 - 1.504}{1.516 - 1} = 0.0465$$

Hence, axial chromatic aberration $= 0.0465 \times 21.5$
 $= 1.0 \text{ cm}$

Example 38: A flint glass convex lens of focal length 20 cm is placed in contact with a crown glass lens. Crown glass has $\mu_v = 1.523$ and $\mu_r = 1.515$ and flint glass has $\mu'_v = 1.664$ and $\mu'_r = 1.644$.

- (a) Find the nature and focal length of the crown glass lens which will form an achromatic combination with the flint glass lens.
 (b) What is the focal length of the combination?

Solution: For crown glass

$$\mu_v = 1.523, \mu_r = 1.515 \text{ and } \mu = \frac{1}{2}(\mu_v + \mu_r) = 1.519$$

$$\therefore \omega = \frac{\mu_v - \mu_r}{\mu - 1} = 0.0154$$

For flint glass

$$\mu'_v = 1.664, \mu'_r = 1.644 \text{ and } \mu' = \frac{1}{2}(\mu'_v + \mu'_r) = 1.654$$

$$\therefore \omega' = \frac{\mu'_v - \mu'_r}{\mu' - 1} = 0.0306$$

(a) Let f' be the focal length of the flint glass convex lens and f be the focal length of the crown glass lens. For achromatism

$$\frac{f}{f'} = -\frac{\omega}{\omega'}$$

where $f = +20 \text{ cm}$ (convex lens)

$$\therefore f' = -f \frac{\omega}{\omega'} = -20 \times \frac{0.0306}{0.0154} = -39.7 \text{ cm}$$

The negative sign indicates that the crown glass lens must be concave.

(b) The focal length of the combination is given by

$$\frac{1}{F} = \frac{1}{f} + \frac{1}{f'} = \frac{1}{20} - \frac{1}{39.7}$$

$$\therefore F = 40.3 \text{ cm}$$

Example 39: Two glasses have dispersive powers in the ratio of 2 : 3. These glasses are used in the manufacture of an achromatic objective of focal length 20 cm. What are the focal lengths of the two lenses of the objective?

Solution: For an achromatic combination

$$\frac{f}{f'} = -\frac{\omega}{\omega'}$$

$$\therefore \frac{1}{f'} = -\frac{\omega}{\omega'} \frac{1}{f}$$

where $\frac{\omega}{\omega'} = \frac{2}{3}$. Therefore,

$$\frac{1}{f'} = -\frac{2}{3f} \quad (i)$$

The focal length of the combination is

$$\frac{1}{F} = \frac{1}{f} + \frac{1}{f'}$$

where $F = +20$ cm. Therefore

$$\frac{1}{20} = \frac{1}{f} + \frac{1}{f'} \quad (ii)$$

From Eqs (i) and (ii) we get

$$f = 6.67 \text{ cm} \quad \text{and} \quad f' = -\frac{3f}{2} = -10 \text{ cm}$$

Example 40: A compound microscope consists of objective and eyepiece of focal lengths 5 cm and 10 cm respectively. A small object is placed at a distance of 6 cm from the objective. Find the magnification of the final image formed at a distance of 25 cm from the eyepiece.

Solution: Refer to Fig. 16.12. We have

$$u_1 = -6 \text{ cm}$$

$$v_2 = -25 \text{ cm}$$

$$f_o = +5 \text{ cm (convex lens)}$$

$$f_e = +10 \text{ cm (convex lens)}$$

$$\text{Magnifying power (m p)} = \frac{v_1}{u_1} \times \left(1 - \frac{v_2}{f_e}\right)$$

To find v_1 we use the formula

$$\frac{1}{v_1} - \frac{1}{u_1} = \frac{1}{f_o}$$

which gives $v_1 = 30$ cm

$$\begin{aligned} \text{Hence } \text{m p} &= \frac{30}{-6} \times \left(1 + \frac{25}{10}\right) \\ &= -17.5 \end{aligned}$$

The negative sign indicates that the final image is inverted.

Example 41: A telescope has an objective of focal length 50 cm and an eyepiece of focal length 5 cm. The least distance of distinct vision is 25 cm. The telescope is focussed for distinct vision on a scale 200 cm away from the objective. Calculate (a) the separation between the objective and the eyepiece and (b) the magnification produced. (IIT 1980)

Solution: Refer to Fig. 16.13. We have

$$u_1 = -200 \text{ cm}, f_0 = +50 \text{ cm}, f_e = +5 \text{ cm and}$$

$$v_2 = -25 \text{ cm}$$

(a) Let the separation between the objective and the eyepiece be x . Then

$$x = |v_1| + |u_2|$$

To evaluate v_1 , we use the formula

$$\frac{1}{v_1} = \frac{1}{f_0} + \frac{1}{u_1} = \frac{1}{50} - \frac{1}{200} = \frac{3}{200}$$

which gives $v_1 = \frac{200}{3} \text{ cm}$

To evaluate u_2 , we use the formula

$$\frac{1}{u_2} = \frac{1}{v_2} - \frac{1}{f_e} = -\frac{1}{25} - \frac{1}{5} = -\frac{6}{25}$$

which gives $u_2 = -\frac{25}{6} \text{ cm}$

or $|u_2| = \frac{25}{6} \text{ cm}$

(b) Total magnification $= m_o \times m_e$

$$= \frac{v_1}{u_1} \times \frac{v_2}{u_2}$$

$$= \frac{200/3}{-200} \times \frac{-25}{25/6} = -2$$

The negative sign shows that the final image is inverted.

16.4 TRUE-FALSE STATEMENTS WITH REASONS

A few statements are given below. We are to decide whether they are true or false, giving, in brief, reasons in support of our answers.

1. A ray of light incident normally on a refracting surface does not suffer any refraction.
2. The refractive index of a prism depends upon the refracting angle of the prism and the angle of minimum deviation.
3. The value of refractive index of a given glass slab depends only on the kind of glass of which the slab is made.
4. The critical angle for total internal reflection is greater when a ray of light travels from glass to water than when it travels from glass to air.
5. A hollow glass prism can produce a spectrum of white light just as a solid glass prism does.
6. In glass, violet light travels faster than red light.
7. A convex lens becomes less convergent when it is immersed in water.
8. The focal length of a lens is the same for lights of all colours.
9. Two convex lenses, each of focal length f , are separated by a distance equal to $2f$. A parallel beam of light, after refraction through the lenses, remains parallel.

10. A convex lens of focal length 1.0 m and a concave lens of focal length 0.25 m are kept 0.75 m apart. A parallel beam of light first passes through the convex lens, then through the concave lens and comes to focus 0.5 m away from the convex lens. (IIT 1983)

ANSWERS

1. *True* Angle of refraction r is given by $\sin r = \frac{\sin i}{\mu}$. Since angle $i = 0$, $\sin r = 0$ or $r = 0$ implying that the ray incident normally on a surface goes through undeviated.
2. *False* The refractive index of a prism depends only on the kind of glass of which it is made and the colour of light.
3. *False* The value of refractive index depends also on the colour of light undergoing refraction.
4. *True* The critical angle i_c is given by

$$\sin i_c = \frac{\mu_1}{\mu_2}$$

When the ray travels from glass to water, $\sin i_c = \frac{\mu_g}{\mu_w}$. When it travels

from glass to air, $\sin i'_c = \frac{\mu_g}{\mu_a}$. Thus $\frac{\sin i'_c}{\sin i_c} = \frac{\mu_w}{\mu_a} = \frac{3}{4}$ showing that $i'_c < i_c$ or $i_c > i'_c$.

5. *False* A hollow prism does not produce a spectrum of white light because air does not disperse white light into its constituent colours.
6. *False* The refractive index of glass for red light is less than that for violet light. Thus red light travels faster in glass than violet light.
7. *True* When a convex lens is immersed in water, its focal length increases. Hence the lens becomes less convergent.
8. *False* The focal length of a lens depends upon its refractive index. Since the refractive index is different for lights of different colours, the focal length of a lens will be different for lights of different colours.
9. *True* The path of rays is shown in Fig. 16.42.

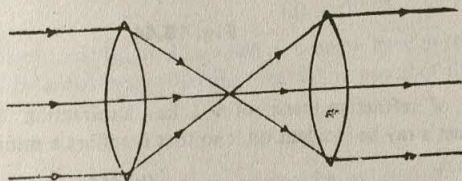


Fig. 16.42

10. *False* The path of rays is shown in Fig. 16.43. The image formed by the convex lens serves as the virtual object for the concave lens. It is clear that the virtual object is at the focus of the concave lens, hence it produces the image

at infinity. Thus a parallel beam, after refractions through the two lenses, remains parallel and will, therefore, not come to focus at any finite distance from the concave lens.

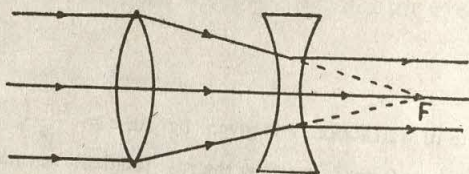


Fig. 16.43

16.5 MULTIPLE-CHOICE QUESTIONS

Choose the correct answer from the given alternatives.

- Which of the following parameters does not change when light travels from air into a glass slab?
 - Velocity
 - Wavelength
 - Frequency
 - Amplitude
- The refractive index of a glass prism depends upon
 - the angle of the prism
 - the angle through which it deviates an incident beam of light
 - the colour of the incident light
 - the intensity of the incident light
- The angle of deviation (δ) by a prism is measured for various angles of incidence (i) of monochromatic light. Which one of the graphs shown in Fig. 16.44 correctly represents the variation of δ with i ?

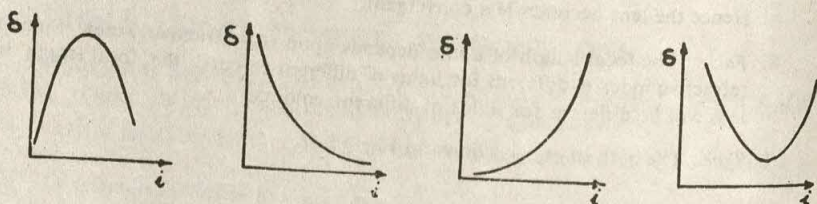


Fig. 16.44

- A prism of refractive index of $\sqrt{2}$ has a refracting angle of 60° . At what angle must a ray be incident on it so that it suffers a minimum deviation?
 - 30°
 - 45°
 - 60°
 - 75°
- A ray of light in a medium of refractive index μ_1 is partly reflected and refracted at the boundary of a medium of refractive index μ_2 , as shown in Fig. 16.45. If $\angle BOC = 90^\circ$. The value of angle i is given by

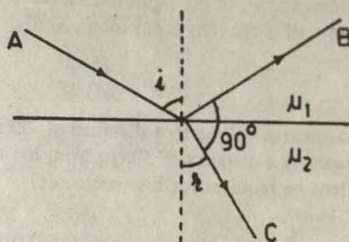


Fig. 16.45

- (a) $\tan^{-1} (\mu_1/\mu_2)$ (b) $\tan^{-1} (\mu_2/\mu_1)$
 (c) $\sin^{-1} (\mu_2/\mu_1)$ (d) $\cos^{-1} (\mu_1/\mu_2)$
6. A glass slab of thickness 8 cm contains the same number of waves as 10 cm of water when both are traversed by the same monochromatic light. If the refractive index of water is $4/3$, the refractive index of glass is
 (a) $5/3$ (b) $5/4$
 (c) $16/15$ (d) $3/2$
7. A rectangular block of glass of refractive index μ and thickness d is placed over an ink dot made on a piece of paper lying on a table. When viewed from above, the dot will appear to be raised by an amount given by
 (a) $\left(1 + \frac{1}{\mu}\right) d$ (b) $\left(1 - \frac{1}{\mu}\right) d$
 (c) $\frac{d}{\left(1 + \frac{1}{\mu}\right)}$ (d) $\frac{d}{\left(1 - \frac{1}{\mu}\right)}$
8. Two transparent media A and B are separated by a plane boundary. The speed of light in medium A is $2.0 \times 10^8 \text{ m s}^{-1}$ and in medium B $2.5 \times 10^8 \text{ m s}^{-1}$. The critical angle for which a ray of light going from A to B is totally internally reflected is
 (a) $\sin^{-1} (1/2)$ (b) $\sin^{-1} (2/5)$
 (c) $\sin^{-1} (4/5)$ (d) $\sin^{-1} (1/2)$
9. A convex glass lens ($\mu_g = 1.5$) has a focal length of 8 cm when placed in air. What is the focal length of the lens when it is immersed in water ($\mu_w = 4/3$)?
 (a) 4 cm (b) 8 cm
 (c) 16 cm (d) 32 cm
10. A plano-concave lens is made of glass of refractive index 1.5 and the radius of curvature of the curved face is 50 cm. The power of the lens is
 (a) -1.0 D (b) -0.5 D
 (c) $+1.0 \text{ D}$ (d) $+0.5 \text{ D}$
11. A convex lens of focal length 50 cm and a concave lens of focal length 100 cm are separated by a distance of 10 cm. What is the power of the combination?
 (a) $+1.5 \text{ D}$ (b) $+1.2 \text{ D}$
 (c) -0.6 D (d) zero
12. How will the image formed by a convex lens be affected if the central portion of the lens is wrapped in black paper, as shown in Fig. 16.46?
 (a) No image is formed by the remaining portion of the lens
 (b) The central portion of the image will be absent
 (c) The full image is formed but it will be less bright
 (d) There will be two images, one due to each exposed half

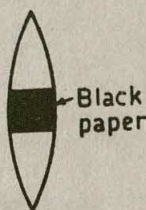


Fig. 16.46

13. The equi-convex lens, shown in the figure, has a focal length f . What will be the focal length of each half if the lens is cut along A/B ? [see Fig. 16.47]

(a) $f/2$ (b) f
(c) $3f/2$ (d) $2f$

14. A person can see clearly only upto a distance of 30 cm. He wants to read a book placed at a distance of 50 cm from his eyes. What is the power of the lens he requires for his spectacles?

(a) -1.0 D (b) -1.33 D
(c) -1.67 D (d) -2.0 D

15. A person with defective eyesight is unable to see objects clearly nearer than 1.5 m. He wants to read a book placed at a distance of 30 cm from his eyes. Find the power of the lens he requires for his spectacles.

(a) $+2.67$ D (b) -2.67 D
(c) $+2.5$ D (d) -2.5 D

16. An object, a convex lens of focal length 20 cm and a plane mirror are arranged as shown in Fig. 16.48. How far behind the mirror is the final image formed?

(a) 20 cm (b) 30 cm
(c) 40 cm (d) 50 cm

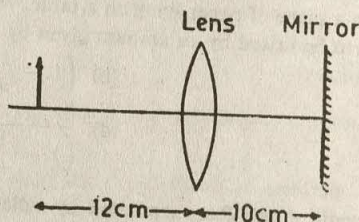


Fig. 16.48

17. A small plane mirror is rotating at a constant frequency of n rotations per second. With what linear velocity (in m s^{-1}) will a light spot move along a spherical screen of radius of curvature of R metres if the mirror is at the centre of curvature of the screen?

(a) $\pi n R$ (b) $2\pi n R$
(c) $4\pi n R$ (d) $2 n R$

18. A ray of light incident normally on face AB of an isosceles prism travels as shown in Fig. 16.49. The least value of the refractive index the prism must have should be

(a) $\sqrt{2}$ (b) 1.5
(c) $\sqrt{3}$ (d) 2.0

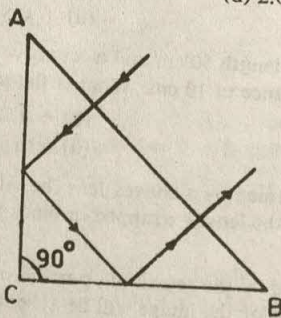


Fig. 16.49

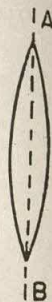


Fig. 16.47

19. The principal section of a glass prism ($\mu = 1.5$) is an equilateral triangle. A ray is incident normally on one of the faces. The angle between this ray and the ray that leaves the prism is
- (a) 60° (b) 90°
(c) 120° (d) 180°
20. A plano-convex lens is made of glass of refractive index 1.5. The focal length f of the lens and radius of curvature R of its curved face are related as
- (a) $f = R/2$ (b) $f = R$
(c) $f = 2R$ (d) $f = 3/2 R$
21. A thin convergent glass lens ($\mu_g = 1.5$) has a power of +5.0 D. When this lens is immersed in a liquid of refractive index μ_l it acts as a divergent lens of focal length 100 cm. The value of μ_l must be
- (a) $4/3$ (b) $5/3$
(c) $5/4$ (d) $6/5$
22. The distance between an object and a divergent lens is m times the focal length of the lens. The linear magnification produced by the lens will be equal to
- (a) m (b) $1/m$
(c) $(m + 1)$ (d) $\frac{1}{(m + 1)}$
23. An object is placed at a distance of 20 cm from the curved face of a plano-convex lens; the plane surface of the lens is silvered. It is found that the real image of the object coincides with the object itself. If the refractive index of the material of the lens is 1.5, the radius of curvature of the curved surface of the lens is
- (a) 10 cm (b) 20 cm
(c) 30 cm (d) 40 cm
24. The dispersion of light in a medium implies that
- (a) lights of different wavelengths travel with different speeds in the medium.
(b) lights of different frequencies travel with different speeds in the medium
(c) the refractive index of the medium is different for different wavelengths of light
(d) all of the above
25. When a telescope is in normal adjustment, the distance of the objective from the eye-piece is found to be 100 cm. If the magnifying power of the telescope, at normal adjustment, is 24, the focal lengths of the lenses are
- (a) 96 cm, 4 cm (b) 90 cm, 10 cm
(c) 80 cm, 20 cm (d) 50 cm, 50 cm
26. Monochromatic light is refracted from air into glass of refractive index μ . The ratio of the wavelengths of the incident and refracted waves is
- (a) $1 : 1$ (b) $1 : \mu$
(c) $\mu : 1$ (d) $\mu^2 : 1$
27. The layered lens shown in Fig. 16.50 is made of two kinds of glass. A point source of light is placed on its principal axis. If reflections from the boundaries between layers are ignored, the lens will form
- (a) only one image (b) two images
(c) no image at all (d) infinite images
28. How much time will light take to traverse a glass slab of thickness 10 cm and refractive index 1.5?
- (a) 0.5×10^{-9} s (b) 1.5×10^{-9} s
(c) 2×10^{-8} s (d) 3×10^{-8} s



Fig. 16.50

29. A glass prism ABC of refractive index 1.5 is immersed in water of refractive index $4/3$ as shown in Fig. 16.51. A ray of light incident normally on face AB is totally reflected at face AC if

(a) $\sin \theta \geq 8/9$

(c) $\sin \theta = \sqrt{3}/2$

(b) $\sin \theta \leq 2/3$

(d) $\frac{2}{3} < \sin \theta < 8/9$

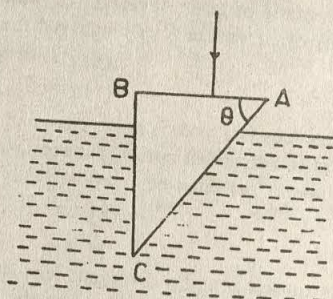


Fig. 16.51

30. What is the relation between refractive indices μ , μ_1 and μ_2 if the behaviour of light rays is as shown in Fig. 16.52?

(a) $\mu > \mu_2 > \mu_1$

(c) $\mu < \mu_2$; $\mu = \mu_1$

(b) $\mu < \mu_2 < \mu_1$

(d) $\mu_2 < \mu_1$; $\mu = \mu_2$

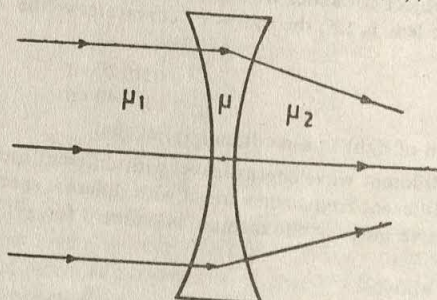


Fig. 16.52

31. A lens forms a sharp image on a screen. On inserting a parallel sided glass slab between the lens and the screen, it is found necessary to move the screen a distance d away from the lens in order to focus the image sharply. If the refractive index of glass relative to air is μ , the thickness of the glass slab is given by

(a) d/μ

(c) $\frac{d}{\left(1 - \frac{1}{\mu}\right)}$

(b) μd

(d) $\left(1 - \frac{1}{\mu}\right) d$

32. A convex lens is placed between an object and a screen which are a fixed distance apart. For one position of the lens the magnification of the image obtained on the screen is m_1 . When the lens is moved by a distance d , the magnification of the image obtained on the same screen is m_2 . The focal length of the lens is ($m_1 > m_2$)

(a) $\frac{d}{(m_1 - m_2)}$

(c) $d \frac{m_1}{m_2}$

(b) $\frac{d}{(m_1 + m_2)}$

(d) $d \frac{m_2}{m_1}$

NUMERICAL EXERCISES

1. When a ray of light enters a glass slab, it is partly reflected and partly refracted. If the angle between the incident and refracted rays is 150° and that between the reflected and refracted rays is 80° , find the refractive index of glass.
2. One face of a rectangular glass slab of thickness 6 cm and refractive index 1.5 is silvered. An object is placed at a distance of 8 cm from the unsilvered face. Find the distance of the image from the silvered face.
3. A person looking through a telescope T just sees the point A on the rim at the bottom of a cylindrical vessel when the vessel is empty (see Fig. 16.53).

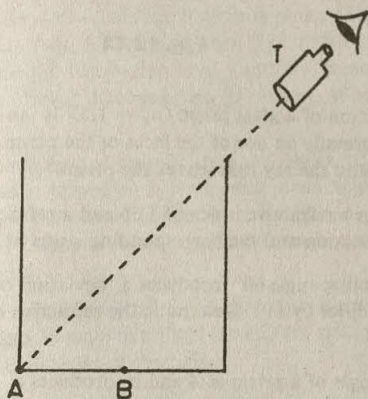


Fig. 16.53

When the vessel is completely filled with a liquid ($\mu = 1.5$), he observes a mark at the centre B , of the bottom of the vessel, without moving the telescope or the vessel. What is the height of the vessel if the diameter of its cross-section is 10 cm? (IIT 1977)

4. A glass plate of thickness 8 cm and refractive index 1.5 is placed in air. At what angle of incidence should a ray strike the plate so that the reflected and refracted rays are perpendicular to each other? For this angle of incidence, what is the displacement of the ray due to refraction through the plate?
5. A rectangular block of glass is placed on a printed page lying on a horizontal surface. Find the minimum value of the refractive index of glass for which the letters on the page are not visible from any of the vertical sides of the block. (IIT 1971)
6. A glass prism ABC of refractive index 1.5 is lowered into water shown in Fig. 16.54. What is the minimum value of $\angle ABC$ so that a ray of light falling nor-

mally on face AB is totally reflected at face BC ?

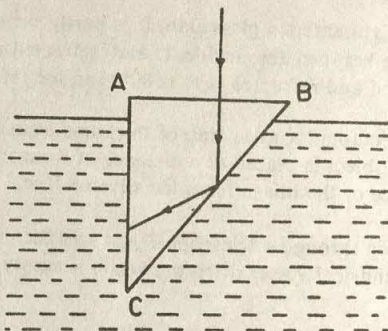


Fig. 16.E4

7. The principal section of a glass prism ($\mu_g = 1.5$) is an equilateral triangle. A ray is incident normally on one of the faces of the prism. Find the angle between the incident ray and the ray that leaves the prism.
8. A glass prism has a refractive index of 1.66 and a refracting angle of 60° . Find the minimum deviation and the corresponding angle of incidence.
9. A prism of refracting angle 60° produces a deviation of 40° for two angles of incidence which differ by 11° . Determine the refractive index of the material of the prism.
10. The refracting angle of a prism is A and it produces a minimum deviation of $(180^\circ - 2A)$. Show that the refractive index of the prism is given by

$$\mu = \frac{1}{\tan \left(\frac{A}{2} \right)}.$$
11. A ray of light falls at grazing incidence on one of the faces of a prism of refractive index μ and refracting angle A . Show that the angle of emergence is given by

$$\sin e = (\mu^2 - 1)^{1/2} \sin A - \cos A$$
12. One face of a 60° prism of refractive index 1.5 is covered with a film of a liquid of refractive index 1.3. Find the angle of incidence of a ray on the uncovered face which will just be totally reflected at the liquid-covered face.
13. A mark made on the surface of a spherical glass vessel of radius 20 cm, filled with water ($\mu = 4/3$), is viewed along a diameter through the opposite side of the vessel. Neglecting the effect of glass, find the distance of the image from the mark.
14. A glass hemisphere of radius 5 cm and refractive index 1.5 is placed centrally over a mark (i) with the flat surface and (ii) with the curved surface of the hemisphere in contact with the object. In each case, find the position of the image when viewed from above.
15. The two surfaces of a concave lens are of radii of curvature 10 cm and 30 cm. If the refractive index of the lens is 1.5, find its focal length when it is immersed in water ($\mu = 4/3$).

16. An object and a screen are set up 80 cm apart. There are two positions at which a convex lens forms a real image on the screen, the magnifications in the two cases being $\frac{2}{3}$ and $\frac{3}{2}$. Find the focal length of the lens.
17. A convex lens forms an image of magnification 1.8 on a screen. When the screen is moved through 5 cm, the image is re-focussed, the new magnification being 2.0. Find the focal length of the lens.
18. An object gives a real image magnified four times at 120 cm from the second principal focus of a lens. Find the focal length of the lens.
19. A plano-convex lens of focal length 15 cm, is fixed on the side of a box with the convex surface outside. An object 5 cm-high is placed at 25 cm, from the lens. Find the position and size of the image when the box is (a) empty and (b) filled with water. Refractive index of lens = 1.5 and that of water = $\frac{4}{3}$.
20. Show that the focal length of an equi-convex lens of refractive index 1.5 is equal to the radius of curvature of either face of the lens.
21. A converging lens *A* of focal length 6 cm is mounted vertically at a distance of 10 cm from a screen *S*. A diverging lens *B* of focal length 12 cm, is mounted coaxially between the converging lens *A* and the screen *S* so that an image of an object 24 cm from *A* is focussed on the screen. What is the distance between *A* and *B*.
22. A thin equi-convex lens of glass of refractive index 1.5 and radius of curvature of surfaces 24 cm is placed on a horizontal plane mirror. When the space between the lens and the mirror is filled with a liquid, a pin held 40 cm vertically above the lens is found to coincide with its own image. Calculate the refractive index of the liquid.
23. A thin plano-convex lens of radius of curvature 30 cm acts like a concave mirror of focal length 30 cm when its plane surface is silvered. What is the refractive index of the material of the lens.
24. A convex lens of focal length 15 cm is placed coaxially at a distance of 5 cm from a convex mirror. When an object is placed on the axis at a distance of 20 cm from the lens, its image is found to coincide with the object. Calculate the radius of curvature of the mirror.
25. The dispersive powers of crown and flint glasses are 0.015 and 0.030, respectively. Calculate the focal lengths of the lenses (made of crown and flint glass) which form an achromatic doublet of focal length 60 cm when placed in contact.
26. A compound microscope has an objective of focal length 5 cm and an eyepiece of focal length 10 cm separated by a distance of 16 cm. Find the magnification of the image formed at a distance of 25 cm from the eyepiece.
27. An astronomical telescope has an objective of focal length 100 cm and an eyepiece of focal length 20 cm. The diameter of the objective is 5 cm. Find the magnifying power in normal adjustment, the separation between the lenses and the diameter of the image.
28. The distance between two point sources of light is 2.4 cm. Where should a converging lens of focal length 9 cm be placed between them to obtain the images of both sources at the same point?
29. A lens of focal length 30 cm produces on a screen a sharp image of an object that is at a distance of 40 cm from the lens. A parallel-plane plate of thickness 9 cm is placed between the lens and the object perpendicular to the axis of the

lens. Through what distance should the screen be shifted for the image to remain distinct? The refractive index of the glass of the plate is 1.8.

30. A parallel beam of rays is incident on a converging lens of focal length 40 cm. Where should a diverging lens of focal length 15 cm be placed so that the beam remains parallel after passing through the two lenses? Draw a ray diagram to show how the rays pass through the lenses.

ANSWERS

MULTIPLE CHOICE QUESTIONS

- | | | | |
|---------|---------|---------|---------|
| 1. (c) | 2. (c) | 3. (d) | 4. (b) |
| 5. (b) | 6. (a) | 7. (b) | 8. (c) |
| 9. (d) | 10. (a) | 11. (b) | 12. (c) |
| 13. (d) | 14. (b) | 15. (a) | 16. (c) |
| 17. (c) | 18. (a) | 19. (c) | 20. (c) |
| 21. (b) | 22. (d) | 23. (a) | 24. (d) |
| 25. (a) | 26. (c) | 27. (b) | 28. (a) |
| 29. (a) | 30. (c) | 31. (c) | 32. (a) |

NUMERICAL EXERCISES

- | | |
|-------------------------------------|------------------------------------|
| 1. 1.58 | 16. 19.2 cm |
| 2. 10 cm | 17. 25 cm |
| 3. 8.45 cm | 18. 30 cm |
| 4. 56.3° , 3.69 cm | 19. (a) 37.5 cm, 7.5 cm |
| 5. 1.41 | (b) 50 cm, 10 cm |
| 6. 62.5° | 21. 4 cm |
| 7. 120° | 22. 1.4 |
| 8. 52.2° , 56.1° | 23. 1.5 |
| 9. 1.53 | 24. 55 cm |
| 12. Zero (normal incidence) | 25. 30 cm, -60 cm, |
| 13. 20 cm | 26. 2.7 |
| 14. (i) at the centre of hemisphere | 27. 5, 120 cm, 1 cm |
| (ii) 10/3 cm from top | 28. 6 cm from one source and 18 cm |
| 15. 60 cm | from the other |
| | 29. 60 cm |
| | 30. 25 cm from converging lens |

Wave Nature & Interference of Light

17.1 REVIEW OF BASIC CONCEPTS

17.1.1 Wave Nature of Light Light is an electromagnetic wave which does not require a material medium for propagation. The electric and magnetic fields vary in space and time resulting in the propagation of an electromagnetic wave even in free space. What is propagated is the electromagnetic energy, i.e. the energy associated with the oscillating electric and magnetic fields of the wave. The electric field varies in space and time as

$$E = A \sin \left\{ \frac{2\pi}{\lambda} (ct - x) \right\} = A \sin 2\pi \left(nt - \frac{x}{\lambda} \right)$$

which represents a wave of amplitude A , wavelength λ , wave velocity c , travelling in the $+x$ direction. In free space (or air) the value of

$$c = 3 \times 10^8 \text{ m s}^{-1}.$$

Amplitude Amplitude A is the maximum value of the electric field of the wave.

Wavelength Wavelength λ is the shortest distance between two points on a wave which differ in phase by 2π radians.

Phase Phase ϕ of a wave at a point x and a given time t is given by the argument of the harmonic function (sine or cosine) representing the wave, i.e.

$$\phi = \frac{2\pi}{\lambda} (ct - x) = 2\pi \left(nt - \frac{x}{\lambda} \right)$$

Phase Difference Suppose two waves meeting at a point P are represented by

$$E_1 = A_1 \sin \left\{ 2\pi \left(n_1 t - \frac{x_1}{\lambda_1} \right) \right\}$$

and

$$E_2 = A_2 \sin \left\{ 2\pi \left(n_2 t - \frac{x_2}{\lambda_2} \right) \right\}$$

where x_1 and x_2 are paths of the waves upto point P where they meet. The phase difference $\Delta\phi$ between them at that point P and at time t is given by

$$\Delta\phi = (\phi_1 - \phi_2) = 2\pi (n_1 - n_2)t - 2\pi \left(\frac{x_1}{\lambda_1} - \frac{x_2}{\lambda_2} \right)$$

1. If the two waves have different frequencies, i.e. $n_1 \neq n_2$, then $\lambda_1 \neq \lambda_2$ and $\Delta \phi$ depends on time t .
2. If $n_1 = n_2$, then $\lambda_1 = \lambda_2$. In this case

$$\Delta \phi = \frac{2\pi}{\lambda} (x_2 - x_1)$$

or Phase difference = $\frac{2\pi}{\lambda} \times (\text{path difference})$

i.e. the phase difference is independent of time and depends only on the path difference ($x_2 - x_1$). This holds only if the two sources of waves are 'coherent', i.e. they have a constant fixed phase relationship.

Intensity The intensity of a wave at any point in its path is proportional to the square of its amplitude at that point. If the medium is non-dispersive, the amplitude of the wave is constant throughout its path.

17.1.2 Reflection and Refraction of Light When a light wave falls on a reflecting surface, it is reflected obeying the usual laws of reflection. When a wave travels from one medium into another, its velocity and wavelength undergo a change and the wave is said to suffer refraction. The frequency of the wave does not undergo any change in refraction (and reflection). If v_1 is the velocity of the wave in the medium in which the incident wave propagates and v_2 is the velocity of the wave in the medium in which the refracted wave propagates, then ${}^1\mu_2$, the refractive index of the second medium with respect to the first, is defined as

$${}^1\mu_2 = \frac{v_1}{v_2} = \frac{\lambda_1}{\lambda_2}$$

where λ_1 and λ_2 are the wavelengths of the same wave in the two media. The frequency of the refracted wave remains the same as that of the incident wave.

When a wave, travelling in a rarer medium, is reflected at the boundary of a denser medium, the reflected wave suffers a phase change of 180° (or 2π radians) in relation to that of the incident wave. No phase change occurs if a wave, travelling in a denser medium, is reflected at the boundary of a rarer medium. The refracted wave, in both cases, does not undergo any phase change.

17.1.3 Interference of Light When two or more light waves meet (or superpose) at a point in a medium, the electric field of the resultant wave can be obtained by using the *principle of superposition* which states that *the resultant electric field is given by the algebraic sum of the individual electric fields, at that point, due to the individual waves, i.e.*

$$E = E_1 + E_2 + \dots$$

resulting in a change in amplitude (and hence in intensity) at that point. The phenomenon in which the intensity of light at a point is modified by the superposition of two or more waves is known as *interference*.

17.1.4 Young's Double Slit Interference Experiment Figure 17.1 illustrates the simple experiment devised by Thomas Young to show interference of light waves. Light from a source S illuminates two narrow slits S_1 and S_2 which are very close together and equi-distant from S .

Slits S_1 and S_2 serve as the two virtual light sources. Since these sources are derived from the same primary source S and since they are equidistant from S , the two virtual sources are *coherent*, i.e. waves diverging from them have the same phase and the same wavelength. If λ is the wavelength of the monochromatic light emitted by S , the waves diverging from S_1 and S_2 , on reaching the point P on the screen, can be written as

$$E_1 = A_1 \sin \frac{2\pi}{\lambda} (ct - x_1) = A_1 \sin (\omega t - \phi_1)$$

and

$$E_2 = A_2 \sin \frac{2\pi}{\lambda} (ct - x_2) = A_2 \sin (\omega t - \phi_2)$$

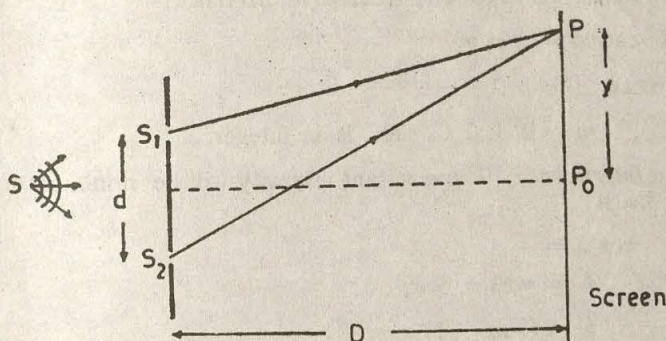


Fig. 17.1

where $x_1 = S_1P$ and $x_2 = S_2P$. $(S_2P - S_1P) = (x_2 - x_1)$ is the path difference between them at point P and $(\phi_2 - \phi_1) = \phi$ is the phase difference between them. From the principle of superposition, the resultant wave at point P is given by

$$\begin{aligned} E &= E_1 + E_2 \\ &= A_1 \sin (\omega t - \phi_1) + A_2 \sin (\omega t - \phi_2) \end{aligned}$$

which may be recast in the form

$$E = A \sin (\omega t - \delta)$$

where the resultant amplitude A is given by

$$A^2 = A_1^2 + A_2^2 + 2 A_1 A_2 \cos \phi \quad (17.1)$$

The resultant intensity $I \propto A^2$

Constructive Interference The resultant intensity I will be maximum if A^2 is maximum. From Eq. (17.1) it is clear that A^2 is maximum if

$$\cos \phi = +1$$

or $\phi = 0, 2\pi, 4\pi, \dots$ etc.

i.e. $\phi = 2m\pi; m = 0, 1, 2, \dots$ etc.

$$\text{or } \frac{2\pi \Delta}{\lambda} = 2m\pi$$

$$\text{or } \Delta = m\lambda$$

where $\phi = \frac{2\pi\Delta}{\lambda}$; $\Delta = x_2 - x_1 = S_2P - S_1P$ being the path difference.

The maximum value of A^2 is

$$A_{\max}^2 = (A_1 + A_2)^2$$

If S_1 and S_2 are equidistant from S , $A_1 = A_2 = A$ (say). Then

$$A_{\max}^2 = 4A^2$$

and $I_{\max} = 4I_0$

where I_0 is the intensity of each interfering beam. The interference is said to be *constructive*. Thus, for constructive interference

Phase difference $\phi = 2m\pi$ (17.2)

Path difference $\Delta = m\lambda$ (17.3)

where $m = 0, 1, 2, \dots$ etc. is an integer.

Destructive Interference The resultant intensity will be minimum if A^2 is minimum, i.e. if

$$\cos \phi = -1$$

or $\phi = \pi, 3\pi, 5\pi, \dots$

or $\phi = (2m + 1)\pi$

The minimum value is given by

$$A_{\min}^2 = (A_1 - A_2)^2$$

If $A_1 = A_2$, then $A_{\min}^2 = 0$. Hence $I_{\min} = 0$. The interference is said to be *destructive*. Hence, for destructive interference,

Phase difference $\phi = (2m + 1)\pi$ (17.4)

or Path difference $\Delta = (m + \frac{1}{2})\lambda$ (17.5)

where $m = 0, 1, 2, \dots$ etc.

At maxima, the waves reinforce each other and at minima they cancel out each other. These maxima and minima constitute the bright and dark fringes.

Fringe Width The distance of the m th bright fringe from the centre of the fringe system is given by (see Fig. 17.1)

$$y_m = m\lambda \frac{D}{d}$$
 (17.6)

The dark fringes satisfy the relation:

$$y_m = (m - \frac{1}{2})\lambda \frac{D}{d}$$
 (17.7)

The separation between two consecutive bright or dark fringes is the same and is called the fringe width (β)

$$\text{Fringe width } \beta = \frac{\lambda D}{d}$$
 (17.8)

The fringe width is directly proportional to wavelength (λ) and the slits-screen distance (D) and inversely proportional to the distance (d) between the slits.

Coherent Sources We do not observe any interference pattern if the two slits S_1 and S_2 in Young's experiment are illuminated by two separate sources of monochromatic light. The reason is that two independent sources are not coherent, i.e. they do not emit light waves having a constant phase. The phase of light waves emitted by an ordinary source remains constant only during a very short time interval $\sim 10^{-8}$ to 10^{-10} second. This is called the *coherence time* of source.

Equation (17.1) tells us that the resultant intensity of light at a point on the screen depends on the phase difference (ϕ) between the two interfering waves. This phase difference depends upon two factors—(1) the initial phase difference between the waves emitted by the two sources and (2) the phase difference resulting from the path difference for that point. The initial phase difference depends upon the time and hence remains constant only for about 10^{-8} to 10^{-10} second. Thus the resultant intensity changes so rapidly with time that, due to persistence of vision, we are unable to see the interference pattern. Thus, non-coherent sources cannot produce sustained interference effects. We conclude that, for a steady interference pattern, the following two conditions must be satisfied.

1. The two sources must be coherent
2. The wavelength of the interfering waves must be the same. Thus, *only monochromatic coherent light sources produce observable interference pattern.*

17.1.6 Displacement of Fringes If a thin transparent plate is introduced in the path of one of the interfering waves, it is observed that the entire fringe system (or pattern) is shifted through a distance given by (see Fig. 17.2)

$$y_0 = \frac{D}{d} (\mu - 1) t$$

where μ = refractive index of the plate for the light used
and t = thickness of the plate.

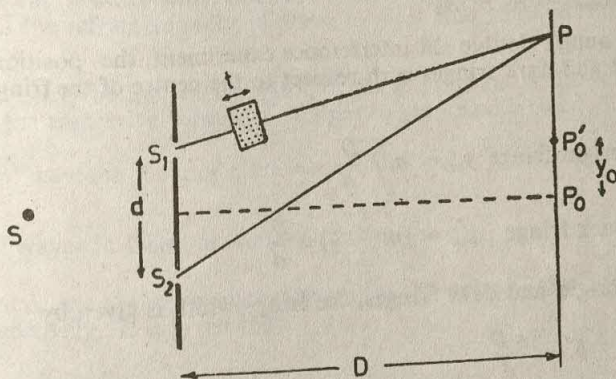


Fig. 17.2

The shift takes place towards the wave in the path of which the plate is introduced, as shown in the figure. The introduction of the plate has no

effect on the fringe width. The shift occurs because the original path difference ($S_2P - S_1P$) is changed by $(\mu - 1) t$.

17.2 FORMULAE

1. If v_1 is the velocity of the incident wave in medium 1 and v_2 the velocity of the refracted wave in medium 2, then the refractive index μ of medium 2 with respect to medium 1 is given by

$$\mu = \frac{v_1}{v_2} = \frac{\lambda_1}{\lambda_2}$$

where λ_1 and λ_2 are the wavelengths of the same light in medium 1 and medium 2 respectively.

2. If two coherent waves of amplitudes A_1 and A_2 superpose at a point where the phase difference between them is ϕ , the resultant amplitude A is given by

$$A = (A_1^2 + A_2^2 + 2 A_1 A_2 \cos \phi)^{1/2}$$

3. Phase difference $= \frac{2\pi}{\lambda} \times (\text{path difference})$

or
$$\phi = \frac{2\pi \Delta}{\lambda}$$

4. For constructive interference

$$\phi = 2m\pi$$

or
$$\Delta = m\lambda \quad ; \quad m = 0, 1, 2, \dots \text{etc.}$$

and
$$A_{\max} = A_1 + A_2$$

5. For destructive interference,

$$\phi = (2m + 1)\pi$$

or
$$\Delta = (m + \frac{1}{2})\lambda$$

and
$$A_{\min} = A_1 - A_2$$

6. In Young's double slit interference experiment, the positions of the m th bright and dark fringes with respect to the centre of the fringe pattern, are

For m th bright fringe
$$y_m = m\lambda \frac{D}{d}$$

For m th dark fringe
$$y'_m = (m - \frac{1}{2})\lambda \frac{D}{d}$$

For both bright and dark fringes, the fringe width is given by

$$\beta = \frac{\lambda D}{d}$$

where λ = wavelength of monochromatic light used

D = distance between slits and the screen

d = separation between the two slits

7. If a transparent plate of thickness t and refractive index μ is introduced in the path of one of the interfering waves, the entire fringe pattern shifts by

$$y_0 = \frac{D}{d} (\mu - 1) t$$

17.3 NUMERICAL EXAMPLES

Example 1: Sodium light of wavelength 5890 Å is incident on a boundary separating air and glass. If the velocity of light in air is $3 \times 10^8 \text{ m s}^{-1}$, calculate the velocity and wavelength of sodium light in glass. The refractive index of glass for sodium light is 1.6.

Solution: Refractive index is defined as

$$\mu = \frac{c}{v}$$

where c = velocity of light in vacuum (or air) and v is the velocity of light in the medium. Hence

$$v = \frac{c}{\mu} = \frac{3 \times 10^8}{1.6} = 1.875 \times 10^8 \text{ m s}^{-1}$$

Wavelength of sodium light in air is 5890 Å. Since the frequency of light remains unchanged on refraction, we have

$$\begin{aligned} \text{Frequency} &= \frac{v_{\text{air}}}{\lambda_{\text{air}}} = \frac{v_{\text{glass}}}{\lambda_{\text{glass}}} \\ \lambda_{\text{glass}} &= \frac{v_{\text{glass}} \lambda_{\text{air}}}{v_{\text{air}}} \\ &= \frac{1.875 \times 10^8 \times 5890}{3 \times 10^8} = 3681.25 \text{ Å} \end{aligned}$$

Example 2: A glass slab of thickness 8 cm contains the same number of waves as 9 cm of water when both are traversed by the same monochromatic light. If the refractive index of water is $4/3$, find the refractive index of glass.

Solution: Let λ_g be the wavelength of light in glass and λ_w in water and v_g and v_w be the respective velocities of light in glass and water.

$$\text{Number of waves in 8 cm of glass} = n_g = \frac{8}{\lambda_g}$$

$$\text{number of waves in 9 cm of water} = n_w = \frac{9}{\lambda_w}$$

here λ_g and λ_w are the wavelengths (in cm) of the same light in glass and water respectively. It is given that

$$n_g = n_w$$

$$\text{or } \frac{8}{\lambda_g} = \frac{9}{\lambda_w}$$

$$\text{or } \frac{\lambda_w}{\lambda_g} = \frac{9}{8}$$

If μ_g and μ_w are the refractive indices of glass and water respectively, then we have

$$\frac{\mu_w}{\mu_g} = \frac{v_g}{v_w} = \frac{\lambda_g}{\lambda_w}$$

or
$$\mu_g = \mu_w \times \frac{\lambda_w}{\lambda_g} = \frac{4}{3} \times \frac{9}{8} = 1.5$$

Example 3: Monochromatic light from a narrow slit illuminates two narrow slits 0.3 mm apart, producing an interference pattern with bright fringes 1.5 mm apart on a screen 75 cm away. Find the wavelength of the light. How will the fringe width be altered if (a) the distance of the screen is doubled and (b) the separation between the slits is doubled?

Solution: $d = 0.3 \text{ mm} = 0.3 \times 10^{-3} \text{ m}$

$$\beta = 1.5 \text{ mm} = 1.5 \times 10^{-3} \text{ m}$$

$$D = 75 \text{ cm} = 0.75 \text{ m}$$

Wavelength λ is given by

$$\begin{aligned} \lambda &= \frac{\beta d}{D} \\ &= \frac{1.5 \times 10^{-3} \times 0.3 \times 10^{-3}}{0.75} \\ &= 6000 \times 10^{-10} \text{ m} = 6000 \text{ \AA} \end{aligned}$$

(a)
$$\beta = \frac{\lambda D}{d}$$

when D is doubled, fringe width β is also doubled.

$$\beta = 1.5 \times 2 = 3.0 \text{ mm}$$

(b) When d is doubled, β is reduced to half, i.e.

$$\beta = \frac{1.5}{2} = 0.75 \text{ mm}$$

Example 4: Monochromatic light of wavelength 6000 Å illuminates two narrow slits 0.3 mm apart producing an interference pattern on a screen 75 cm away. Calculate the separation between (a) the second bright fringe and the central bright fringe and (b) the second dark fringe and the central bright fringe.

Solution: (a) The separation between the m th bright fringe and the central fringe is given by

$$y_m = m \lambda \frac{D}{d}$$

$$y_2 = 2 \lambda \frac{D}{d}$$

$$\begin{aligned} &= 2 \times (6000 \times 10^{-10}) \times \frac{75 \times 10^{-2}}{0.3 \times 10^{-3}} \\ &= 3 \times 10^{-3} \text{ m} = 3 \text{ mm} \end{aligned}$$

(b) The separation between the m th dark fringe and the central fringe is given by

$$y'_m = (m - \frac{1}{2}) \frac{\lambda D}{d}$$

$$\begin{aligned} y'_2 &= (2 - \frac{1}{2}) \times (6000 \times 10^{-10}) \times \frac{75 \times 10^{-2}}{0.3 \times 10^{-3}} \\ &= 2.25 \times 10^{-3} \text{ m} = 2.25 \text{ mm} \end{aligned}$$

Example 5: In Young's double slit experiment the slits are 0.5 mm apart and interference is observed on a screen placed at a distance of 100 cm from the slits. It is found that the 9th bright fringe is at a distance of 8.835 mm from the second dark fringe from the centre of the fringe pattern. Find the wavelength of light used.

Solution: The distance of the m th bright fringe from the central fringe is

$$y_m = m \lambda \frac{D}{d} = m \beta$$

where $\beta = \frac{\lambda D}{d}$ is the fringe width.

$$y_9 = 9\beta$$

(i)

The distance of the m th dark fringe from the central fringe is

$$y'_m = (m - \frac{1}{2}) \frac{\lambda D}{d} = (m - \frac{1}{2}) \beta$$

$$y'_2 = \frac{3}{2} \beta$$

(ii)

From (i) and (ii) we get

$$y_9 - y'_2 = 9\beta - \frac{3}{2}\beta = \frac{15}{2}\beta$$

It is given that $y_9 - y'_2 = 8.835 \text{ mm}$. Hence

$$\beta = \frac{8.835 \times 2}{15} = 1.178 \text{ mm} = 1.178 \times 10^{-3} \text{ m}$$

Now

$$\begin{aligned} \lambda &= \frac{\beta d}{D} \\ &= \frac{(1.178 \times 10^{-3}) \times (0.5 \times 10^{-3})}{1} \\ &= 5.89 \times 10^{-7} \text{ m} = 5890 \text{ \AA} \end{aligned}$$

Example 6: When one of the slits in Young's experiment is covered with a transparent sheet of thickness $3.6 \times 10^{-3} \text{ cm}$, the central fringe shifts to a position originally occupied by the 30th bright fringe. If $\lambda = 6000 \text{ \AA}$, find the refractive index of the sheet.

Solution The position of the 30th bright fringe is given by

$$y_{30} = 30 \frac{\lambda D}{d}$$

Hence the shift of the central fringe is

$$y_0 = 30 \frac{\lambda D}{d}$$

But $y_0 = \frac{D}{d} (\mu - 1) t$

$$\therefore 30 \frac{\lambda D}{d} = \frac{D}{d} (\mu - 1) t$$

or $(\mu - 1) = \frac{30 \lambda}{t} = \frac{30 \times (6000 \times 10^{-10})}{(3.6 \times 10^{-5})} = 0.5$

$$\therefore \mu = 1.5$$

Example 7: In Young's double slit experiment using monochromatic light, the fringe pattern shifts by a certain distance on the screen when a mica sheet of refractive index 1.6 and thickness 1.964 microns is introduced in the path of one of the interfering waves. The mica sheet is then removed and the distance between the slits and the screen is doubled. It is found that the distance between successive maxima (or minima) now is the same as the observed fringe shift upon the introduction of the mica sheet. Calculate the wavelength of the monochromatic light used in the experiment.

(IIT 1983)

Solution: When a mica sheet of thickness t and refractive index μ is introduced in one of the interfering waves, the distance y_0 through which the fringes shift is given by

$$y_0 = (\mu - 1) t \frac{D}{d} \quad (i)$$

The fringe width β , i.e. the distance between successive maxima (or minima) is given by

$$\beta = \frac{\lambda D}{d}$$

When the distance D between the slits and the screen is doubled, the new fringe width becomes

$$\beta' = 2 \frac{\lambda D}{d} \quad (ii)$$

It is given that $y_0 = \beta'$. Equating (i) and (ii) we get

$$\frac{2 \lambda D}{d} = (\mu - 1) t \frac{D}{d}$$

or $\lambda = \frac{1}{2} (\mu - 1) t$

$$= \frac{(1.6 - 1) \times (1.964 \times 10^{-6})}{2}$$

$$= 5.892 \times 10^{-7} \text{ m} = 5892 \text{ \AA}$$

Example 8: A beam of light consisting of two wavelengths 6500 Å and 5200 Å, is used to obtain interference fringes in a Young's double slit experiment.

- (i) Find the distance of the third bright fringe on the screen from the central maximum for wavelength $6500 \text{ \AA}$.
 (ii) What is the least distance from the central maximum where the bright fringes due to both the wavelengths coincide?

The distance between the slits is 2 mm and the distance between the plane of the slits and the screen is 120 cm . (IIT 1985)

Solution: (i) The distance of the m th bright fringe from the central maximum is given by

$$y_m = \frac{m \lambda D}{d}$$

$$y_3 = 3 \lambda \frac{D}{d} = \frac{3 \times (6500 \times 10^{-10}) \times 1.20}{2 \times 10^{-3}} \\ = 1.17 \times 10^{-3} \text{ m} = 1.17 \text{ mm}$$

(ii) Let the n th bright fringe of wavelength λ_n and the m th bright fringe of wavelength λ_m coincide at a distance y from the central maximum, then

$$y = \frac{m \lambda_m D}{d} = \frac{n \lambda_n D}{d}$$

or
$$\frac{m}{n} = \frac{\lambda_n}{\lambda_m} = \frac{6500}{5200} = \frac{5}{4}$$

The least integral values of m and n which satisfy the above condition are

$$m = 5 \quad \text{and} \quad n = 4$$

i.e. the 5th bright fringe of wavelength $5200 \text{ \AA}$ coincides with the 4th bright fringe of wavelength $6500 \text{ \AA}$. The smallest value of y at which this happens is

$$y_{\min} = \frac{m \lambda_m D}{d} \\ = \frac{5 \times (5200 \times 10^{-10}) \times 1.20}{2 \times 10^{-3}} \\ = 1.56 \times 10^{-3} \text{ m} = 1.56 \text{ mm}$$

Example 9: In Young's double slit experiment, the intensity of light at a point on the screen where the path difference is λ is K units; λ being the wavelength of light used. Find the intensity at a point where the path difference is $\lambda/4$.

Solution:

$$\text{Phase difference} = \frac{2\pi}{\lambda} \times (\text{path difference})$$

or
$$\phi = \frac{2\pi \Delta}{\lambda}$$

The intensity at a point on the screen is given by

$$I = 2 I_0 (1 + \cos \phi)$$

where I_0 is the intensity of each interfering beam.

At a point where $\Delta = \lambda$, $\phi = 2\pi$. Hence, at this point,

$$I = 2 I_0 (1 + \cos 2\pi) = 4 I_0 = K \text{ units (given)}$$

Thus $I_0 = \frac{K}{4} \text{ units.}$

Therefore $I = \frac{K}{2} (1 + \cos \phi)$

When $\Delta = \frac{\lambda}{4}$, $\phi = \frac{\pi}{2}$. Therefore, at this point,

$$I = \frac{K}{2} \left(1 + \cos \frac{\pi}{2} \right) = \frac{K}{2} \text{ units}$$

Example 10: Two coherent light sources of intensity ratio 25 : 4 are employed in an interference experiment. What is the ratio of the intensities of the maxima and minima in the interference pattern?

Solution: Let I_1 and I_2 be the intensities of the two coherent beams and A_1 and A_2 their respective amplitudes. Now

Intensity ratio $\frac{I_1}{I_2} = \frac{25}{4}$, therefore

Amplitude ratio $\frac{A_1}{A_2} = \sqrt{\frac{25}{4}} = \frac{5}{2}$

i.e. $A_1 = 5 \text{ units}$ and $A_2 = 2 \text{ units.}$

At maxima: $A_{\max} = A_1 + A_2 = 7 \text{ units}$

At minima: $A_{\min} = A_1 - A_2 = 3 \text{ units}$

Hence $\frac{I_{\max}}{I_{\min}} = \frac{A_{\max}^2}{A_{\min}^2} = \frac{(7)^2}{(3)^2} = \frac{49}{9}$

17.4 TRUE-FALSE STATEMENTS WITH REASONS

A few statements are given below. We have to decide whether they are true or false giving, in brief, reasons in support of our answers.

STATEMENTS

1. Like sound, light cannot propagate in vacuum.
2. A filament lamp emits light having a constant phase.
3. Only coherent light sources produce a sustained interference pattern.
4. When a light wave travelling in air, enters water, its wavelength, velocity and frequency all undergo a change.
5. When light is reflected from a plane mirror, the reflected wave is identical with the incident wave, except that it travels in a different direction.
6. In Young's double slit experiment the fringes become indistinct if one of the slits is covered with cellophane paper.

ANSWERS

1. *False* Sound is a mechanical wave. It propagates in a medium by virtue of particle oscillations. Hence sound cannot propagate in vacuum. But light is an electromagnetic wave with associated oscillating electric and magnetic fields. Since these fields can exist even in a vacuum, light can propagate in a vacuum.
2. *False* A filament lamp emits light waves of different wavelengths and random phases. The phases change very rapidly with time. The phase of the emitted light remains constant only during a very short time interval of the order of 10^{-8} to 10^{-10} second.
3. *True* Coherent light sources emit light waves of the same wavelength and with a constant phase difference. Hence the resultant intensity in the interference pattern is independent of time, depending only on the path difference. Consequently, the interference pattern is permanent or sustained.
4. *False* The frequency remains unchanged, the wavelength as well as the velocity undergo a change depending on the value of the refractive index of water with respect to air.
5. *False* The intensity (hence the amplitude), the phase and the state of polarization of the reflected wave are, in general, different from those of the incident wave.
6. *True* If one of the slits in Young's experiment is covered with a cellophane paper, the intensity of light emerging from that slit decreases. Thus the two interfering beams no longer have equal intensities. Hence their amplitudes are also different. Consequently the resultant intensity at minima in the interference pattern is finite; different from zero. The pattern, therefore, becomes indistinct.

17.5 MULTIPLE CHOICE QUESTIONS

Choose the correct answer from the given alternatives in each of the following

1. Monochromatic light is refracted from air into glass of refractive index n . The ratio of the wavelengths of the incident and refracted waves is

(a) $1 : 1$	(b) $1 : n$
(c) $n : 1$	(d) $n^2 : 1$
2. Which one of the following phenomena cannot be explained by the wave theory of light ?

(a) Refraction	(b) Total internal reflection
(c) Diffraction	(d) Photoelectric effect
3. Two light sources are said to be coherent if they emit

(a) lights of the same intensity	(b) waves of the same frequency
(c) waves of the same velocity	(d) waves of the same wavelength having a constant phase difference.
4. The wave theory of light, in its original form, was first postulated by

(a) Isaac Newton	(b) Christian Huygens
(c) Thomas Young	(d) Albert Einstein

5. Figure 17.3 illustrates Young's double slit experiment performed with sodium light of wavelength λ . If $d \ll D$, the fringe width β in the interference pattern is given by

(a) $\beta = \frac{\lambda D}{d}$

(b) $\beta = \frac{\lambda d}{D}$

(c) $\beta = \frac{d D}{\lambda}$

(d) $\beta = \frac{\lambda D^2}{d^2}$

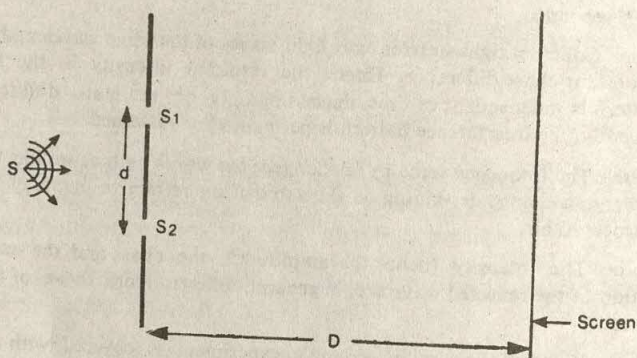


Fig. 17.3

6. How is the interference pattern in Young's double slit experiment affected if the sodium (yellow) light is replaced by red light of the same intensity?
- The fringes will vanish
 - The fringes will become brighter
 - The fringe width will decrease
 - The fringe width will increase.
7. In Young's double slit experiment, if the distance between the slits and the screen is doubled and the separation between the slits is reduced to half, the fringe width
- is doubled
 - becomes four times
 - is halved
 - remains unchanged
8. What happens if the monochromatic light used in Young's double slit experiment is replaced by white light?
- All bright fringes become white
 - All bright fringes have colours between violet and red
 - Only the central fringe is white, all other fringes are coloured
 - No fringes are observed.
9. What will happen if one of the slits, say S_1 , in the experiment described in Fig. 17.3 is covered with a cellophane paper which absorbs a fraction of the intensity of light from S_1 ?
- The fringe width will decrease
 - The fringes will become more distinct
 - The bright fringes will become less bright and the dark fringes will not be completely dark
 - No fringes will be observed.

10. What happens to the interference pattern if the two slits in Young's experiment are illuminated by two independent sources such as two sodium lamps S and S' as shown in Fig. 17.4?
- Two sets of interference fringes overlap
 - No fringes are observed
 - The intensity of the bright fringes is doubled
 - The intensity of the bright fringes becomes four times.

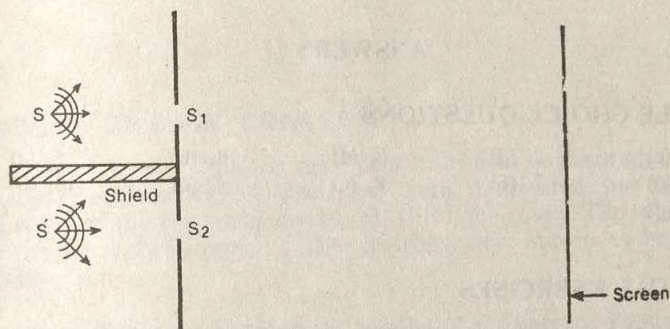


Fig. 17.4

11. In Young's double slit experiment, if the slit widths are in the ratio of $1 : 2$, the ratio of the intensities at minima and maxima will be
- $1 : 2$
 - $1 : 3$
 - $1 : 4$
 - $1 : 9$

17.6 NUMERICAL EXERCISES

- Calculate the wavelength of yellow light in glass of refractive index 1.5 if its wavelength in air is 5.9×10^{-7} m.
- The refractive index of a sample of glass for violet light is 1.665 and for red light is 1.618. Calculate the difference in the velocity of light in glass for the two colours.
- The wavelength of yellow light in air is $6000 \text{ \AA}$. Calculate its wavelength in water and glass; the refractive indices of water and glass being $4/3$ and 1.6 respectively.
- Determine the depth of water that contains the same number of waves as 16 cm of glass when each is traversed by the same monochromatic light. The refractive indices of water and glass are $4/3$ and 1.5 respectively.
- In Young's two-slit experiment the separation between 10 fringes is found to be 5.5 mm on a screen placed at a distance of 2.0 m from the slits. If the separation between the slits is 2.0 mm, determine the wavelength of light.
- What is the distance between two consecutive bright fringes of Ex. 5 if the separation between the slits is reduced to 1.0 mm and the distance of the screen from the slits is reduced to 1.0 m?
- Using the data given in Ex. 5 above, calculate the fringe width for (a) violet light of $\lambda = 4.5 \times 10^{-7}$ m, (b) yellow light of $\lambda = 6.0 \times 10^{-7}$ m and (c) red light of $\lambda = 7.5 \times 10^{-7}$ m.
- In Young's double slit experiment the two slits are 0.5 mm apart and the screen-slits distance is 100 cm. The distance of the 11th bright fringe from the first bright fringe is found to be 1.178 cm. Find the wave-length of the light used.

9. In Young's double slit experiment the two slits are 0.3 mm apart. The distance between the screen and the slits is 90 cm. It is observed that the 4th bright fringe is 8 mm from the central bright fringe. Find (a) the wavelength of light used and (b) the distance of the second dark fringe from the central bright fringe.
10. Interference fringes are produced by a double slit arrangement and a piece of plane parallel glass of refractive index 1.5 is interposed in one of the interfering beams. If the fringes are displaced through 30 fringe widths for light of wavelength 6×10^{-5} cm, find the thickness of the plate.

ANSWERS

MULTIPLE CHOICE QUESTIONS

- | | | | | |
|---------|--------|--------|--------|---------|
| 1. (c) | 2. (d) | 3. (d) | 4. (b) | 5. (a) |
| 6. (d) | 7. (b) | 8. (c) | 9. (c) | 10. (b) |
| 11. (d) | | | | |

NUMERICAL EXERCISES

- 3.93×10^{-7} m
- 5.2×10^6 m s⁻¹
- 4500 Å, 3750 Å
- 18 cm
- 5.5×10^{-7} m
- 5.5 mm
- (a) 0.45 mm, (b) 0.6 mm, (c) 0.75 mm
- 5890 Å
- (a) 6667 Å, (b) 3 mm
- 3.6×10^{-3} cm

Physics of the Atom

18.1 REVIEW OF BASIC CONCEPTS

18.1.1 Cathode Rays Cathode rays, now known to be made up of negatively charged particles called electrons, were discovered by making a detailed study of the electrical discharge through gases. The constituent particles of the cathode rays, i.e. the electrons, were shown to be present in all matter.

18.1.2 Effect of Electric Field on the Motion of a Charged Particle When a particle of mass m and charge q moves in an electric field $\vec{E}$, it experiences a force

$$\vec{F} = q \vec{E} \quad (18.1)$$

so that its equation of motion is

$$m \frac{d^2 \vec{r}}{dt^2} = q \vec{E} \quad (18.2)$$

This gives

$$\vec{r} = \vec{r}_0 + \vec{V}_0 t + \frac{1}{2} \frac{q}{m} \vec{E} t^2 \quad (18.3)$$

where $\vec{r}_0$ is the initial position vector and $\vec{v}_0$ the initial velocity of the particle.

Considering the particular case where (i) $\vec{r}_0$ is zero, i.e. the particle starts from the origin, (ii) $\vec{E}$ is along, say, the y -axis and (iii) $\vec{v}_0$ is along, say, the x -axis, we can easily show that the path of the particle is described by the equation

$$y = \frac{1}{2} \frac{q E}{m} \frac{x^2}{v_0^2} \quad (18.4)$$

This is the equation of a parabola. Thus the path of a particle subjected to an electric field at right angles to its initial direction of motion is a parabola.

Quite often, the duration of the particle in the field is so short that its path in the field can also be approximated by an arc of a circle of radius R_E (Fig. 18.1).

Under this approximation we may write

$$q E = \frac{m v^2}{R_E} \quad (18.5)$$

After leaving the region of the field the particle moves in a straight line along the tangent to its path at the point where the field terminates.

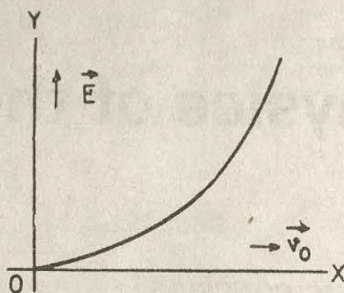


Fig. 18.1

18.1.3 Effect of a Magnetic Field on the Motion of a Charged Particle

When a particle of mass m and charge q moving with a velocity $\vec{v}$ is subjected to a magnetic field $\vec{B}$, it experiences a force $\vec{F}$ given by

$$\vec{F} = q (\vec{v} \times \vec{B}) \quad (18.6)$$

The equation of the particle is

$$m \frac{d\vec{v}}{dt} = q (\vec{v} \times \vec{B}) \quad (18.7)$$

This gives

$$(\vec{v}) \cdot \left(\frac{d\vec{v}}{dt} \right) = \frac{1}{2} \frac{d}{dt} (v^2) = 0 \quad (18.8)$$

$$\text{or} \quad v = \text{constant} \quad (18.9)$$

Thus the speed of the particle remains constant when it moves in a magnetic field.

Let the direction of the magnetic field be labelled as the x -axis. It can be shown that a particle projected with a velocity $\vec{v}$ then moves with a constant speed v_{11} along the x -axis while, simultaneously, it describes a circular path with a constant speed v_t in the y - z plane, i.e. the plane perpendicular to B (Fig 18.2).

If $\vec{v}$ is perpendicular to $\vec{B}$ then $v_{11} = 0$ and the radius R_B of the circular path described by the particle in the plane perpendicular to $\vec{B}$ is given by

$$q v B = \frac{m v^2}{R_B}$$

$$\text{or} \quad R_B = \frac{m v^2}{q B} \quad (18.10)$$

This is so because, with $\vec{v}$ perpendicular to $\vec{B}$, the force acting on the charged particle has a magnitude of $|q (\vec{v} \times \vec{B})| = qvB$.

After the particle leaves the region of the magnetic field, there is no force acting on it and it continues to move in a straight line which is tangential to the circular path at the point where the field ends.

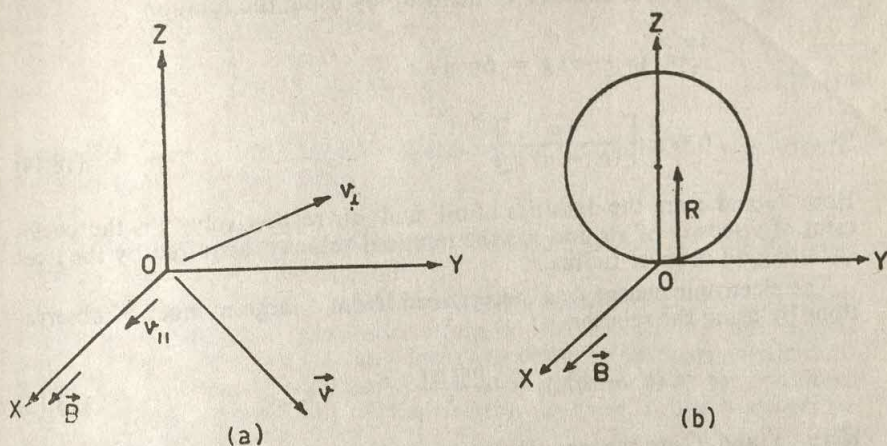


Fig. 18.2

18.1.4. Thomson's Method for e/m Measurement of Cathode Rays J J Thomson determined e/m of the cathode rays by subjecting them to *crossed* electric and magnetic fields, i.e. fields that are at right angles to each other. He also arranged the two fields to be perpendicular to the initial direction of motion of the electrons and adjusted their sense to be such that they produced oppositely directed deflections in the electron beam. If the deflections nullify each other for certain values, say E_c and B_c of the two fields, we must have

$$e E_c = e v B_c \quad (18.11)$$

$$\text{or} \quad v = \frac{E_c}{B_c}$$

If y is the deflection of the electron beam under the action of a perpendicular electric field alone, it is easy to show that

$$y = k \frac{1}{2} \frac{e}{m} E \left(\frac{l}{v} \right)^2 \quad (18.12)$$

where l is the 'length' of the region over which the electric field acts and k , a constant of the apparatus, depends on the distance of the screen from the point where the electric field 'ends'.

Similarly if y' is the deflection of the electron beam under the action of the perpendicular magnetic field alone, it can be shown that

$$y' = a l' \frac{B}{v} \frac{e}{m} \quad (18.13)$$

where l' is the 'length' of the region over which the magnetic field acts and a is a constant of the apparatus dependent on the distance of the screen from the 'centre' of the magnetic field.

We may use either of the Eqs (18.12) and (18.13) along with Eq. (18.11) to get the value of e/m .

18.1.5 Millikan's Determination of the Electronic Charge R A Millikan obtained a value for the charge of the electron by observing the effect of an electric field on the motion of a charged oil drop. He first used Stokes' law to determine the radius r of the drop by using the relation

$$\frac{4\pi}{3} r^3 (\rho - \sigma) g = 6\pi \eta r v$$

or

$$r = \frac{9}{2} \left[\frac{\eta v}{(\rho - \sigma) g} \right]^{1/2} \quad (18.14)$$

Here ρ and σ are the densities of oil and air respectively, η is the coefficient of viscosity of air and v is the terminal velocity acquired by the freely falling oil drop of radius r .

The electronic charge was determined from a large number of observations by using the relation

$$e = (q' - q)_{\min} = \frac{6\pi \eta r}{E} (v'' - v')_{\min} \quad (18.15)$$

Here v' and v'' are the terminal velocities of the oil drop when it has charges q and q' respectively and is moving under the action of a vertical electric field $\vec{E}$ in addition to the force of gravity.

Mass of the Electron The mass of the electron was calculated using the value of e/m obtained by Thomson and of e obtained by Millikan.

$$m = \frac{e}{(e/m)}$$

18.1.6 The Photoelectric Effect The phenomenon of emission of electrons caused by the incidence of electromagnetic radiation of appropriate frequency on a metal surface is known as the photoelectric effect.

Photons The concept of photons, first introduced by Max Planck to explain the black-body spectrum, was used by Einstein to explain the observed features of the phenomenon of photo-electricity. He assumed that the energy of light is concentrated in a number of small packages, each of energy $h\nu$ where ν is the frequency of light h is the Planck's constant. These packages or *quanta* of light energy were later called *photons*. The intensity of a light beam is determined by the number of photons falling normally on a unit area in a unit time.

Einstein's Photoelectric Equation Einstein explained the various observed features of photo-electric emission by assuming that (i) electrons are emitted from the metal by the impact of photons (ii) each photon, on impact gives its full energy to a single electron. He obtained a simple equation

$$\frac{1}{2} m v_{\max}^2 = h\nu - W \quad (18.16)$$

to explain the phenomenon of photoelectricity. Here ν is the frequency of the incident light, W is the work function of the metal used and $\frac{1}{2} m v_{\max}^2$ is the maximum energy of the emitted electrons. Since $W = h\nu_c$ where ν_c is the threshold frequency for the metal used, we also have

$$\frac{1}{2} m v_{\max}^2 = h(\nu - \nu_c) \quad (18.17)$$

The maximum energy of the emitted electrons is determined in terms of the stopping potential V_s . Thus

$$\frac{1}{2} m v_{\max}^2 = e V_s$$

$$\therefore h(\nu - \nu_c) = e V_s$$

$$\text{or } V_s = \frac{h}{e} \nu - \frac{h}{e} \nu_c \quad (18.18)$$

It was this form of the Einstein's photo-electric equation that was experimentally verified by Millikan.

18.1.7 The Bohr Model of the Atom The Bohr model of the atom is based on the following two postulates

(i) An electron revolving round the nucleus can move only in certain special orbits, known as *stationary orbits*, which are such that an electron moving in them does not lose any energy by emitting electromagnetic radiations. These special or stationary orbits are determined by the condition that the angular momentum of the electron in these orbits is always an integral multiple of $h/2\pi$:

$$m v r_n = n \frac{h}{2\pi} \quad (18.19)$$

where n is an integer called the *principal quantum number*.

(ii) Whenever an electron 'jumps' from one stationary orbit of principal quantum number n_2 to another of principal quantum number n_1 , it emits (or absorbs) a quantum of radiation whose energy $h\nu$ equals the difference of the energies of the electron in these two stationary orbits

$$h\nu = E_{n_2} - E_{n_1} \quad (18.20)$$

The Bohr formulae Using the above two postulates and Coulomb's law, Bohr obtained a number of formulae for the Hydrogen atom. Thus, if Ze = charge on the nucleus, m = mass of the electron, e = electronic charge, h = Planck's constant, we have

$$r_n = \frac{1}{k} \frac{n^2 h^2}{4\pi^2 m Z e^2} \quad (18.21)$$

where $k = 1/4 \pi \epsilon_0 = 9 \times 10^9 \text{ N} - \text{m}^2/\text{C}^2$ in the SI units. Here r_n is the radius of the n th stationary orbit specified by the principal quantum number n .

The velocity v_n of the electron in the n th orbit is given by

$$v_n = k \frac{2\pi Z e^2}{n h} \quad (18.22)$$

The total energy E_n of the electron in the n th stationary orbit is given by

$$E_n = -k^2 \frac{2\pi^2 m Z^2 e^4}{n^2 h^2} \quad (18.23)$$

The negative sign indicates that the electron is in a *bound state*, i.e., it is held in check by the attraction of the positive nucleus. Further as $|E_n| \propto 1/n^2$, the negative sign means that higher orbits (greater value of n) have

a greater energy. The energy of the electron in the first orbit of the hydrogen atom is found to be -13.6 eV . Its value in the second, third, fourth ... orbits are, therefore, $-13.6/2^2 \text{ eV}$, $-13.6/3^2 \text{ eV}$, $-13.6/4^2 \text{ eV}$... These are represented on an energy-level diagram of the form shown in Fig. 18.3.

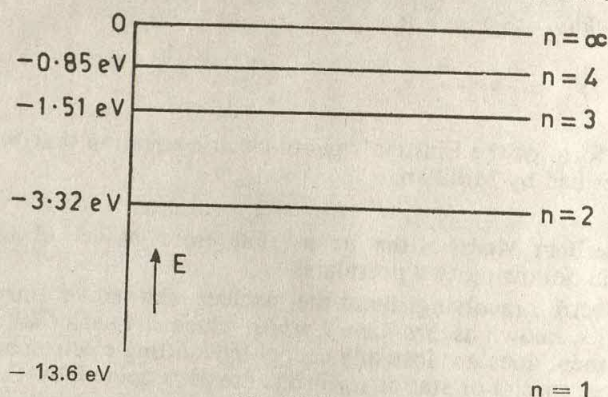


Fig. 18.3

The Spectrum of Hydrogen Using the energy-level formula (Eq. 18.23) and the second postulate of Bohr's theory (Eq. 18.20), it is easy to see that the various lines in the spectrum of hydrogen are given by

$$\frac{1}{\lambda} = \frac{\nu}{c} = R_{\infty} Z^2 \left(\frac{1}{n_1^2} - \frac{1}{n_2^2} \right) \quad (18.24)$$

$$\text{where } R_{\infty} = k^2 \frac{2\pi^2 m e^4}{c h^3} \quad (18.25)$$

is known as Rydberg's constant. Eq. (18.24) is nothing but a generalised form of the empirical *Balmer formula* for the spectral lines of hydrogen. Bohr's theory gives us the constant R in terms of fundamental constants and one of the immediate reasons for acceptance of Bohr's theory was the remarkable agreement between the theoretical and experimental values of the constant R .

The Main Series in the Hydrogen Spectrum The various lines in the hydrogen spectrum have been grouped into the following main series.

(i) *Lyman series* : This series is given by the formula

$$\frac{1}{\lambda} = R_{\infty} Z^2 \left(\frac{1}{1^2} - \frac{1}{n^2} \right)$$

where $n = 2, 3, 4, \dots$

(ii) *Balmer series* : This is given by

$$\frac{1}{\lambda} = R_{\infty} Z^2 \left(\frac{1}{2^2} - \frac{1}{n^2} \right)$$

where $n = 3, 4, 5, \dots$

(iii) *Paschen series* This series is given by

$$\frac{1}{\lambda} = R_{\infty} Z^2 \left(\frac{1}{3^2} - \frac{1}{n^2} \right)$$

where $n = 4, 5, 6, \dots$

(iv) *Brackett series*: For this series, we have

$$\frac{1}{\lambda} = R_{\infty} Z^2 \left(\frac{1}{4^2} - \frac{1}{n^2} \right)$$

where $n = 5, 6, 7, \dots$ etc.

The *series limits* for the various series (corresponding to $n = \infty$) are given by

$$\frac{1}{\lambda} = R_{\infty} Z^2, \quad \frac{1}{\lambda} = R_{\infty} \frac{Z^2}{4}, \quad \frac{1}{\lambda} = R_{\infty} \frac{Z^2}{9}$$

and $\frac{1}{\lambda} = R_{\infty} = \frac{Z^2}{16}$ respectively.

The transitions responsible for these different series are shown in Fig. 18.4.

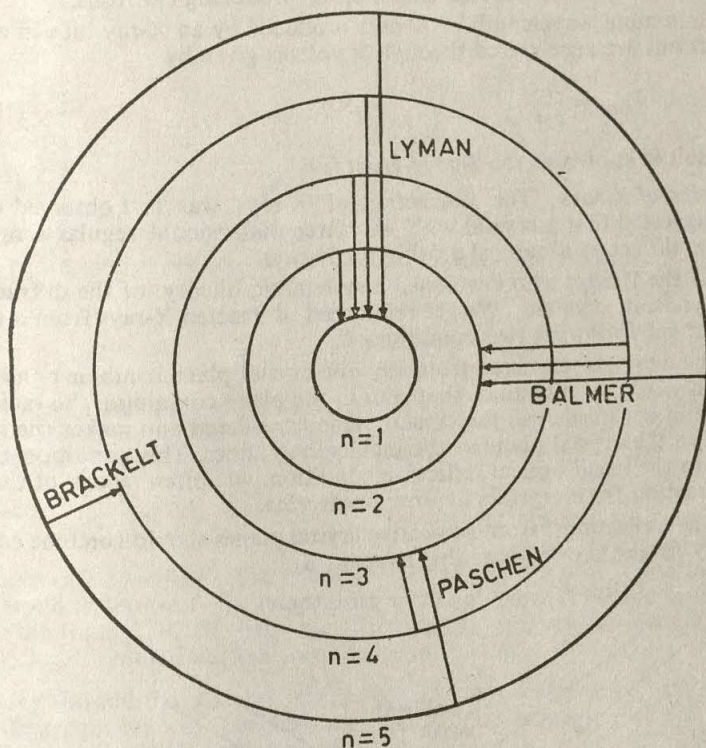


Fig. 18.4

The Reduced mass of the Electron The concept of the reduced mass of the electron takes into account the fact that the mass of the nucleus is not infinite, and strictly speaking, both the electron and the nucleus must be considered as rotating about their centre of mass. The 'reduced mass' of the electron is given by

$$\mu = \left(\frac{m}{1 + \frac{m}{M}} \right)$$

(18.26)

where M is the mass of the nucleus. In all formulae of the Bohr theory, the mass m of the electron must be replaced by its reduced mass.

The introduction of the idea of reduced mass makes the Rydberg constant $\left(R = k^2 \frac{2\pi^2 \mu e^4}{c h^3}\right)$ dependent on the mass of the nucleus. This enables us to have a better agreement between theory and experiment for the spectra of other one electron systems. Such electron systems are deuterium, tritium, singly ionised helium, doubly ionised lithium, etc.

18.1.8 X-rays X-ray are produced whenever high speed electrons are stopped abruptly by allowing them to impinge upon a suitable target. We, therefore, need a source of electrons, arrangement for accelerating these electrons, and a suitable target in an X-ray tube.

The X-ray tube used in the early stages was the so-called 'gas tube'. This has now been almost exclusively replaced by the *Coolidge tube* which uses the phenomenon of thermionic emission for producing electrons.

The minimum wavelength of X-rays produced by an X-ray tube in which the electrons are accelerated through V volts is given by

$$\lambda_{\min} = \frac{hc}{eV} \quad (18.27)$$

This result is known as the *Duane-Hunt law*.

Diffraction of X-rays The diffraction of X-rays was first observed when Laue suggested that a crystal with its three-dimensional regular arrays of atoms could act as a natural grating for X-rays.

It was the Braggs who developed a systematic theory of the diffraction of X-rays from crystals. We get reinforced diffracted X-rays from a crystal under the following two conditions

(i) The wavelets scattered from any one crystal plane combine constructively only in the direction that lies in the plane containing the incident X-rays and the normal to the crystal plane considered and makes the same angle with the crystal plane as the incident ray does. This condition being similar to the usual optical reflection condition, we often speak of the X-ray diffraction from crystals as *Bragg reflection*.

(ii) The 'reflections' from successive crystal planes should combine constructively. Bragg showed that this happens if

$$2d \sin \theta = n\lambda \quad (n = \text{an integer})$$

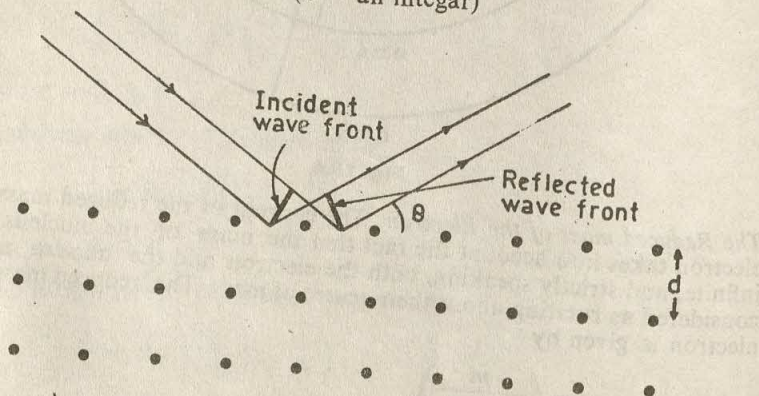


Fig. 18.5

where ' d ' is the spacing of the crystal planes and λ is the wavelength of X-rays. This equation is known as the *Bragg equation*.

X-ray Spectra The field of X-ray spectroscopy was opened up by the discovery of Laue and the invention of the Bragg spectrometer. We now find that the spectrum of X-rays produced by the impact of fast moving electrons on a target consists of a series of 'bright' lines superimposed on a continuous 'luminous' background. This background part of the spectrum is the same for all target materials and is known as the *continuous spectrum*. The 'bright' lines, however, depend on the target material and make up the *characteristic spectrum* of the target.

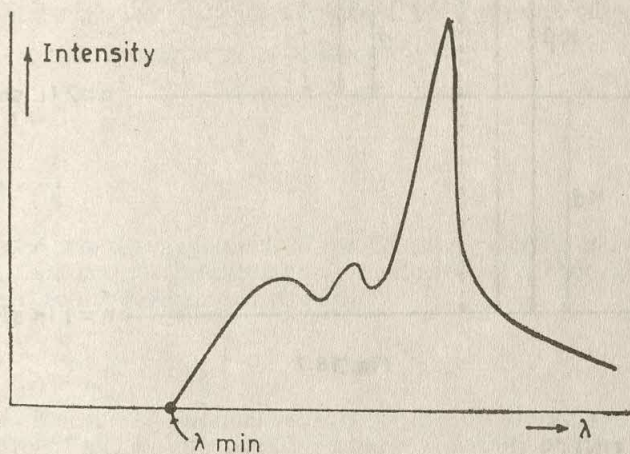


Fig. 18.6

The continuous part of the spectrum is explained in terms of the 'braking' or retardation of the impinging electrons by the target. The shortest wavelength of the X-rays produced is given by Duane-Hunt law:

$$\lambda_{\min} = \frac{hc}{eV}$$

Characteristic spectrum The characteristic spectrum of a given target is understood in terms of the transitions of the electron from the higher orbits to the inner K , L , M shells etc. The various lines are designated as the K_α , K_β , K_γ , ... radiations and so on (Fig. 18.7).

Moseley showed that for any given line of a particular series (say the K_α line), the frequency ν of the line and the atomic number Z of the target material are related by a relation of the type

$$\sqrt{\nu} = a(Z - b)$$

where a and b are constants.

This relation is known as *Moseley's law* and proved very useful in improving our understanding of the periodic table.

Absorption of X-rays The absorption of X-rays by any substance takes place according to the exponential law:

$$I = I_0 e^{-(\mu_m)m}$$

Here I_0 and I are the 'initial' and 'final' intensities of the X-ray beam, m is the mass per unit area of the absorbing substance and μ_m is referred to as the mass absorption coefficient of the absorber.

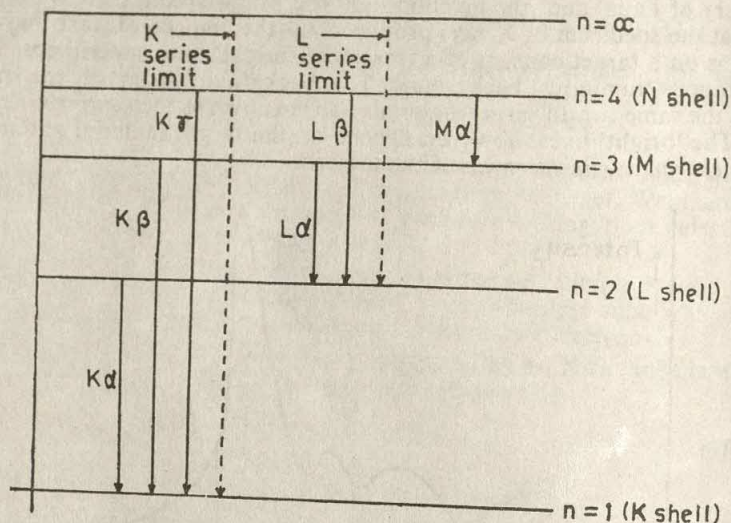


Fig. 18.7

18.2 SI UNITS

Table 18.1

Physical Quantity	Name	SI Unit	Symbol
Electric field	newton per coulomb or volt per metre	N C^{-1}	
Magnetic field	weber per (metre) ² or tesla	Wb m^{-2}	T
Specific charge (e/m)	coulomb per kilogram	C kg^{-1}	
Work function	joule or electron volt	J or eV	
Planck's constant	joule second	J-s	
$1/4\pi\epsilon_0$	$\frac{\text{newton} \cdot (\text{metre})^2}{(\text{coulomb})^2}$	$\text{N-m}^2 \text{C}^{-2}$	
Rydberg constant	per metre	m^{-1}	
Moseley's constant 'a'	$\sqrt{\text{Hertz or (second)}^{-1/2}}$	$(\text{Hz})^{1/2}$ or $(\text{s})^{-1/2}$	
Moseley's constant 'b'	Dimensionless number	—	
Mass absorption coefficient μ_a	$\frac{(\text{metre})^2}{\text{kilogram}}$	$\text{m}^2 \text{kg}^{-1}$	

18.3 FORMULAE

1. Force due to an electric field $\vec{E}$ on a charge e

$$\vec{F} = e \vec{E}$$

2. Force due to a magnetic field $\vec{B}$ on a charge q moving with a velocity $\vec{v}$ is given by

$$\vec{F} = q (\vec{v} \times \vec{B})$$

3. Radius of the circular path of a particle of charge q , mass m in a magnetic field $\vec{B}$ that is normal to its velocity $\vec{v}$

$$R_B = \frac{q v}{m B}$$

$$4. \quad v = \frac{E_c}{B_c}$$

where E_c and B_c are the magnitudes of the 'crossed' electric and magnetic fields which are both perpendicular to the velocity v of a charged particle and produce no net deflection in its path.

$$5. \quad r = \frac{9}{2} \left[\frac{\eta v}{(\rho - \sigma) g} \right]^{1/2}$$

r = radius of oil drop v = terminal velocity of the oil drop, η = coefficient of viscosity of air, ρ = density of oil and σ = density of air.

$$6. \quad e = \frac{6\pi \eta r}{E} (\Delta v)_{\min}$$

where e = electronic charge, E = electric field, η = coefficient of viscosity of air, r = radius of oil drop and $(\Delta v)_{\min}$ is the least change in the terminal velocity of the charged oil drop in the presence of an electric field when the charge on the drop is allowed to change.

$$7. \quad h (\nu - \nu_c) = e V_s$$

h = Planck's constant, e = electronic charge, ν_c = threshold or critical frequency of the irradiated surface, V_s = stopping potential needed to stop even the most energetic emitted electrons when the surface is irradiated with light of frequency ν .

$$8. \quad m v_n r_n = \frac{n h}{2\pi}$$

m = mass of electron, v_n = velocity of electron when moving in a stationary orbit of radius r_n , h = Planck's constant and n = principal quantum number of the orbit considered.

$$9. \quad h \nu = E_2 - E_1$$

ν = frequency of emitted radiation and E_2 and E_1 are the energies in the higher and lower orbits of principal quantum numbers n_2 and n_1 ($n_2 > n_1$).

$$10. \quad r_n = \frac{4\pi \epsilon_0 n^2 h^2}{(4\pi^2 m Z e^2)}$$

Z = atomic number of the nucleus of the one-electron system, r_n = radius of n th stationary orbit

$$11. \quad v_n = \frac{1}{4\pi\epsilon_0} \frac{(2\pi Z e^2)}{nh}$$

v_n = velocity of the electron in n th stationary orbit

$$12. \quad E_n = - \left(\frac{1}{4\pi\epsilon_0} \right)^2 \frac{2\pi^2 m Z^2 e^4}{n^2 h^2}$$

E_n = electron energy in n th stationary orbit

$$13. \quad R_\infty = \left(\frac{1}{4\pi\epsilon_0} \right)^2 \frac{2\pi^2 m e^4}{c h^3}$$

R_∞ = Rydberg constant for a nucleus of 'infinite' mass

$$14. \quad \frac{1}{\lambda} = \frac{\nu}{c} = R_\infty Z^2 \left(\frac{1}{n_1^2} - \frac{1}{n_2^2} \right)$$

λ = wavelength of the spectral line emitted when an electron 'jumps' from a stationary orbit of principal quantum number n_2 to one of principal quantum number n_1 ($n_2 > n_1$).

$$15. \quad \mu = \frac{m}{\left(1 + \frac{m}{M} \right)}$$

μ = 'reduced mass' of the electron, m and M are the masses of the electron and the nucleus respectively.

16. Duane-Hunt Law

$$\lambda_{\min} = \frac{hc}{eV}$$

$\lambda_{\min}$ = wavelength of the shortest X-rays produced when electrons accelerated by a potential V are suddenly stopped, c = velocity of light.

17. Bragg equation

$$2d \sin \theta = n\lambda$$

n = an integer, d = spacing between the crystal planes considered, θ = 'grazing angle' of incidence, λ = wavelength of X-rays used.

18. Moseley's Law

$$\sqrt{\nu} = a(Z - b)$$

ν = frequency of a particular line (say the K_α line) in the characteristic X-ray spectrum of a target material of atomic number Z , a and b are constants.

$$19. \quad I = I_0 e^{-\mu x}$$

I and I_0 are respectively the final and initial intensities of X-rays, μ = linear absorption coefficient of the absorbing material, x = length of the path of the absorbing material covered by the x-ray beam.

$$20. \quad I = I_0 e^{-(\mu_m)m}$$

where μ_m = mass absorption coefficient of the absorber
 m = mass per unit area of the absorbing substance

18.4 NUMERICAL EXAMPLES

Example 1: A proton is shot with a speed of 1.0×10^6 m/s along a line at an angle of 30° to the x -axis, parallel to which a uniform field of 0.20 Wb/ m^2 exists. Describe the motion of the proton.

Solution: We know that a magnetic field acting parallel to the direction of a moving charged particle does not exert a force on it. We can therefore, resolve the velocity vector $\vec{v}$ into components v_x and v_y along x and y axis respectively. As mentioned above, since the field is along x axis, v_x would not be affected by the field. Hence the proton will move in the x direction with a constant velocity.

$$v_x = 1.0 \times 10^6 \times \cos 30^\circ \text{ m/s} = 0.867 \times 10^6 \text{ m/s}$$

Further, as the applied field B is perpendicular to y component of velocity v_y , the proton would travel in a circle lying in the Y - Z plane. Hence we have

$$\frac{m v_y^2}{r} = B e v_y$$

$$\text{or } r = \frac{m v_y}{B e} = \frac{1.67 \times 10^{-27} \times 10^6 \times \frac{1}{2}}{0.20 \times 1.6 \times 10^{-19}} \text{ m}$$

$$= 2.6 \times 10^{-2} \text{ m}$$

$$\therefore \text{Time period } T = \frac{2\pi r}{v_y} = \frac{2 \times 3.142 \times 2.6 \times 10^{-2}}{0.5 \times 10^6} \text{ s}$$

$$= 3.26 \times 10^{-7} \text{ s}$$

During this time the proton will have travelled a distance parallel to the x -axis. This distance

$$= v_x t = 0.867 \times 10^6 \times 3.26 \times 10^{-7} \text{ m} = 0.28 \text{ m}$$

Thus the proton will follow a *helix like path* with a radius 0.026 m and pitch 0.28 m.

Example 2: Two electron beams having their velocities in the ratio of $1:2$ are subjected to identical magnetic fields acting at right angles to the direction of motion of electron beams. Calculate the ratio of deflections produced.

Solution: We know that an electron moving with a velocity v when subjected to a magnetic field B acting at right angles to its direction of motion bends along the arc of a circle such that

$$B e v = \frac{m v^2}{R_B}$$

where R_B = the radius of circular arc along which the electron have bent. Let D_1 and D_2 be the deflections produced in two cases and let R_{B_1} and R_{B_2}

denote the radii of circular arc along which two beams bend in the applied field.

$$\therefore D_1 = \frac{K}{R_{B_1}}$$

$$\text{and } D_2 = \frac{K}{R_{B_2}}$$

where K is a geometrical constant of the apparatus.

$$\therefore \frac{D_1}{D_2} = \frac{K}{R_{B_1}} \frac{R_{B_2}}{K} = \frac{R_{B_2}}{R_{B_1}}$$

$$\text{Also } \frac{m v^2}{R_{B_1}} = B e v$$

$$\text{or } R_{B_1} = m v / B e$$

$$\text{Further } \frac{m (2v)^2}{R_{B_2}} = B e (2v) \quad \text{or } R_{B_2} = (2v) \frac{m}{B e}$$

$$\frac{R_{B_2}}{R_{B_1}} = \frac{2}{1}$$

$$\text{Hence } \frac{D_1}{D_2} = \frac{2}{1}$$

Example 3: In Thomson's method of determining the specific charge of an electron, an electric field is maintained over a length of 0.05 m between two parallel metal plates kept at a distance of 0.08 m with a potential difference of 450 V between them. The deflection on a fluorescent screen at a distance of 0.25 m from the field is 0.04 m. A magnetic field of flux density 8×10^{-4} tesla applied over the same length as the electric field and at right angles to it causes a deflection of 0.06 m on the screen while acting alone. Calculate the value of e/m .

Solution: An electron in an electric field E experiences an acceleration eE/m in the direction of field which persists for a time l/v where l is the length of electric field. The vertical deflection suffered by the electron is given by

$$\frac{E e l}{m} \frac{l}{v} = \frac{E e l L}{m v^2}$$

where L is the distance of screen from the centre of the field. Substituting the values we get

$$0.04 = \frac{450}{0.03} \times \frac{0.05 \times 0.25}{v^2} \times \frac{e}{m} \quad (i)$$

Similarly the deflection suffered by the electron in a magnetic field of flux density B is given by

$$B e l L / m v$$

$$\text{Hence } 0.06 = \frac{8 \times 10^{-4} \times 0.05 \times 0.25}{v} \times \frac{e}{m} \quad (ii)$$

Dividing square of Eq. (ii) by Eq. (i), we get

$$\frac{0.0036}{0.04} = \frac{64 \times 10^{-8} \times (0.05)^2 \times (0.25)^2}{450 \times 0.05 \times 0.25} \times 0.03 \left(\frac{e}{m} \right)$$

or

$$\frac{e}{m} = \frac{450 \times 0.0036}{0.04 \times 0.03} \times \frac{1}{64 \times 10^{-8} \times 0.05 \times 0.25} \frac{C}{kg}$$

$$= 1.688 \times 10^{11} C/kg$$

Example 4: The following are the observations in Millikan's method of determining " e ".

Distance between discs = 15 mm

P.D between discs = 4500 V

Radius of droplet = 10^{-6} m

Coefficient of viscosity = 1.83×10^{-5} SI units

Successive velocities of the droplet in the field are $12.508 \times 10^{-4} \text{ m s}^{-1}$, $8.34 \times 10^{-4} \text{ m s}^{-1}$, $5.56 \times 10^{-4} \text{ m s}^{-1}$, $11.12 \times 10^{-4} \text{ m s}^{-1}$.

Using above data, calculate the value of the electronic charge.

Solution: In accordance with the theory of Millikan's method, the electron charge e is given by the relation

$$e = \frac{6\pi \eta r}{E} (\Delta v)_{\min}$$

From the given data, we find that the minimum change in velocity is given by

$$(\Delta v)_{\min} = 1.388 \times 10^{-4} \text{ m s}^{-1}$$

Also $E = \frac{4500}{1.5 \times 10^{-2}} \text{ V/m} = 3 \times 10^5 \text{ V/m}$

$$\therefore e = \frac{6 \times 3.142 \times 1.83 \times 10^{-5} \times 10^{-6} \times 1.388 \times 10^{-4}}{3 \times 10^5} \text{ C}$$

$$= 1.597 \times 10^{-19} \text{ C}$$

Example 5: A radio station operates at a frequency of 108.7 MHz with a power output of 250 kW. Determine the rate of emission of quanta from the station.

Solution: Let E be the energy of each quantum.

$$E = h\nu = (6.63 \times 10^{-34} \text{ J-s}) \times (108.7 \times 10^6 \text{ s}^{-1})$$

$$= 7.20 \times 10^{-26} \text{ J}$$

Let the no. of quanta emitted per unit time be N

Further the rate of emission of quanta, i.e.

$$\frac{\text{number of quanta}}{\text{time}} = \frac{\text{power output}}{\text{energy of each quanta}}$$

Thus $N = 250 \times 10^3 \text{ J s}^{-1} \times \frac{1 \text{ quantum}}{7.20 \times 10^{-26} \text{ J}}$

$$= 3.4 \times 10^{30} \text{ quanta/second}$$

Example 6: Light of wavelength $2000 \text{ \AA}$ falls on an aluminium surface (work function of aluminium 4.2 eV). Calculate

- the kinetic energy of the fastest and slowest emitted photo-electrons
- stopping potential
- cut off wavelength for aluminium.

Solution: (a) We know that

$$\begin{aligned}\text{Energy of photon} &= \frac{h c}{\lambda} \\ &= \frac{6.62 \times 10^{-34} \times 3 \times 10^8}{2000 \times 10^{-10}} \text{ J} \\ &= \frac{6.62 \times 3 \times 10^{-26}}{2 \times 10^{-7} \times 1.6 \times 10^{-19}} \text{ eV} = 6.20 \text{ eV}\end{aligned}$$

$$\text{Energy of the fastest emitted photo-electron} = h \nu - w_0$$

(where w_0 is the work function)

$$= (6.2 - 4.2) \text{ eV} = 2.0 \text{ eV}$$

Since the emitted electrons from a metal surface have an energy distribution, the minimum energy in this distribution being zero, the energy of slowest photoelectrons is also zero.

(b) Since $eV_s = \frac{1}{2} m v_{\text{max}}^2$ where $\frac{1}{2} m v_{\text{max}}^2$ is the maximum energy of the emitted photo electrons and V_s is the stopping potential, the stopping potential is 2 V .

(c) The threshold frequency is related to the work function by the relation

$$w_0 = h \nu_0 = \frac{h c}{\lambda_0}$$

$$\therefore \frac{1}{\lambda_0} = \frac{w_0}{h c} = \frac{4.2 \times 1.6 \times 10^{-19} \text{ J}}{6.62 \times 10^{-34} \text{ J-s} \times 3 \times 10^8 \text{ m s}^{-1}}$$

$$\therefore \lambda_0 = 3 \times 10^{-7} \text{ m} = 3000 \text{ \AA}$$

Example 7: Calculate the strength of the transverse magnetic field required to bend all the photo electrons within a circle of radius 50 cm when light of wavelength $4000 \text{ \AA}$ is incident on a barium emitter. Given that work function of barium is 2.5 eV .

Solution: We know that

$$\frac{1}{2} m v_{\text{max}}^2 = h \nu - \phi = \frac{h c}{\lambda} - \phi$$

$$\therefore v_{\text{max}}^2 = \frac{2}{m} \left[\frac{h c}{\lambda} - \phi \right]$$

$$= \frac{2}{9.11 \times 10^{-31}} \left[\frac{6.63 \times 10^{-34} \times 3 \times 10^8}{4 \times 10^{-7}} - 2.5 \times 1.6 \times 10^{-19} \right]$$

$$= \frac{2 \times 10^{31}}{9.11} [4.97 - 4] \times 10^{-19} (\text{m/s})^2$$

$$= 0.2129 \times 10^{12} (\text{m/s})^2$$

$$v_{\text{max}} = 4.62 \times 10^5 \text{ m s}^{-1}$$

When electrons moving with $v_{\max} = 4.62 \times 10^5$ m/s enter the magnetic field, we have

$$Be v_{\max} = \frac{m v_{\max}^2}{R_{\max}}$$

$$\begin{aligned} B &= \frac{m v_{\max}}{e R_{\max}} \\ &= \frac{9.11 \times 10^{-31} \times 4.62 \times 10^5}{1.6 \times 10^{-19} \times 0.5} \text{ tesla} \\ &= 5.26 \times 10^{-6} \text{ tesla} \end{aligned}$$

Example 8: Determine the region of the electron spectrum which liberates photo-electrons from potassium. Given that the work function of potassium is 2.24 eV.

Solution: The photo-electric equation is

$$h\nu = \frac{1}{2} m v_{\max}^2 + \phi$$

where ϕ is the work function of the photo sensitive surface. When electrons are just liberated, $v = 0$ and the corresponding frequency is

$$\begin{aligned} \frac{\phi}{h} &= \frac{2.24 \times 1.6 \times 10^{-19}}{6.6 \times 10^{-34}} \text{ s}^{-1} \\ &= 5.43 \times 10^{14} \text{ Hz} \end{aligned}$$

But since $\lambda = \frac{c}{\nu}$, the wavelength is

$$\lambda = \frac{3 \times 10^8}{5.43 \times 10^{14}} \text{ m} = 5.520 \times 10^{-7} \text{ m} = 5520 \text{ \AA}$$

Thus we find that all wavelengths shorter than 5520 Å can liberate electrons from potassium.

Example 9: Ultraviolet light of wavelength 800 Å and 700 Å when allowed to fall on hydrogen atoms in their ground state is found to liberate electrons with kinetic energy 1.8 eV and 4 eV respectively. Find the value of the Planck's constant.

Solution: We know that when a photon of a given energy $E = h\nu = hc/\lambda$ is allowed to be incident on a hydrogen atom, its energy is partly used to supply the ionisation energy E_i of the hydrogen atom and the remaining part appears as the kinetic energy K of the free electron.

Thus we have

$$\frac{hc}{\lambda} = E_i + K$$

For photons of wavelength $\lambda_1 = 800 \text{ \AA}$ and $\lambda_2 = 700 \text{ \AA}$, the energy of the liberated electrons is $K_1 = 1.8 \text{ eV}$ and $K_2 = 4.0 \text{ eV}$ respectively.

Thus we have

$$\frac{hc}{\lambda_1} = E_i + K_1$$

and
$$\frac{hc}{\lambda_2} = E_i + K_2$$

Subtracting

$$h c \left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1} \right) = K_2 - K_1$$

or
$$h = \frac{(K_2 - K_1) \lambda_1 \lambda_2}{c (\lambda_1 - \lambda_2)}$$

substituting the values, we get

$$h = \frac{(4.0 - 1.8) \times 1.6 \times 10^{-19} \times 8 \times 10^{-8} \times 7 \times 10^{-8}}{3 \times 10^8 \times (8 - 7) \times 10^{-8}} \text{ J-s}$$

$$= 6.37 \times 10^{-34} \text{ J s}$$

Example 10: Calculate the nearest distance of approach of an α -particle of energy 2.5 eV being scattered by a gold nucleus ($Z = 79$).

Solution: We know that the electrostatic potential at a distance x due to the nucleus is given by $Ze/4 \pi \epsilon_0 x$ where Ze is the charge on the nucleus.

The potential energy of an α particle when it is at a distance x , from the nucleus is given by

$$\left(\frac{Ze}{4 \pi \epsilon_0 x} \right) 2e = \frac{2Ze^2}{4 \pi \epsilon_0 x}$$

$2e$ being the charge on α particle.

Since the α particle is momentarily stopped at a distance x , its initial kinetic energy is completely changed into potential energy

$$\therefore \frac{1}{2} m v^2 = \frac{2Ze^2}{4 \pi \epsilon_0 x}$$

or
$$x = \frac{2Ze^2}{4 \pi \epsilon_0} \times \frac{1}{(\frac{1}{2} m v^2)}$$

Now energy of α particle $= \frac{1}{2} m v^2 = 2.5 \text{ MeV}$

$$= 2.5 \times 10^6 \times 1.62 \times 10^{-19} \text{ J}$$

$$= 2.5 \times 1.62 \times 10^{-13} \text{ J}$$

Substituting the values we get

$$x = \frac{2 \times 79 \times 1.6 \times 1.6 \times 10^{-38} \times 9 \times 10}{2.5 \times 1.62 \times 10^{-13}} \text{ m}$$

$$= 8.984 \times 10^{-14} \text{ m}$$

Example 11: The Rydberg constant for hydrogen is 10967700 m^{-1} . Calculate the short and long wave length limits of Lyman series.

Solution: For Lyman series, the wave number is given by

$$\bar{\nu} = \frac{1}{\lambda} = R_H \left(\frac{1}{1^2} - \frac{1}{n^2} \right)$$

For the short wavelength limit ($\lambda = \lambda_s$), $n = \infty$

$$\text{or } \bar{v}_s = \frac{1}{\lambda_s} = R_H \left(\frac{1}{1^2} - \frac{1}{\infty^2} \right) = R_H$$

$$\therefore \lambda_s = \frac{1}{R_H} = \frac{1}{10967700} \text{ m}$$

$$= 9.116 \times 10^{-8} \text{ m} = 911.6 \text{ \AA}$$

For long wavelength limit ($\lambda = \lambda_L$) $n = 2$

$$\therefore \bar{v}_L = \frac{1}{\lambda_L} = R_H \left(\frac{1}{1^2} - \frac{1}{2^2} \right) = \frac{3}{4} R_H$$

$$\therefore \lambda_L = \frac{4}{3 R_H}$$

$$= \frac{4}{3} \times 911.6 \text{ \AA} = 1215 \text{ \AA}$$

Example 12: How many times does the electron go round the first Bohr orbit of hydrogen in 1 s?

Solution: We know that the distance travelled by electron in traversing the first Bohr's orbit is given by $2\pi r$ where r is the radius of the first Bohr orbit.

Now the radius of the first Bohr orbit is given by

$$r = \frac{h^2 (4 \pi \epsilon_0)}{4 \pi^2 m e^2}$$

$$= (6.6 \times 10^{-34})^2 \times \frac{1}{9 \times 10^9}$$

$$\times \frac{1}{4 \times (3.142)^2 \times 9.1 \times 10^{-31} \times (1.6 \times 10^{-19})^2}$$

$$= 0.5292 \times 10^{-10} \text{ m}$$

If T is the time taken by the electron to traverse the first Bohr orbit, then

$$T = \frac{2 \pi r}{v}$$

where v is the velocity of electron in the first Bohr orbit and is given by

$$\frac{2 \pi e^2}{(4 \pi \epsilon_0) h} = \frac{2 \times 3.142 \times (1.6 \times 10^{-19})^2 \times 9 \times 10^9}{6.6 \times 10^{-34}} \text{ m s}^{-1}$$

$$= 2.19 \times 10^6 \text{ m s}^{-1}$$

$$\therefore T = \frac{2 \pi \times 0.5292 \times 10^{-10}}{2.19 \times 10^6} \text{ s}$$

The frequency of revolution of electron round the first Bohr orbit is given by $1/T$

$\therefore$ Number of times the electron goes round the first Bohr orbit in one second is given by

$$n = \frac{1}{T} = \frac{2.19 \times 10^6}{2 \times 3.142 \times 0.5292 \times 10^{-10}} \text{ Hz}$$

$$= 6.57 \times 10^{15} \text{ Hz}$$

Example 13: Electrons of energy 12.2 eV are fired at the hydrogen atoms in a gas discharge tube. Determine the wavelengths of the lines that can be emitted by hydrogen.

Solution: When electrons of energy 12.2 eV are fired at the hydrogen atom, the hydrogen atom would absorb this amount of energy. Assuming that the hydrogen atom was initially at ground state, the absorption of this energy would excite the atom into the energy state E given by

$$E_{\text{excited}} = E_0 + 12.2 \text{ eV}$$

where $E_0 = -13.6 \text{ eV}$ is the energy of the electron in the ground state.

$$E_{\text{excited}} = -13.6 \text{ eV} + 12.2 \text{ eV} = -1.4 \text{ eV}$$

The value of n corresponding to this state is given by the relation

$$E_{\text{excited}} = \frac{E_0}{n^2}$$

$$\text{or} \quad -1.4 \text{ eV} = -\frac{13.6 \text{ eV}}{n^2}$$

$$\text{or} \quad n = 3.12$$

Since n must be an integer, the highest state that can be reached corresponds to $n = 3$. Therefore three transitions corresponding to states $3 \rightarrow 2$; $2 \rightarrow 1$ and $3 \rightarrow 1$ are possible. Denoting by λ_1 , λ_2 and λ_3 the wavelengths corresponding to these transitions we have

$$\begin{aligned} \frac{1}{\lambda_1} &= R \left(\frac{1}{2^2} - \frac{1}{3^2} \right) = \frac{5}{36} R \\ &= \frac{5}{36} \times 1.097 \times 10^{-3} \text{ \AA}^{-1} \end{aligned}$$

$$\therefore \lambda_1 = \frac{36}{5 \times 1.097} \times 10^3 \text{ \AA} = 6563 \text{ \AA}$$

$$\text{Similarly} \quad \frac{1}{\lambda_2} = R \left(\frac{1}{1^2} - \frac{1}{2^2} \right) = \frac{3}{4} R$$

$$\text{or} \quad \lambda_2 = \frac{4}{3} R = 1215 \text{ \AA}$$

$$\text{and} \quad \lambda_3 = \frac{1}{1.097 \times 10^{-3} \left(\frac{1}{1^2} - \frac{1}{3^2} \right)} = 1026 \text{ \AA}$$

Example 14: Calculate the wavelength difference between the second Balmer line ($\lambda = 4800 \text{ \AA}$) of hydrogen and that of heavy hydrogen. Rydberg constant for hydrogen = 10967760 m^{-1} and $R_\infty = 10973740 \text{ m}^{-1}$.

Solution: The wave number of 2nd line of Balmer series for hydrogen is

$$\bar{\nu}_{H_1} = R_H \left(\frac{1}{2^2} - \frac{1}{4^2} \right) = \frac{3}{16} R_H$$

$$\therefore \lambda_{H_2} = \frac{16}{3 R_H}$$

Similarly the wavelength of 2nd line of Balmer series for heavy hydrogen is given by

$$\lambda_{D_2} = \frac{16}{3 R_D}$$

$$\therefore \lambda_{H_2} - \lambda_{D_2} = \frac{16}{3} \left(\frac{1}{R_H} - \frac{1}{R_D} \right)$$

Now we know that

$$\frac{R_\infty}{R_H} = \left(1 + \frac{m}{M_H} \right)$$

$$\therefore \frac{m}{M_H} = \frac{R_\infty - R_H}{R_H}$$

Similarly $\frac{m}{M_D} = \frac{R_\infty - R_D}{R_D}$

$$\therefore \frac{M_D}{M_H} = 2 = \frac{R_\infty - R_H}{R_\infty - R_D} \times \frac{R_D}{R_H}$$

This gives

$$\frac{1}{R_D} = \frac{R_\infty + R_H}{2 R_\infty R_H}$$

$$\begin{aligned} \text{or } R_D &= \frac{2 R_\infty R_H}{R_\infty + R_H} \\ &= \frac{1.0967760 \times 1.0973740}{1.0970750} \times 10^7 \text{ m}^{-1} \\ &= 1.0970749 \times 10^7 \text{ m}^{-1} \end{aligned}$$

$$\begin{aligned} \therefore \lambda_{H_2} - \lambda_{D_2} &= \frac{16}{3} \left[\frac{1}{1.0967760} - \frac{1}{1.0970749} \right] \times 10^{-7} \text{ m} \\ &= \frac{16}{3} \times \frac{.0002990}{1.0967760 \times 1.0970749} \times 10^{-7} \text{ m} \\ &= 1.325 \text{ \AA} \end{aligned}$$

Example 15: An X-ray tube works at a potential difference of 100,000 V. Only 0.1% of the energy of Cathode rays is converted into X-ray radiation and heat is generated in the target at the rate of 120 cal per second. What current does the tube pass and what is the energy and velocity of an electron when it reaches the target?

Solution: Operating potential = 10^5 V

$$\text{Electron energy} = 10^5 \text{ eV} = 10^5 \times 1.6 \times 10^{-19} \text{ J}$$

$$\text{Velocity of the electron} = \left(\frac{2 \times 1.6 \times 10^{-14}}{9.1 \times 10^{-31}} \right)^{1/2} \text{ ms}^{-1}$$

$$= \left(\frac{32}{9.1} \right)^{1/2} \times 10^8 \text{ m s}^{-1}$$

$$= 1.875 \times 10^8 \text{ ms}^{-1}$$

As the velocity is quite high, relativistic correction has to be applied. The relativistic expression for KE is

$$E = c^2 (m - m_0) = c^2 m_0 (\beta - 1)$$

when we have put $\beta = 1/\sqrt{1 - (v^2/c^2)}$

$$\begin{aligned} \therefore \beta - 1 &= \frac{E}{m_0 c^2} = \frac{1.6 \times 10^{-14}}{9.1 \times 10^{-31} \times 9 \times 10^{16}} = \frac{16}{81.9} \\ &= \frac{1}{5.12} \end{aligned}$$

$$\therefore \beta = \frac{1}{5.12} + 1 = 1.1953$$

$$\therefore 1 - \frac{v^2}{c^2} = \left(\frac{1}{\beta}\right)^2 = \left(\frac{1}{1.1953}\right)^2 \approx 0.7$$

$$\therefore \frac{v^2}{c^2} = 0.3$$

or $v = \sqrt{0.3} \times c = 1.643 \times 10^8 \text{ m/s}$

Let us suppose that n electrons reach the target in one second.

Amount of energy of those n electrons converted into heat $= n \times 1.6 \times 10^{-14} \times 99.9 \text{ J} = 159.84 \times 10^{-14} \text{ J}$

Also $120 \text{ Cal/s} = 120 \times 4.2 \text{ J/s}$

$$n \times 159.84 \times 10^{-14} = 120 \times 4.2$$

$$n = \frac{120 \times 4.2 \times 10^{14}}{159.84}$$

$$= 3.161 \times 10^{14} / \text{s}$$

Electronic charge $= 1.6 \times 10^{-19} \text{ C}$

$$\text{Current} = 3.161 \times 10^{14} \times 1.6 \times 10^{-19} \text{ C/s}$$

$$= 5.056 \times 10^{-5} \text{ A}$$

$$= 50.6 \mu \text{ A}$$

Example 16: The K_α line from the anti-cathode of an X-ray tube is reflected in the first order at an angle of 6° by a crystal of NaCl mounted on a Bragg Spectrometer. Calculate the wavelength of the K_α line. Given density of NaCl $= 2.17 \times 10^3 \text{ kg/m}^3$; $N = 6.023 \times 10^{26} / \text{kg-mole}$; molecular weight of NaCl $= 58.5$.

Solution: We know that

$$N = \frac{M}{2\rho d^3}$$

where N is the Avogadro's number. M , molecular weight, ρ the density and d the atomic spacing

$$\therefore d^3 = \frac{M}{2\rho N} \quad \text{or} \quad d = \left(\frac{M}{2\rho N} \right)^{1/3}$$

$$\begin{aligned} \therefore d &= \left[\frac{58.5}{2 \times 2.17 \times 10^3 \times 6.023 \times 10^{26}} \right]^{1/3} \text{ m} \\ &= 2.818 \times 10^{-10} \text{ m} = 2.818 \text{ \AA} \end{aligned}$$

Further we know that

$$2d \sin \theta = n\lambda$$

$$\begin{aligned} \therefore \lambda &= \frac{2 \times 2.818 \times \sin 6^\circ}{n} \text{ \AA} \\ &= 5.636 \times 0.1045 \text{ \AA} \\ &\approx 0.59 \text{ \AA} \end{aligned}$$

Example 17: The K_α line from molybdenum (atomic number = 42) has a wavelength of 0.7078 Å. Calculate the wavelength of the K_α line of Zinc (atomic number = 30).

Solution: We know from Moseley's law that for K series

$$\nu \propto (Z - 1)^2$$

$$\text{or} \quad \frac{c}{\lambda} \propto (Z - 1)^2$$

$$\text{or} \quad (Z - 1)^2 \lambda = \text{constant}$$

Let λ' be the wavelength of K_α line of Zinc.

Then we must have

$$(42 - 1)^2 \lambda = \lambda' (30 - 1)^2$$

$$\begin{aligned} \text{or} \quad \lambda' &= \left(\frac{41}{29} \right)^2 \times 0.7078 \text{ \AA} \\ &= 1.4148 \text{ \AA} \end{aligned}$$

Example 18: If the series limit of the Balmer series for hydrogen is 3646 Å, calculate the atomic number of the element which gives X-ray wavelengths down to 1.0 Å. Identify the element.

Solution: The limit of the Balmer series is given by

$$\begin{aligned} \bar{\nu} &= \frac{1}{\lambda} = R \left(\frac{1}{2^2} - \frac{1}{\infty^2} \right) \\ &= \frac{R}{4} \end{aligned}$$

$$\therefore R = \frac{4}{\lambda} = \frac{4}{3646} \times 10^{10} \text{ m}^{-1}$$

Further the wavelengths of the K_α series are given by the relation

$$\bar{\nu} = \frac{1}{\lambda} = R (Z - 1)^2 \left(\frac{1}{1^2} - \frac{1}{n^2} \right)$$

The maximum wave number corresponds to $n = \infty$ and, therefore, we must have

$$\bar{\nu} = \frac{1}{\lambda} = R (Z - 1)^2$$

$$\text{or} \quad (Z - 1)^2 = \frac{1}{R\lambda} = \frac{3645 \times 10^{-10}}{4 \times 1 \times 10^{-10}} = 911.5$$

$$\therefore (Z - 1) = \sqrt{911.5} \\ \simeq 30.2$$

$$\text{or} \quad Z = 31.2 \simeq 31$$

Thus the atomic number of the element concerned is 31.

The element having atomic number $Z = 31$ is *Gallium*.

Example 19: A material whose K absorption edge is $0.15 \text{ \AA}$ is irradiated with $0.13 \text{ \AA}$ X-rays. What is the maximum kinetic energy of photo-electrons that are emitted from the K shell?

Solution: We know that K shell binding energy (E_K) is given by the relation

$$E_K = \frac{h c}{\lambda_K}$$

where λ_K is the wavelength of K absorption edge.

$$\therefore E_K = \frac{12.4}{0.15} \text{ keV} = 82.7 \text{ keV}$$

The energy of the incident photon is

$$E_\nu = \frac{h c}{\lambda} = \frac{12.4}{0.13} \text{ keV} = 95.38 \text{ keV}$$

Hence the maximum energy of the photoelectrons that are emitted from the K shell is given by

$$E_\nu - E_K \\ = (95.38 - 82.7) \text{ keV} \\ = 12.68 \text{ keV}$$

Example 20: For a certain metal the K absorption edge is at $0.172 \text{ \AA}$. The wavelengths of K_α , K_β and K_γ lines of K series are $0.210 \text{ \AA}$, $0.192 \text{ \AA}$ and $0.180 \text{ \AA}$ respectively. Calculate the energies of K , L , M and N shells for the metal.

Solution: We know that the energy of a photon is given by

$$E = \frac{h c}{\lambda} \text{ eV}$$

The energy of K edge

$$= \frac{6.6 \times 10^{-34} \times 3 \times 10^8}{0.172 \times 10^{-10} \times 1.6 \times 10^{-19}} \text{ eV}$$

$$= 71.95 \text{ k eV}$$

Energy of K_α line

$$= \frac{6.6 \times 10^{-34} \times 3 \times 10^8}{0.210 \times 10^{-10} \times 1.6 \times 10^{-19}} \text{ eV} = 58.93 \text{ k eV}$$

Energy of K_β line

$$= \frac{6.6 \times 10^{-34} \times 3 \times 10^8}{0.192 \times 10^{-10} \times 1.6 \times 10^{-19}} \text{ eV} = 64.45 \text{ k eV}$$

Energy of K_γ line

$$= \frac{6.6 \times 10^{-34} \times 3 \times 10^8}{0.180 \times 10^{-10} \times 1.6 \times 10^{-19}} \text{ eV} = 68.75 \text{ k eV}$$

$$\therefore \text{Energy of } K \text{ shell} = -71.95 \text{ keV}$$

$$\text{Energy of } L \text{ shell} = -71.95 + 58.93 = -13.02 \text{ keV}$$

$$\text{Energy of } M \text{ shell} = -71.95 + 64.45 = -7.50 \text{ keV}$$

$$\text{Energy of } N \text{ shell} = -71.95 + 68.75 = -3.20 \text{ keV}$$

18.5 TRUE-FALSE STATEMENTS WITH REASONS

We are given below some statements. We have to decide whether they are true or false giving reasons in support of our choice

1. The charge carriers in a discharge tube at very low pressures are electrons.
2. The trajectory followed by an electron when subjected to a magnetic field acting at right angles to its direction of motion is a parabola.
3. The lateral deflection suffered by an electron when subjected to magnetic field at right angles to its direction of motion is directly proportional to the radius of the circular arc along which it bends in the magnetic field.
4. Photons do not have a finite rest mass.
5. The kinetic energy of photo-electrons emitted by a photo-sensitive surface depends on the intensity of the incident radiation.
6. Photo emission from a photo-sensitive surface is possible only if the incident radiation has a frequency above the threshold frequency.
7. The different lines in the Lyman series have their wavelengths lying between 911 Å and 1215 Å.
8. If the angular momentum of the earth due to its motion around the sun were quantized according to Bohr's relation $L = nh/2\pi$, then the quantum number corresponding to this quantization would be 2.5×10^{74} .
9. If elements with principal quantum number $n > 4$ were not allowed in nature, then the number of possible elements would be 64.
10. Rydberg's constant varies with the mass number of a given element.
11. The ratio of the wave number of H_α line of Balmer series for hydrogen and that of H_α line of Balmer series for singly ionized helium is 4.
12. The shortest wavelength of X-rays emitted from an X-ray tube depends on the current in the tube.

13. Characteristic X-ray spectra is simple compared to optical spectra.
14. X-rays cannot be diffracted by means of a grating.
15. There exists a sharp limit on the short wavelength side for each continuous X-ray spectrum.

ANSWERS

1. *False* In addition to the electrons, we also have positive ions. At very low pressures, the mean free path is large and hence the negative ions are not formed. The charge carriers are, therefore, electrons and positive ions only.
2. *False* A charged particle subjected to a uniform magnetic field perpendicular to its direction of motion, moves entirely in the plane perpendicular to the magnetic field $\vec{B}$ along a circular path of radius mv/eB , where m , v and e are respectively the mass, velocity and charge of the particle.
3. *False* The deflection produced is inversely proportional to the radius of the circular arc along which it bends in the magnetic field.
4. *True* We cannot assign any finite 'rest mass' to the photons as they always move with the speed of light. If we were to assign a non-zero rest mass to them, their observed mass should be infinite since

$$m = \frac{m_0}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}}$$

so that $m \rightarrow \infty$ as $v \rightarrow c$ for a finite non-zero value of m_0 .

5. *False* The kinetic energy of the emitted photo-electrons changes only with a change in the frequency of the incident radiation. A change in intensity only changes the number of the photo-electrons emitted.
6. *True* Unless $h\nu > W$, the work function of the photo-sensitive surface, no photo-emission is possible.
7. *True* The short and long wavelength limits of Lyman series are $\lambda_s = 1/R$ and $\lambda_l = 4/3R$ respectively. With $R = 10967700 \text{ m}^{-1}$, we get the values given in the statement.
8. *True* The angular momentum of the earth ($= m r^2 \omega$) has to be equal to $nh/2\pi$. This gives $n = m r^2 2\pi/h$. Substituting the numerical values of the earth's mass, radius and the angular velocity, we get the given value of n .
9. *False* The maximum number of electrons allowed in an orbit being $2n^2$, the required number is $2(1^2 + 2^2 + 3^2 + 4^2) = 60$.
10. *True* The 'reduced mass' of the electron [$\mu = m M/(m + M)$] being dependent on the mass of the nucleus (M), the Rydberg constant also varies with the mass number of the given element.
11. *False* The ratio is not exactly equal to four but slightly different from four because of the dependence of the Rydberg constant on the mass number of the element concerned.
12. *False* The shortest wavelength of the X-rays emitted depends on the energy of the electrons incident on the target. This, in turn, depends on the potential through which they have 'fallen'. In fact

$$\lambda_{\min} = hc/eV$$

13. *True* While discussing the details of the characteristic X-rays spectrum, we can assume that the electrons with a particular value of n differ very little among themselves. Also the energy differences between the energy of the N shell ($n = 4$) and the energy of the free electron can be neglected. This leads to the result that the X-ray spectra of all heavy elements are similar in character.
14. *True* The short wavelength of the X-rays (compared to the grating constant of optical grating) makes it difficult to observe X-ray diffraction with gratings.
15. *True* This sharp limit on the short wavelength side is dependent on the voltage applied to the incident electrons, being given by

$$\lambda_{\min} = \frac{hc}{eV}$$

18.6 MULTIPLE CHOICE QUESTIONS

Select the best alternative in each of the following:

- Cathode rays of velocity 10^6 m/s describe an approximate circular path of radius 1 meter in an electric field of 300 volt/cm. If the velocity of cathode rays is doubled the value of electric field needed so that the rays describe the same circular path is
 - 1200 volt/cm
 - 1200 volt/m
 - 600 volt/m
 - 2400 volt/cm
- An electron is not deflected on passing through a certain region, because
 - there is no magnetic field in that region
 - there may be magnetic field but velocity of the electron may be parallel to the direction of magnetic field.
 - electron is a chargeless particle
 - none of the above
- The frequency of the charge circulating at right angles to a uniform magnetic field does not depend upon
 - the speed of the charge
 - mass of the charge
 - charge of the particle
 - magnetic field
- The mass of a proton is 1847 times that of an electron. An electron and a proton are projected into a uniform electric field in a direction at right angles to the direction of the field with the same initial kinetic energy. Then
 - the electron trajectory will be less curved than the proton trajectory.
 - the proton trajectory will be less curved than the electron trajectory.
 - both the trajectories will be equally curved
 - none of the above
- Two electron beams having velocities in the ratio of 1 : 2 are subjected separately to identical magnetic fields. The ratio of deflections produced is
 - 4 : 1
 - 1 : 2
 - 1 : 4
 - 2 : 1
- You are sitting in a room in which uniform magnetic field is present in vertically downward direction. When an electron is projected in horizontal direction, it will be moving in a circular path with constant speed
 - clockwise in the vertical plane
 - clockwise in the horizontal plane
 - anti-clockwise in the horizontal plane
 - anti-clockwise in the vertical plane

7. The work function for aluminium surface is 4.2 eV. The cut off wavelength for the photoelectric effect for the surface is
 - (a) 2955 Å
 - (b) 3100 Å
 - (c) 4200 Å
 - (d) 1500 Å
8. The work function for aluminium surface is 4.2 eV and that for sodium surface is 2.0 eV. The two metals were illuminated with appropriate radiations so as to cause photo emission. Then
 - (a) both aluminium and sodium will have same threshold frequency
 - (b) the threshold frequency of aluminium will be more than that of sodium
 - (c) the threshold frequency of aluminium will be less than that of sodium
 - (d) none of the above
9. X-rays of frequency ν are used to irradiate sodium and copper surface in two separate experiments and the stopping potential determined. Then
 - (a) the stopping potential is more for copper than for sodium
 - (b) the stopping potential is more for sodium than for copper
 - (c) the stopping potential is same for sodium and copper
 - (d) none of the above
10. A Caesium photo cell with a steady potential difference of 60 volt across it, is illuminated by a small bright light placed 50 cm away. When the same light is placed one meter away the electrons crossing the photo cell
 - (a) each carry one quarter of their previous energy
 - (b) each carry one quarter of their previous momentum
 - (c) are half as numerous
 - (d) are one quarter as numerous
11. An image of the sun is formed by a lens of focal length 30 cm on the metal surface of a photo-electric cell and it produces a current I . The lens forming the image is then replaced by another of the same diameter but of focal length 15 cm. The photo-electric current in this case will be
 - (a) $I/2$
 - (b) $2 I$
 - (c) I
 - (d) $4 I$
12. In a Rutherford α -scattering experiment with thin gold foil, 238 scintillations per minute are observed at an angle of 30° . The number of scintillations per minute at an angle of 120° will be
 - (a) 1.900
 - (b) 0.950
 - (c) 3.800
 - (d) 2.136
13. The different lines in the Lyman series have their wavelengths lying between
 - (a) 0 to ∞
 - (b) 911 Å to 1215 Å
 - (c) 1000 Å to 1500 Å
 - (d) 500 Å to 1000 Å
14. The shortest wavelength in the Lyman Series is 911.6 Å. Then the longest wavelength in the Lyman's Series is
 - (a) 1215 Å
 - (b) ∞
 - (c) 2430 Å
 - (d) 600 Å
15. Which of the following transitions in a hydrogen atom emits the photon of lowest frequency
 - (a) $n = 2$ to $n = 1$
 - (b) $n = 4$ to $n = 2$
 - (c) $n = 4$ to $n = 3$
 - (d) $n = 3$ to $n = 1$
16. How many times does the electron go round the first Bohr orbit in a second
 - (a) 6.57×10^8
 - (b) 6.57×10^{10}
 - (c) 6.57×10^{13}
 - (d) 6.57×10^{15}

17. If elements with principal quantum number $n = 3$ were not allowed, the number of possible elements would be
 - (a) 28
 - (b) 90
 - (c) 32
 - (d) 64
18. An X-rays tube produces X-rays whose shortest wavelength is $0.2475 \text{ \AA}$. The operating voltage of the tube must be
 - (a) 5 kV
 - (b) 25 kV
 - (c) 50 kV
 - (d) 100 kV
19. Electromagnetic theory demands that the acceleration of any charged particle must be accompanied by the emission of electro-magnetic waves. The target in an X-ray tube is bombarded only with high speed electrons because
 - (a) electrons have the smallest possible charge
 - (b) electronic mass is very small
 - (c) electrons are common constituent of all matter
 - (d) electrons can be produced very conveniently
20. When the accelerating voltage applied on the electrons is increased beyond a critical value
 - (a) the spectrum of white radiation is unaffected
 - (b) only the intensities of various wavelengths is increased
 - (c) only the wavelength of characteristic radiation is affected
 - (d) the intensities of characteristic lines relative to the white spectrum are increased but there is no change in their wavelength
21. X-rays can be diffracted from crystals like the diffraction of light from a grating. The crystal grating for X-rays has a grating element
 - (a) equal to the distance between any two nearest atoms
 - (b) equal to the size of crystal
 - (c) which is an integral multiple of the wavelength of X-rays
 - (d) none of the above
22. In a continuum spectrum the limiting frequency is
 - (a) inversely proportional to the potential through which electrons have been accelerated
 - (b) is directly proportional to the accelerating potential
 - (c) not dependent upon the accelerating potential
 - (d) is dependent upon the nature of the target material
23. The frequency of K_{α} line of a source of atomic number Z is proportional to
 - (a) Z^2
 - (b) $(Z - 1)^2$
 - (c) $1/Z$
 - (d) Z
24. The wavelength of K_{α} line from an element of atomic number 41 is λ . Then the wave length of K_{α} line of an element of atomic number 21 is
 - (a) 4λ
 - (b) $\lambda/4$
 - (c) 3.08λ
 - (d) 0.26λ
25. An X-ray photo has a wavelength of $0.02 \text{ \AA}$. Its momentum is
 - (a) $3.313 \times 10^{-22} \text{ Kg m/s}$
 - (b) $6.626 \times 10^{-21} \text{ Kg m/s}$
 - (c) $6.626 \times 10^{-24} \text{ Kg m/s}$
 - (d) none of the above.

18.7 NUMERICAL EXERCISES

1. A 50 MeV proton is falling vertically downward in a uniform field of magnetic induction 1.5 Wb/m^2 pointing horizontally from south to north. Find the force exerted on it.

2. A narrow beam of electrons, accelerated by a potential difference of 250 V, passes between two parallel metal plates 0.030 m long and 0.005 m apart in an evacuated tube, and then strikes a fluorescent screen 0.175 m from the centre of the plates. A potential difference of 10 V is established between the plates. What is the deflection produced on the screen?
3. In a Cathode ray tube a potential difference of 3000 V is maintained between the electrodes 2 cm apart. A magnetic field of 2.5×10^{-4} Wb/m² at right angles to the electric field gives no deflection of the electron beam which received an initial acceleration by a potential difference of 10000 V. Calculate the value of e/m .
4. A charged oil drop is held stationary between two horizontal plates of a capacitor when a field of 576 kV/m is applied between them. When the field is removed the drop falls freely with a steady velocity of 0.120×10^{-3} m/s. Calculate the radius and charge on the drop. Given that $\eta = 0.109 \times 10^{-4}$ kg m⁻¹ s⁻¹ and the difference between the densities of oil and air may be taken as 800 kg m⁻³.
5. The most energetic electrons emitted from a surface by 3500 Å photons are found to be bent in a 0.18 m circle by a magnetic field of 1.5×10^{-5} Wb/m². Calculate the work function for the material.
6. Photo-electrons, of charge e and constant mass m are liberated in a vacuum and with negligible velocities from the negative plate of a parallel plate capacitor. A magnetic field with an induction B is applied parallel to the plates. The separation between the plates is x . Show that a potential difference

$$V = \frac{1}{2} x^2 \left(\frac{e}{m} \right) B^2$$

when applied across the capacitor will enable the electrons to just reach the positive plate.

7. Light of wavelength 4500 Å is incident on two photo-electric tubes. The emitter in the first tube has a threshold wavelength of 6000 Å and the emitter in the second tube has a work function twice as large as in the first tube. Find the stopping potential in each of the tubes.
8. The ionization energy of a hydrogen atom is 13.6 eV. A photon of wavelength 600×10^{-10} m is absorbed by the hydrogen atom. Calculate the kinetic energy of the ejected electron.
9. The kinetic energies of photo electrons range from zero to 4.0×10^{-19} J, when light of wavelength 3000 Å falls on a surface. What is the stopping potential for this light?
10. The wave numbers of a spectral series are 4857 Å, 19932 Å, 25215 Å, 27665 Å and 27997 Å. Calculate the limit of this series.
11. The convergence limit for an element is 43488 cm⁻¹. Calculate the ionisation potential.
12. Determine the minimum energy that must be given to a hydrogen atom so that it can emit the $H\beta$ line of the Balmer series.
13. The life time of the first excited state of hydrogen is 10^{-8} s. How many revolutions will an electron make in this excited state of hydrogen?
14. Determine the Rydberg constant and the ionization potential for positronium.
15. What should be the potential difference through which an electron must be accelerated from rest so that when it falls on a target the short wave limit of its continuous spectrum shall be 1.00×10^{-10} m? Also calculate the speed of the electron.
16. Calculate the smallest grazing angle at which the K line of copper ($\lambda = 1.449$ Å) will be reflected from a quartz crystal which has an atomic spacing = 4.2555 Å. What is the highest reflecting order that could be observed with this radiation?

17. Stopping potentials of 24, 100, 110 and 115 kV are measured for photo electrons emitted from a certain element when it is irradiated with monochromatic X-rays. If this element is used as a target in an X-ray tube, what will be the wavelength of the K_{α} line?
18. The wave length for Cobalt ($Z = 27$) for K_{α} line is 1.785 Å. The impurities present give two lines at 2.285 Å and 1.537 Å. Identify these elements using Moseley's law.

ANSWERS

MULTIPLE CHOICE QUESTIONS

- | | | | | |
|---------|---------|---------|---------|---------|
| 1. (a) | 2. (b) | 3. (a) | 4. (b) | 5. (d) |
| 6. (b) | 7. (a) | 8. (c) | 9. (b) | 10. (d) |
| 11. (c) | 12. (a) | 13. (b) | 14. (a) | 15. (c) |
| 16. (d) | 17. (a) | 18. (c) | 19. (b) | 20. (d) |
| 21. (a) | 22. (b) | 23. (b) | 24. (a) | 25. (a) |

NUMERICAL EXERCISES

1. 7.4×10^{-12} N, the force is in horizontal plane in west east direction.

2. 0.02 m

3. 1.8×10^{-11} C/kg

4. 1.12×10^{-4} m 7.28×10^{-18} C

5. 2.90 eV

7. 0.69 V

there is no photo electric emission from the second tube

8. 11.36×10^{-18} J

9. 2.5 V

10. 32033

11. 54 V

12. 2.55 eV.

13. 8.3×10^6 revolutions

14. 0.5485×10^{-2} Å⁻¹

6.8 eV

15. 12.4 kV

6.61×10^7 m/s

16. $10^\circ 30'$

$n \times 0.1821 < L$

17. 0.163 Å

18. Chromium ($Z = 24$)

Copper [$Z = 29$]

Radioactivity and Atomic Nucleus

19.1 REVIEW OF BASIC CONCEPTS

19.1.1 Radioactivity The phenomenon of radioactivity was a chance discovery by Henry Becquerel who observed that photographic plates wrapped in black paper were affected by radiations from some uranium salts placed nearby. The phenomenon involves the spontaneous disintegration of the nuclei of heavy elements by the emission of highly penetrating radiations.

Rutherford and Soddy first suggested that radioactivity is due to the spontaneous transformation of one element into another. Elements that are unstable emit radiations to transform themselves into less unstable elements. We call an element as having *natural radioactivity*, if it is naturally unstable. We also have artificial radioactivity or induced radioactivity in which the instability is induced by an external agency.

19.1.2 Radioactive Radiations The radiations emitted by a radioactive element are found to be of three kinds.

(i) *Alpha-rays* They consist of alpha particles. An alpha particle is nothing but a helium nucleus (${}_2\text{He}^4$) having 2 protons and 2 neutrons. It has a positive charge equal to the charge of 2 protons. Alpha rays are the most ionizing but the least penetrative of the three radiations. They have velocities of the order of $10^6 - 10^7 \text{ m s}^{-1}$ and energies from 4 MeV to 9 MeV. They are also characterized by a well-defined range in air.

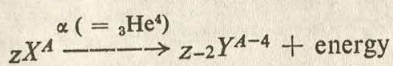
(ii) *Beta-rays* They are negative (or positive) electrons whose energy as well as ionizing power are much less than those of α -rays. Their velocity is much higher than that of α -rays. They are 100 times more penetrating than α -rays. They, however, do not have a well-defined range in air.

(iii) *Gamma-rays* They are electromagnetic waves of very short wavelengths. They originate in the nucleus, have a very low ionizing power but a very high penetrating power.

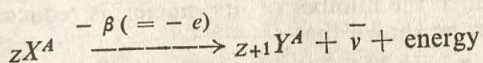
19.1.3 Laws of Radioactivity The following two basic laws of radioactivity, the *displacement law* and the *statistical law*, are valid for both natural and artificial radioactivity.

(i) *The displacement law* This law enables us to relate the product nucleus with the parent nucleus as a result of the emission of an α -particle, a β -particle or a γ -ray photon.

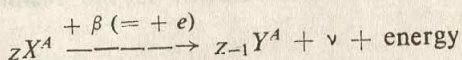
(a) When a nucleus X with atomic number Z and mass number A (${}_Z X^A$) emits an α -particle, it is transformed into an element Y with atomic number $(Z-2)$ and mass number $(A-4)$



(b) When an element emits a negative β -particle (an electron), its atomic number goes up by 1 while its mass number remains unchanged



where $\bar{\nu}$ stands for an anti-neutrino. On the other hand, when a positive β -particle (a positron) is emitted, the atomic number goes down by 1 leaving the mass number unchanged



where ν stands for a neutrino.

(c) On the emission of a γ -ray photon, the mass number as well as the atomic number of the product nucleus remain the same as those of the parent nucleus.

(ii) *The statistical law* This law may also be called the *law of radioactivity decay*. According to this law, the rate of decay of a given radioactive element at any instant is proportional to the total number (N) of its nuclei present at that instant

$$-\frac{dN}{dt} \propto N$$

$$\text{or} \quad -\frac{dN}{dt} = \lambda N$$

The constant of proportionality, λ , is known as the *disintegration constant* for the given radioactive nucleus.

If we start with N_0 nuclei of the given radioactive element (i.e. $N = N_0$ at $t = 0$), the above law can be integrated to give

$$N = N_0 e^{-\lambda t}$$

showing that radioactive decay follows an exponential law (see Fig. 19.1).

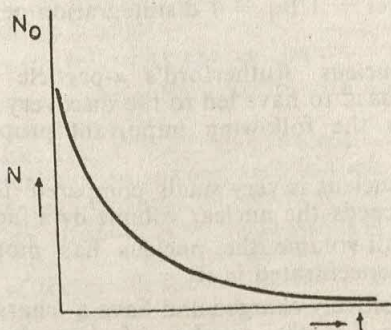


Fig. 19.1

The number N of the nuclei present in a mass m gram of a radioactive element of mass number A can be obtained from the knowledge of Avogadro's number = 6.02×10^{23} per gram mole:

$$N = 6.02 \times 10^{23} \frac{m}{A}$$

19.1.4 Half Life and Average Life The half-life (T) of a radioactive element is the time in which the number of its nuclei is reduced to one-half its initial value. It can be shown that

$$T = \frac{\log_e 2}{\lambda} = \frac{0.693}{\lambda}$$

The average life T_{av} of a radioactive element is the reciprocal of the disintegration constant.

$$T_{av} = \frac{1}{\lambda}$$

The activity ($= \left| \frac{dN}{dt} \right|$) of a radioactive element at any instant is, of course, just λN :

$$\left| \frac{dN}{dt} \right| = \lambda N$$

19.1.5 Units of Radioactivity The following units have been used for 'measuring' the radioactivity of a radioactive element.

(i) *The curie* The curie is the activity of a radioactive element that is disintegrating at the rate of 3.7×10^{10} disintegrations per second, i.e.

$$1 \text{ curie} = 3.7 \times 10^{10} \text{ disintegrations per second}$$

(ii) *The rutherford* The rutherford is the activity of a radioactive element that is disintegrating at the rate of 1 million disintegrations per second, i.e.

$$1 \text{ rutherford} = 10^6 \text{ disintegrations per second}$$

(iii) *The becquerel* This is the unit of radioactivity in the SI units.

$$1 \text{ becquerel} = 1 \text{ Bq} = 1 \text{ disintegration per second}$$

19.1.6 The Atomic Nucleus Rutherford's α -particle scattering experiments in 1911 can be said to have led to the discovery of the atomic nucleus. We now assign the following important properties to the atomic nucleus:

1. The size of the nucleus is very small compared to that of the atom. The atomic volume exceeds the nuclear volume by a factor of about 10^{15} .
2. In spite of its small volume, the nucleus has more than 99.99% of the mass of the atom concentrated in it.
3. All nuclei are positively charged and have a charge Z times that of an electron. The integer Z is the number of the orbital electrons in a neutral atom.

4. Except for small asymmetries, the positive charge of the nucleus is uniformly distributed in it. This permits us to think of the nucleus as being nearly spherical in shape.

5. The nuclear volume is (substantially) proportional to the number of the nucleons (protons and neutrons) in it. This implies a constant density of nuclear matter and a relation

$$R = R_0 A^{1/3}$$

for the nuclear radius. The 'constant' R_0 varies slightly from one nucleus to another; the range of variation being about 1.2 to 1.5 fermi (1 fermi = 10^{-15} m). It is nearly constant for elements with $A > 20$.

6. The nucleus of an element of mass number A and atomic number Z is made up of Z protons and $(A - Z)$ neutrons.

19.1.7 Binding Energy The mass M of an atomic nucleus made up of Z protons and $(A - Z)$ neutrons is always found to be less than the sum of the masses of these particles in the free state. The difference is known as the *mass defect* (Δm) of the given nucleus

$$\Delta m = Zm_p + (A - Z)m_n - M$$

The existence of a mass defect means that we need a minimum 'external' energy to tear apart the individual nucleons composing the nucleus. In terms of the Einstein's mass energy relation, we have

$$\text{B.E} = (\Delta m) c^2 = [Zm_p + (A - Z)m_n - M] c^2$$

where c is the velocity of light in free space. Since the nucleus has a total of A nucleons,

$$\text{Binding energy per nucleon} = \frac{[Zm_p + (A - Z)m_n - M] c^2}{A}$$

Using mass spectrometers, we generally determine the mass M of the atom rather than the mass of the corresponding nucleus. We can then still use the above relations for B.E and B.E per nucleon, provided we understand m_p to represent the mass of the normal neutral hydrogen atom. This replacement will result in a cancellation of the (extra) mass of the Z orbital electrons.

19.1.8 Variation of Binding Energy The binding energy per nucleon is found to be different for different nuclei. Figure 19.2 shows the variation of this quantity with mass number.

This graph shows the following characteristics:

1. The binding energy per nucleon starts with a low value for deuteron and then increases rapidly.
2. Some nuclei like He^4 , C^{12} , O^{16} have exceptionally high values of binding energy per nucleon as compared to their neighbours.
3. The binding energy per nucleon increases to a maximum value of nearly 8.5 MeV at about $A = 60$. It again decreases slowly and drops to a value of nearly 7.5 MeV, at about $A = 240$.
4. An analysis of the graph shows that the higher the binding energy per nucleon the more stable is the atom and vice versa.

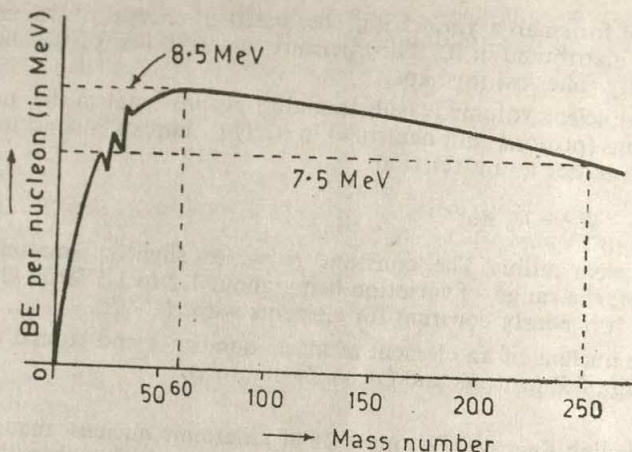


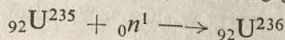
Fig. 19.2

19.1.9 Energy Release from Nuclei A closer look at the binding energy per nucleon versus mass number curve (Fig. 19.2) reveals that we can release energy from the nuclei by the following two general processes.

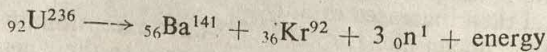
1. By combining two lighter nuclei (with A below 20) to form a more massive nucleus.
2. By splitting up a heavy nucleus (with A above 200) into two or more lighter fragments of comparable masses.

In both these processes the total binding energy of the products exceeds that of the reactants. The 'excess energy' is consequently released. These processes are called (1) nuclear fusion and (2) nuclear fission, respectively. Whereas fusion energy is believed to be the source of energy of the sun and stars, fission energy is the source of energy in atomic reactors and the atom bomb.

19.1.10 Nuclear Fission The discovery of the process of nuclear fission followed a great deal of drama and confusion following the study of nuclear reactions brought about by neutrons. Otto Hahn and Fritz Strassman, in 1939, discovered that when neutrons were 'captured' by uranium nuclei, the uranium nucleus seemed to be splitting apart into two nearly equal fragments. The 'proper' interpretation of this phenomenon of nuclear fission was given by two Austrian physicists, Lise Meitner and R Frisch, who viewed the splitting of a uranium nucleus, as a result of the bombardment of neutrons on them, as a two stage nuclear reaction:



and



The energy released per fission is about 200 MeV—much more than the energy released in the usual nuclear reactions. This makes the fission reaction a particularly suitable source of energy.

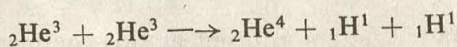
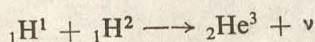
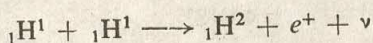
The fission reaction given above has a unique feature. Apart from the fission fragments, the reaction results in the release of 2 to 3 neutrons—the very particles that initiated the reaction. So fission after fission, the neutrons present in a bulk sample of uranium increase in a geometric ratio.

The rate of energy release also increases similarly in a geometric ratio. The fission reaction is thus a *self sustaining chain reaction*.

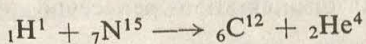
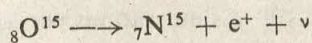
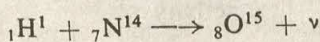
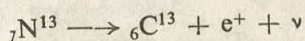
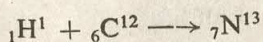
When the number of neutrons released per fission is limited to one per fission by *absorption* of excess neutrons, the chain reaction is a *controlled* one and is used in nuclear reactors. When there is no such control on the number of released neutrons, we have an uncontrolled chain reaction and this is the source of energy in the atom bomb.

An essential part of a controlled fission reaction is what is known as the *moderator*. The role of the moderator is to slow down the neutrons released in fissions so that they may be easily absorbed by another ${}_{92}\text{U}^{235}$ nucleus. Media which contain nuclei of masses comparable to the neutron are found to act as efficient moderators.

19.1.11 Nuclear Fusion The process of nuclear fusion consists in the 'combination' of two light nuclei to form a stable nucleus of mass less than the total initial mass. It is believed to be the main source of energy for the sun and the stars. The fusion reaction in stars is believed to occur either via the *proton-proton cycle* or the *carbon-cycle*. The proton-proton cycle is as follows:



The energy released in this sequence works out to be 24.7 MeV. The carbon-cycle proceeds in the following sequence



The energy released in this sequence also is 24.7 MeV. The initial ${}_6\text{C}^{12}$ is basically acting as a catalyst for the reaction.

The positrons (e^+) produced in the reaction combine with electrons and their mutual annihilation also results in release of energy as per the mass-energy relation. Such fusion reactions are possible only at very high temperatures ($\sim 10^7$ K) and are also known as *thermonuclear reactions*.

Bringing about nuclear fusion is a difficult task as we can fuse two nuclei together only if their mutual repulsive forces can be overcome. The nuclei have to be accelerated to very high energies corresponding to temperatures of about 10^7 K. As such, it has not been possible so far to use 'controlled fusion reactions' in the same way as controlled fission reactions.

19.2 SI UNITS

Table 19.1

Physical Quantity	SI Units
Half life	second (s)
Average life	second (s)
'Radioactivity'	becquerel (Bq)
Binding energy	Joule (J)
Mass defect	kilogram (kg) or atomic mass unit (amu)

19.3 Formulae

- ${}_Z X^A \xrightarrow{\alpha} {}_{Z-2} Y^{A-4} + \text{energy}$
- ${}_Z X^A \xrightarrow{-\beta} {}_{Z+1} Y^A + \bar{\nu} + \text{energy}$
- ${}_Z X^A \xrightarrow{+\beta} {}_{Z-1} Y^A + \nu + \text{energy}$
- $-\frac{dN}{dt} = \lambda N \quad ; \quad \lambda = \text{disintegration constant}$
- $N = N_0 e^{-\lambda t}$
- $T = \frac{\log_e 2}{\lambda} \quad ; \quad T = \text{half life}$
- $T_{\text{av}} = \frac{1}{\lambda} \quad ; \quad T_{\text{av}} = \text{average life}$
- $\left| \frac{dN}{dt} \right| = \lambda N \quad : \quad \left| \frac{dN}{dt} \right| = \text{activity}$
- 1 curie = 3.7×10^{10} disintegrations per second

1 rutherford = 10^6 disintegrations per second

1 becquerel = 1 disintegration per second
- $R = R_0 A^{1/3} \quad ; \quad R = \text{nuclear radius of a nucleus of mass number } A,$

$R_0 = \text{a 'constant' (varies between 1.2 and 1.5 fermi)}$
- $\Delta m = [Zm_p + (A - Z)m_n - M]$

$\Delta m = \text{mass defect of a nucleus of mass } M$

$m_p = \text{mass of a proton} \quad , \quad Z = \text{atomic number}$

$m_n = \text{mass of a neutron} \quad , \quad A = \text{mass number}$

12. Binding energy (BE) $= \Delta mc^2$
 $= [Zm_p + (A - Z)m_n - M] c^2$
 $= [Zm_H + (A - Z)m_n - M_{A,Z}] c^2$
 m_H = mass of the normal hydrogen atom
 $M_{A,Z}$ = mass of the atom of atomic number Z and mass number A .
13. B.E per nucleon $= \frac{1}{A} [Zm_H + (A - Z)m_n - M] c^2$
14. ${}_{92}\text{U}^{235} + {}_0\text{n}^1 \rightarrow {}_{56}\text{Ba}^{141} + {}_{36}\text{Kr}^{92} + 3{}_0\text{n}^1 + \text{energy}$
15. $4{}_1\text{H}^1 \rightarrow {}_2\text{He}^4 + 2e^+ + 4\nu + \text{energy}$
16. $E = mc^2$
 For $m = 1 \text{ kg}$, $E = 9 \times 10^{16} \text{ J}$
17. $1 \text{ eV} = 1.6 \times 10^{-19} \text{ J}$
 $1 \text{ MeV} = 10^6 \text{ eV} = 1.6 \times 10^{-13} \text{ J}$
18. $1 \text{ amu} = 1.66 \times 10^{-27} \text{ kg}$
 $= 1.492 \times 10^{-10} \text{ J} = 931.5 \text{ MeV}$
19. $m_e = 9.11 \times 10^{-31} \text{ kg} = 5.5 \times 10^{-4} \text{ amu} = 0.51 \text{ MeV}$
 $m_p = 1.672 \times 10^{-27} \text{ kg} = 1.00728 \text{ amu} = 938.2 \text{ MeV}$
 $m_n = 1.674 \times 10^{-27} \text{ kg} = 1.008665 \text{ amu} = 939.5 \text{ MeV}$
 $m_H = 1.673 \times 10^{-27} \text{ kg} = 1.007825 \text{ amu} = 938.7 \text{ MeV}$

19.4 NUMERICAL EXAMPLES

Example 1: Calculate the approximate mass of uranium which must undergo fission to produce the same energy as 100 tons of coal. Assume that 1 ton = 10^6 g , heat of combustion of coal = 8000 calories per gram, 1 calorie = 4 joule, 1 fission process of uranium-235 releases 200 MeV.

Solution: Mass of coal = 100 tons = $100 \times 10^6 \text{ g} = 10^8 \text{ g}$

Heat of combustion of coal = 8000 cal g^{-1}

Total heat produced on combustion of 10^8 g of coal

$$= 8000 \times 10^8 = 8 \times 10^{11} \text{ cal}$$

$$= 8 \times 4 \times 10^{11} \text{ J}$$

$$= 32 \times 10^{11} \text{ J}$$

Thus we have to calculate the mass of uranium which will undergo fission to produce $32 \times 10^{11} \text{ J}$ of energy.

Energy released per fission

$$= 200 \times 10^6 \times 1.6 \times 10^{-19} \text{ J}$$

$$= 3.2 \times 10^{-11} \text{ J}$$

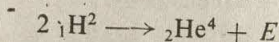
Number of U^{235} atoms required to produce this energy

$$= \frac{32 \times 10^{11}}{3.2 \times 10^{-11}} = 10^{23} \text{ atoms}$$

Since 6.02×10^{23} atoms of U^{235} have a mass of 235 g, the mass of 10^{23} atoms

$$= \frac{235 \times 10^{23}}{6.02 \times 10^{23}} \text{ g} = 39.2 \text{ g}.$$

Example 2: The fusion reaction



is proposed to be used for the production of industrial power. Assuming the efficiency of the process to be 30%, find how many grams of deuterium will be consumed in a day for an output of 50,000 kW. Given

$$\text{Mass of } {}_1\text{H}^2 = 2.01478 \text{ amu}$$

$$\text{Mass of } {}_2\text{He}^4 = 4.00388 \text{ amu}$$

Solution: The proposed fusion reaction involves the fusion of two deuterium atoms to form a helium atom with the release of energy.

$$\text{Mass of 2 deuterium atoms} = 2 \times 2.01478 = 4.02956 \text{ amu}$$

$$\text{Mass of a helium atom} = 4.00388 \text{ amu}$$

$$\text{Mass defect } \Delta m = 4.02956 - 4.00388 = 0.02568 \text{ amu}$$

$$\text{Energy released} = 0.02568 \times 931 = 23.91 \text{ MeV}$$

$$\text{Actual output of energy} = \frac{23.91 \times 30}{100} = 7.173 \text{ MeV}$$

$$\begin{aligned} \text{Average energy output per deuterium atom} &= \frac{7.173}{2} = 3.587 \text{ MeV} \\ &= 3.587 \times 1.6 \times 10^{-13} \text{ J} \end{aligned}$$

$$\text{The output required} = 50,000 \text{ kW} = 5 \times 10^7 \text{ J s}^{-1}$$

Number of deuterium atoms required per second for producing $5 \times 10^7 \text{ J}$ of energy

$$= \frac{5 \times 10^7}{3.587 \times 1.6 \times 10^{-13}} = 871.4 \times 10^7 \text{ atoms per second}$$

Since 6.02×10^{23} atoms of deuterium have a mass of 2.01478 g, the mass of 871.4×10^{17} atoms

$$= \frac{2.01478 \times 871.4 \times 10^7 \text{ g}}{6.023 \times 10^{23}} = 291.7 \times 10^{-6} \text{ g}$$

This is the consumption of deuterium in one second. Therefore, the consumption in one day will be

$$291.7 \times 10^{-6} \times 24 \times 60 \times 60 \text{ g} = 25.22 \text{ g}$$

Example 3: Calculate the electrostatic potential energy between two equal nuclei produced by the fission of ${}_{92}\text{U}^{235}$ at the moment of their separation. Given $R_0 = 1.3 \times 10^{-15} \text{ m}$

Solution: Let R be the radius of either nucleus formed at the moment of the fission of ${}_{92}\text{U}^{235}$. Then

$$R = 1.3 \times 10^{-15} \left(\frac{235}{2} \right)^{1/3} \text{ m} = 1.3 \times 4.88 \times 10^{-15} \text{ m}$$

The electrostatic potential energy is given by

$$E = \frac{(Z/2)^2 e^2}{4\pi \epsilon_0 \times 2 \times R}$$

Thus
$$E = \frac{(46)^2 \times (1.6 \times 10^{-19})^2}{4\pi \times 8.85 \times 10^{-12} \times 2 \times 1.3 \times 4.88 \times 10^{-15}} \text{ J}$$

$$= \frac{384 \times 10^{-13}}{1.6 \times 10^{-13}} \text{ MeV}$$

$$= 240 \text{ MeV}$$

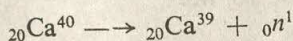
Example 4: What is the energy required to remove the least tightly bound neutron from ${}_{20}\text{Ca}^{40}$? Given that

$$\text{Mass of } {}_{20}\text{Ca}^{40} = 39.962589 \text{ amu}$$

$$\text{Mass of } {}_{20}\text{Ca}^{39} = 38.970691 \text{ amu}$$

$$\text{Mass of neutron} = 1.008665 \text{ amu}$$

Solution: The reaction is represented as



Let M represent the mass of ${}_{20}\text{Ca}^{40}$ and M' that of ${}_{20}\text{Ca}^{39}$. Let E be the energy required to remove the least tightly bound neutron from ${}_{20}\text{Ca}^{40}$. Then, from the conservation of energy, we have

$$Mc^2 + E = M'c^2 + m_n c^2$$

or
$$E = (M' + m_n)c^2 - Mc^2$$

Thus
$$E = (38.970691 \text{ amu} + 1.008665 \text{ amu}) 931.5 \frac{\text{MeV}}{\text{amu}}$$

$$- (39.962589 \text{ amu}) 931.5 \frac{\text{MeV}}{\text{amu}}$$

$$= 15.6 \text{ MeV}$$

Example 5: Calculate the force between two fission fragments of equal masses and sizes that are produced in the fission of Pu^{239} (by a thermal neutron) in which 4 neutrons are emitted.

Solution: Total mass number of Pu^{239} and neutron = 240

$$\text{Mass number of each fragment} = \frac{240 - 4}{2} = 118$$

$$\text{Atomic number of each fragment} = \frac{94}{2} = 47$$

Radius of each nucleus formed by the fission of Pu^{239} is

$$R = R_0 A^{1/3}$$

$$= 1.2 \times 10^{-15} \times (118)^{1/3} = 5.886 \times 10^{-15} \text{ m}$$

Distance between the centres of the two fragments

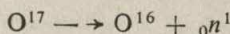
$$= 2 \times 5.886 \times 10^{-15} \text{ m} = 11.77 \times 10^{-15} \text{ m}$$

Electrostatic force between them

$$\begin{aligned} &= \frac{1}{4\pi\epsilon_0} \frac{(47 \times 1.6 \times 10^{-19})^2}{(11.77 \times 10^{-15})^2} \text{ N} \\ &= 3.763 \times 10^3 \text{ N} \end{aligned}$$

Example 6: The binding energy per nucleon of O^{16} is 7.97 MeV and that of O^{17} is 7.75 MeV. Calculate the energy required to remove a neutron from O^{17} .

Solution: The nuclear reaction is represented as



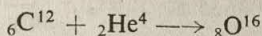
If m_p represents the mass of a proton and m_n that of a neutron, the energy required to remove a neutron from O^{17} is given by

$$\begin{aligned} E &= \text{energy equivalent of (mass of } \text{O}^{16} + m_n - \text{mass of } \text{O}^{17}) \\ &= (8m_p + 8m_n - 7.97 \times 16) \\ &\quad + m_n - (8m_p + 9m_n - 7.75 \times 17) \\ &= [7.75 \times 17 - 7.97 \times 16] \text{ MeV} \\ &= 4.23 \text{ MeV} \end{aligned}$$

Example 7: Assuming that ${}_8\text{O}^{16}$ is obtained by adding the appropriate nucleus to ${}_6\text{C}^{12}$, find the energy released. Given that

Mass of ${}_6\text{C}^{12} = 12.00381 \text{ amu}$, mass of ${}_2\text{He}^4 = 4.00387 \text{ amu}$.

Solution: ${}_8\text{O}^{16}$ can be obtained by adding ${}_2\text{He}^4$ to ${}_6\text{C}^{12}$ in accordance with the reaction represented by the equation



$$\begin{aligned} \text{Total mass of } {}_6\text{C}^{12} \text{ and } {}_2\text{He}^4 &= (12.00381 + 4.00387) \text{ amu} \\ &= 16.00768 \text{ amu} \end{aligned}$$

$$\text{Mass of } {}_8\text{O}^{16} = 16.00000 \text{ amu}$$

$$\text{Mass defect } \Delta m = (16.00768 - 16.00000) = 0.00768 \text{ amu}$$

$$\text{Energy released} = 0.00768 \times 931 = 7.15 \text{ MeV}$$

Example 8: One milligram of thorium emits 22 α -particles per unit solid angle per minute. Calculate the half life of thorium. Given atomic mass of thorium = 232 and Avogadro's number = 6.02×10^{23} .

Solution: Let N be the number of α -particles emitted per second in all directions. Then

$$N = \frac{22 \times 4\pi}{60} = \frac{22\pi}{15}$$

The number of atoms in 1 milligram of thorium is

$$\frac{10^{-3} \times 6.023 \times 10^{23}}{232} = 25.96 \times 10^{17}$$

Decay rate per second = $22\pi/15$

$$\frac{-22\pi}{15} = -\lambda \times 25.96 \times 10^{17}$$

$$\lambda = \frac{22\pi}{15 \times 25.96 \times 10^{17}} = 1.775 \times 10^{-18} \text{ s}^{-1}$$

The half life $T_{1/2}$ is given by

$$T_{1/2} = \frac{0.693}{\lambda} = \frac{0.693}{1.775 \times 10^{-18}} \text{ s} = 1.238 \times 10^{10}$$

Example 9: An accident occurs in a laboratory in which a large amount of radioactive material with a known half life of 20 days becomes embedded in floor and walls etc. Tests show that the level of radiation is 32 times the last statement is correct, after how many days can the laboratory be the last statement is correct, for how many days can the laboratory be safely occupied?

Solution: Since the initial level of radiation is 32 times the permissible limit, we have to find the time t in which the activity drops to $1/32$ of its initial value, i.e.

$$\frac{N}{N_0} = \frac{1}{32}$$

Half life of the sample ($T_{1/2}$) = 20 days

Disintegration constant

$$\lambda = \frac{0.693}{20} (\text{day})^{-1}$$

Let the laboratory become safe for use after t days. Then

$$\frac{N}{N_0} = e^{-\lambda t}$$

$$\frac{1}{32} = \exp\left(-\frac{0.693 t}{20}\right)$$

$$\text{or } \frac{0.693 t}{20} = 2.303 \log 32 = 2.303 \times 1.5051$$

$$\begin{aligned} \text{or } t &= \frac{2.303 \times 1.5051 \times 20}{0.693} \\ &= 100 \text{ days} \end{aligned}$$

Example 10: Show that the distance in free space over which the intensity of a 5 eV neutron beam is reduced by a factor one-half is approximately about 2 earth diameters. Given $T_{1/2} = 12.8$ minute.

Solution: Let v be the velocity of the neutrons in the beam. Then

$$\frac{1}{2} mv^2 = E_k$$

where m is the mass of a neutron and E_k its kinetic energy.

$$\therefore \frac{1}{2} (1.67 \times 10^{-27} \text{ kg}) v^2 = 5 \times 1.6 \times 10^{-19} \text{ J}$$

$$\therefore v^2 = \left(\frac{2 \times 5 \times 1.6 \times 10^{-19}}{1.67 \times 10^{-27}} \right) \text{ m}^2 \text{ s}^{-2}$$

$$v = 31.0 \text{ km s}^{-1}$$

Now in a time span of 12.8 minutes, half of the neutrons in the beam will have decayed. The distance travelled by the undecayed neutrons during this time is

$$d = vt$$

$$= (31.0 \text{ km s}^{-1}) (12.8 \text{ min}) (60 \text{ s/min})$$

$$= 23,800 \text{ km}$$

which is about 2 earth diameters.

Example 11: The activity of ${}^{200}_{79}\text{Au}$ of mass $20 \times 10^{-9} \text{ kg}$ is 60 curie. Calculate the half life of ${}^{200}_{79}\text{Au}$.

Solution: Let N be the number of atoms in $20 \times 10^{-9} \text{ kg}$ of ${}^{200}_{79}\text{Au}$. Then

$$\begin{aligned} N &= (20 \times 10^{-9} \text{ kg}) \left(\frac{1 \text{ k mole}}{200 \text{ kg}} \right) \left(6.025 \times 10^{26} \frac{\text{atoms}}{\text{k mole}} \right) \\ &= 6.025 \times 10^{16} \text{ atoms} \end{aligned}$$

The activity is given by

$$(60 \text{ curie}) \frac{3.7 \times 10^{10} \text{ disintegrations per second}}{1 \text{ curie}}$$

$$= 2.22 \times 10^{12} \text{ disintegrations per second}$$

Let λ be the disintegration constant. Then we have

$$\lambda = \frac{\text{activity}}{N} = \frac{2.22 \times 10^{12}}{6.025 \times 10^{16}} = 0.368 \times 10^{-4} \text{ s}^{-1}$$

Let $T_{1/2}$ be the half life of ${}^{200}_{79}\text{Au}$. Then

$$\begin{aligned} T_{1/2} &= \frac{\ln 2}{\lambda} = \frac{0.693}{\lambda} \\ &= \frac{0.693}{0.368 \times 10^{-4} \text{ s}^{-1}} = 1.883 \times 10^4 \text{ s} \end{aligned}$$

Example 12: 10 mg of carbon from living matter produce 200 counts per minute due to a small proportion of the radioactive isotope carbon-14. A piece of ancient wood of mass 10 mg is found to give 50 counts per minute. Find the age of the wood assuming that carbon-14 content of the atmosphere has remained unchanged. The half-life of carbon-14 is 5700 years.

Solution: Since the amount of C^{14} in the atmosphere remains constant and a living matter represents the amount of C^{14} present in the atmosphere, the ancient wood, which has now 50 counts per minute from 10 mg, in the beginning, must have had the same count rate as from living matter. Thus the

age of the ancient wood is the time in which the C^{14} count rate has decreased from the initial value of 200 counts per minute from 10 mg to the final value of 50 counts per minute from 10 mg.

We know that $T_{1/2}$ for carbon is 5700 years. Hence the disintegration constant λ of C^{14} is given by

$$= \frac{0.693}{T_{1/2}} = \frac{0.693}{5700} (\text{yr})^{-1}$$

Let t represent the age of the wood. Then

$$50 = 200 \exp \left(-\frac{0.693 t}{5700} \right)$$

or

$$4 = \exp \left(\frac{0.693 t}{5700} \right)$$

$$\begin{aligned} t &= \frac{\ln 4 \times 5700}{0.693} \text{ yr} \\ &= \frac{2.303 \times 0.6021 \times 5700}{0.693} \text{ yr} \\ &= 1.41 \times 10^4 \text{ years} \end{aligned}$$

Example 13: A nucleus with $A = 235$ splits into two new nuclei whose mass numbers are in the ratio of 2 : 1. Find the radii of the new nuclei.

Solution: We know that the radius R of a nucleus of mass number A is given by the relation

$$R = R_0 A^{1/3}$$

where experiments show that the value of R_0 is 1.4 fm. Let A_1 and A_2 represent the mass numbers of the new nuclei which are formed when the nucleus with mass number A splits into two halves. Let R_1 and R_2 be the radii of the new nuclei so formed. Then we have

$$A_1 = \frac{1}{3} (235)$$

and

$$A_2 = \frac{2}{3} (235)$$

Using

$$R = R_0 (A)^{1/3} \text{ we have}$$

$$R_1 = R_0 (A_1)^{1/3} = (1.4 \text{ fm}) \left(\frac{235}{3} \right)^{1/3} = 5.99 \text{ fm}$$

and

$$R_2 = R_0 (A_2)^{1/3} = (1.4 \text{ fm}) \left(\frac{235 \times 2}{3} \right)^{1/3} = 7.55 \text{ fm}$$

Example 14: The half life of radium is 1500 yrs. In how many years will 1 g of pure radium (a) lose 1 mg and (b) be reduced to 1 centigram?

Solution: Half life of radium ($T_{1/2}$) = 1500 yrs
The disintegration constant λ of radium is given by

$$= \frac{0.693}{T_{1/2}} = \frac{0.693}{1500} (\text{yr})^{-1}$$

Initial amount of radium (N_0) = 1 g = 1000 mg

(a) Let in t years the radium lost be 1 mg. The remaining amount = 999 mg

$$999 = 1000 \exp \left(-\frac{0.693 t}{1500} \right)$$

or $\log_e \left(\frac{1000}{999} \right) = \left(\frac{0.693 t}{1500} \right)$

$$t = \frac{2.303 \times 0.0004 \times 1500}{0.693} = 1.995 \text{ yrs}$$

(b) Let in τ years the amount of radium left be 1 centigram = 10 mg

$$10 = 1000 \exp \left(-\frac{0.6936}{1500} \right)$$

$$100 = \exp \left(\frac{0.6936}{1500} \right)$$

or $\ln 100 = \frac{0.6936}{1500}$

or $\tau = \frac{\ln 100 \times 1500}{0.693} \text{ years}$

$$= \frac{2.303 \times 2 \times 1500}{0.693} \text{ yrs} = 9.972 \times 10^3 \text{ yrs}$$

Example 15: Calculate the mass of 1 curie of RaB (Pb^{214}) from its half life of 26.8 minutes.

Solution: Activity of RaB = $\frac{dN}{dt} = 1 \text{ curie} = 3.7 \times 10^{10} \text{ disintegration/s}$

Half life of RaB (T) = 26.8 minutes

If λ is the disintegration constant of RaB, we have

$$\lambda = \frac{0.693}{T} = \frac{0.693}{26.8 \times 60} \text{ s}^{-1}$$

Let N be the number of atoms of RaB having an activity of 1 curie. Then we have

$$\left| \frac{dN}{dt} \right| = \lambda N$$

or $N = \left| \frac{dN}{dt} \right| / \lambda$

$$\therefore N = \frac{3.7 \times 10^{10} \times 26.8 \times 60}{0.693}$$

Further we know that $6.02 \times 10^{23} \text{ atoms} = 1 \text{ gram atom} = 214 \text{ g}$. Therefore, the mass of RaB having an activity of 1 curie

$$= \frac{214 \times 3.7 \times 10^{10} \times 26.8 \times 60}{6.02 \times 10^{23} \times 0.693}$$

$$= 30.52 \times 10^{-9} \text{ g}$$

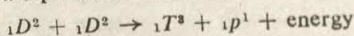
19.5 TRUE-FALSE STATEMENTS WITH REASONS

A few statements are given below. We have to decide whether they are true or false giving, in brief, reasons in support of our answers.

1. Like other light nuclei, the ${}^4\text{He}^+$ nuclei also have a low value of the binding energy per nucleon.
2. A radioactive element of half life 1.5 years completely disintegrates in four and a half years.
3. For similar reasons, the binding energy per nucleon decreases for nuclei with small as well as large A values.
4. During the β -decay of a nucleus, we get the product nucleus, the β -particle and another particle (which may be a neutrino or an anti-neutrino).
5. For stable nuclei, the size does not approach or exceed 10^{-14} m.
6. The energy required to 'tear off' one neutron from ${}^7\text{Li}$ to transform it into the isotope ${}^6\text{Li}$ is nearly 5.6033 MeV, the same as the binding energy per nucleon for the lithium isotope ${}^7\text{Li}$.
7. The net result of both the proton-proton cycle and the carbon cycle fusion reactions in the sun and stars is the fusion of 4 protons to form a helium nucleus and a pair of positrons accompanied by a release of 24.7 MeV of energy.
8. When two deuterium nuclei fuse together, they give rise to a tritium nucleus accompanied by a release of energy.
9. In artificial radioactivity, we sometimes observe decay by 'K-electron capture'. In this process, the nucleus captures one of its own orbital electrons. The atomic number change in this process is different from that in the decay process that involves the emission of a positron.
10. The mass per nucleon in a hydrogen atom is the same as that in an oxygen atom.

ANSWERS

1. *False* The ${}^4\text{He}^+$ nucleus (or the α -particle) is exceptionally stable and has a much higher value of B.E per nucleon than that for most other light nuclei.
2. *False* After four and a half years (i.e. 3 half lives), only $(1/2)^3 = 1/8$ of the radioactive element will have disintegrated.
3. *False* The reason for the decrease in the B.E per nucleon for nuclei with large A is that with an increase in the number of protons, the Coulomb repulsion increases. On the other hand, the decrease in the B.E per nucleon for nuclei with small A is due to a surface effect: the nucleons at the surface being less strongly bound than those in the interior.
4. *True* To ensure that the fundamental laws of conservation of energy and momentum were not violated, it was essential to postulate the emission of a neutrino or anti-neutrino in the β -decay process. Later on a 'direct' evidence of the emission of these particles was obtained through a study of the 'recoil' of the decaying nucleus.
5. *True* The nuclear forces holding the nucleons together are very strong attractive forces that are non-negligible only within a range of about 10^{-14} m.
6. *True* The energy required to tear off one neutron (i.e. one nucleon) is the same as the B.E per nucleon for a given isotope.
7. *True* The carbon isotope ${}^{12}\text{C}$ involved in the carbon cycle acts merely as a 'catalyst' for the reaction and is 're-obtained' after a complete cycle.
8. *False* To ensure both charge and mass number conservation, we must also obtain a proton as a biproduct of the reaction:



We often write this reaction as: $\text{D}(\text{D}, \text{p}) \text{T}$.

9. *False* In both the 'K-electron capture' decay and the positron emission decay, the atomic number decreases by unity since in both cases the nuclear positive charge goes down by unity.
10. *False* The energy released during the formation of nuclei of different atoms from their constituent nuclei is different. Hence the mass per nucleon is dependent upon the atom considered. The oxygen nucleus, being particularly stable, has a much larger value of the binding energy per nucleon. Hence the mass per nucleon in an oxygen atom is slightly less than the mass per nucleon in a hydrogen atom.

19.6 MULTIPLE CHOICE QUESTIONS

Choose the correct answer from the alternatives given in each of the following.

- If the end A of a wire is irradiated with alpha rays and the end B is irradiated with beta rays, then
 - there will be no current in the wire
 - a current will flow from A to B
 - a current will flow from B to A
 - a current will flow from each end to the midpoint of the wire.
- The radioactivity of a sample is X at a time t_1 and Y at a time t_2 . If the mean life of the specimen is τ , the number of atoms that have disintegrated in the time interval $(t_2 - t_1)$ is
 - $X t_1 - Y t_2$
 - $X - Y$
 - $(X - Y)\tau$
 - $(X - Y) \tau$
- After two hours $1/16$ th of the initial amount of a certain radioactive isotope remains undecayed. The half life of the isotope is
 - 15 min
 - 30 min
 - 45 min
 - 1 h
- The half life of a certain radio isotope is 10 minutes. The number of radioactive nuclei at a given instant of time is 10^8 . Then the number of radioactive nuclei left 5 minutes later would be
 - $(\sqrt{2}) \times 10^8$
 - 10^4
 - 5×10^7
 - 2×10^7
- The half life of Pa^{218} is 3 minutes. What fraction of a 10 g sample of Pa^{218} will remain after 15 minutes?
 - $1/5$
 - $1/25$
 - $1/32$
 - $1/64$
- The radioactivity of a given sample of whisky due to tritium (half life 12.5 years) was found to be only about 3% of that measured in a recently purchased bottle marked '7 years old'. The sample must have been prepared about
 - 70 years ago
 - 220 years ago
 - 420 years ago
 - 300 years ago
- In an atom there are p protons, n neutrons and e electrons surrounding the nucleus. For atoms of small mass the relation between p , n and e is
 - $p \approx n \approx e$
 - $p = e \approx n$
 - $p = n = e$
 - $n = p - 1 = e$

8. In any nuclear reaction the reactants and the resultants must always be in conformity with the law of conservation of
- (a) energy
 - (b) charge number
 - (c) mass number
 - (d) both charge and mass numbers

9. The mass per nucleon in a heavy hydrogen atom is
- (a) the same as the mass per nucleon in an oxygen atom
 - (b) slightly greater than the mass per nucleon in an oxygen atom
 - (c) slightly smaller than the mass per nucleon in an oxygen atom
 - (d) $1/16$ th of the mass per nucleon in an oxygen atom.

10. The radius of a nucleus is
- (a) proportional to its mass number
 - (b) inversely proportional to its mass number
 - (c) proportional to the cube root of its mass number
 - (d) not related to its mass number.

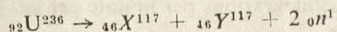
11. The approximate density of nuclear matter is
- (a) $1.5 \times 10^{17} \text{ kg m}^{-3}$
 - (b) 1000 kg m^{-3}
 - (c) $10^{36} \text{ kg m}^{-3}$
 - (d) none of the above.

12. The kinetic energy of a 300 K thermal neutron is
- (a) 300 eV
 - (b) 300 MeV
 - (c) 0.026 eV
 - (d) 0.026 MeV

13. The average number of neutrons released by the fission of one uranium atom is
- (a) 1
 - (b) 2.5
 - (c) 3
 - (d) none of the above.

14. In fission the percentage of mass converted into energy is about
- (a) 1%
 - (b) 0.1%
 - (c) 0.01%
 - (d) 100%

15. What is the energy released in the fission reaction



given that the binding energy per nucleon of X and Y is 8.5 MeV and that of U^{236} is 7.6 MeV?

- (a) 20 MeV
 - (b) 180 MeV
 - (c) 200 MeV
 - (d) 2000 MeV
16. Slow neutrons are incident on a sample of uranium containing both ${}_{92}\text{U}^{235}$ and ${}_{92}\text{U}^{238}$ isotopes. Then
- (a) both isotopes will undergo fission and break up
 - (b) only ${}_{92}\text{U}^{235}$ atoms undergo fission
 - (c) only ${}_{92}\text{U}^{238}$ atoms undergo fission
 - (d) neither of the isotopes will undergo fission and break up.
17. Which of the following is a good nuclear fuel?
- (a) Uranium-236
 - (b) Plutonium-239
 - (c) Neptunium-239
 - (d) Thorium-236
18. The energy released by the fission of a single uranium atom is 200 MeV. What is the number of fissions per second required to produce 3.2 W of power?
- (a) 10^7
 - (b) 10^{10}
 - (c) 10^{11}
 - (d) 10^{13}

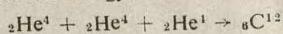
19. The temperature required to produce fusion of light nuclei is of the order of
 (a) 10^6 K (b) 10^7 K
 (c) 10^8 K (d) 10^9 K
20. Radio carbon dating is done by estimating in the specimen
 (a) the amount of ordinary carbon still present
 (b) the amount of radio carbon still present
 (c) the ratio of the amounts of C^{14} to C^{12} still present
 (d) none of the above.

19.7 NUMERICAL EXERCISES

- Find the volume at NTP of radon ${}_{86}\text{Rn}^{222}$, whose half life is 3.825 day, which will have an activity of 0.1 curie.
- Radioactive isotopes P^{32} and P^{33} are mixed in the ratio 2 : 1 by the number of atoms. The activity is 2 microcuries. Find the activity after 30 days. Given half life of $\text{P}^{32} = 14$ days and that of $\text{P}^{33} = 25$ days.
- Calculate the number of alpha decays that occur in a one gram sample of Th^{232} in one year. The disintegration constant of $\text{Th}^{232} = 1.58 \times 10^{-18} \text{ s}^{-1}$. What is the activity of the given sample?
- Two consecutive radiations from a long lived parent have decay constants λ_1 and λ_2 . Show that the apparent mean life of the second radiation is

$$\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} \right)$$

- A dose of 5 mc (millicuries) of ${}_{15}\text{P}^{32}$ is administered intravenously to a patient whose blood volume is 3.5 litres. At the end of one hour it is assumed that the phosphorous is uniformly distributed. What would be the count rate per ml of the withdrawn blood if the counter had an efficiency of 10% (a) one hour after injection and (b) 28 days after injection?
- Assuming that the average disintegration rate of radioactive carbon ${}_{6}\text{C}^{14}$ in a living organism is 1.53×10^4 disintegrations per minute for each kg of carbon, estimate the age of an archaeological specimen of some dung for which the disintegration rate is 1.10×10^4 disintegrations per minute per kg. The half life of radioactive carbon is 5600 years.
- Calculate the energy released when thermal neutrons enter the nucleus of Pu^{239} . The masses of Pu^{239} and Pu^{240} are 239.1270 amu and 240.1291 amu respectively.
- The fission of ${}_{92}\text{U}^{235}$ produces a mean energy of 195 MeV per nucleon. Calculate the mass of ${}_{92}\text{U}^{235}$ that would be required per day to run a generating station working at 30% overall efficiency and producing 100 MW of power.
- The fission of ${}_{92}\text{U}^{235}$ produces a mean energy of 185 MeV. If ${}_{92}\text{U}^{235}$ in a reactor is continuously generating 100 MW of power, how long will it take for 1 kg of uranium to be used up?
- Calculate the power output of a ${}_{92}\text{U}^{235}$ reactor if it takes one month to use up 2 kg of uranium fuel.
- The volume of water in the Great Lakes is 5000 m^3 . Calculate the amount of energy released, if all the deuterium atoms in the water are used up in fusion. The abundance of deuterium is 0.0156% and the fusion of 6 deuterium atoms gives an energy release of 43 MeV.
- Calculate the amount of energy released when two deuterium nuclei fuse into an alpha particle. The masses of deuterium and alpha particle are 2.014102 amu and 4.002603 amu respectively.
- Calculate the energy released in the fusion reaction



The mass of an alpha particle = 4.002603 amu.

14. The sun and the stars are believed to be 'getting' their energy from the fusion of four protons to form a helium nucleus and a pair of positrons. Calculate the release of energy per fusion and the rate at which hydrogen must be consumed to produce 1 kilo-joule of energy per second. Given

$${}_1\text{H}^1 = 1.007825 \text{ amu (hydrogen atom)}$$

$${}_2\text{H}^4 = 4.002603 \text{ amu (helium atom)}$$

$$m_{\beta^+} = m_e = 5.5 \times 10^{-4} \text{ amu (positron)}$$

The energy carried away by the accompanying neutrinos is assumed to be negligible.

ANSWERS

MULTIPLE CHOICE QUESTIONS

- | | | | | |
|---------|---------|---------|---------|---------|
| 1. (b) | 2. (d) | 3. (b) | 4. (a) | 5. (c) |
| 6. (a) | 7. (b) | 8. (d) | 9. (b) | 10. (c) |
| 11. (a) | 12. (c) | 13. (b) | 14. (b) | 15. (c) |
| 16. (b) | 17. (b) | 18. (c) | 19. (c) | 20. (c) |

NUMERICAL EXERCISES

- $0.0656 \times 10^{-13} \text{ cm}^3$
- 0.598 curie
- 2.60×10^{-21} per gram; 0.11 curie
- $5.0 \times 10^3, 1.32 \times 10^3$ disintegrations per second
- 2680 years
- 6.349 MeV
- 0.36 kg
- 8.78 days
- 62.5 MW
- 1.6×10^{39} MeV
- 23.8 MeV
- 7.27 MeV
- 26.731 MeV and 1.565×10^{-12} kg per second.

The Diode and the Triode

20.1 REVIEW OF BASIC CONCEPTS

20.1.1 The Diode The diode is the simplest electron tube and has two electrodes placed inside an evacuated glass or ceramic envelope. The two electrodes are known as the *cathode* and the *plate* (or anode). The cathode serves as the electron emitter and the plate 'collects' a part or all of the emitted electrons.

The cathode is simply a heater coil in *directly heated* type of diodes. It is a heater-cathode combination in the *indirectly heated* diodes which are more often in use.

20.1.2 The Diode Characteristics The characteristic curve of a diode is a graph showing the variation of the plate current with the applied plate voltage. It is a non-linear curve that can be subdivided into two main parts: the *space charge limited* region and the *saturation* region.

The Space Charge Limited Region—The Child-Langmuir's Law In the space charge limited region the applied plate potential is not sufficient to collect electrons as fast as they are emitted by the cathode (Fig. 20.1). We, therefore, observe a 'rise' in plate current with increasing plate voltage. The rise, however, is not linear, i.e. it is not governed by Ohm's law.

The relation between the plate current (I_p) and the plate potential (V_p) in the space charge limited region is given approximately by the so-called 'Child-Langmuir law' according to which

$$I_p = k V_p^{3/2}$$

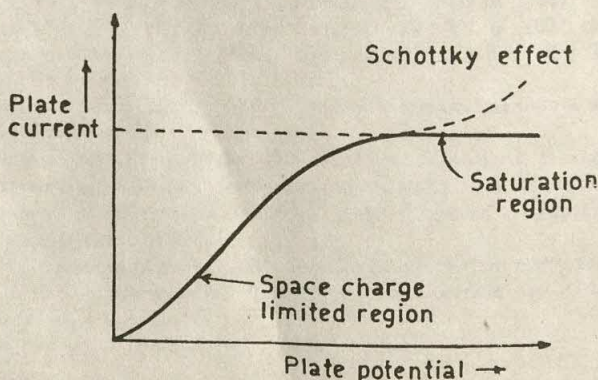


Fig. 20.1

The Saturation Region The saturation region 'begins' at a plate potential that is just sufficient to collect electrons as fast as they are emitted. In this region (Fig. 20.1), the plate current remains practically constant even though the plate potential is being continuously increased. The saturation current value, dependent as it is on the number of electrons emitted per second by the cathode, is governed mainly by the cathode temperature.

20.1.3 The Schottky Effect The current in the saturation region is not strictly constant but does show a small increase with the applied potential. This small increase is due to the so-called *field emission* or *cold-emission*.

20.1.4 The Rectifier A rectifier is a device that is capable of converting an alternating current into a unidirectional or pulsating form of direct current. The process of conversion of alternating currents into direct currents is known as *rectification*.

20.1.5 The Diode as a Rectifier Plate current can flow in a diode only when the plate is positive with respect to the cathode. A diode thus permits a unidirectional current flow only for positive values of its plate potential. We, therefore, can use a diode as a rectifier.

The Half-wave Rectifier The half-wave rectifier circuit is shown in Fig. 20.2. It is the basic circuit for converting ac into dc. The AC voltage is applied through a power transformer in series with the diode tube and the load resistance R_L . The current flows in the circuit only during the positive half cycles of the applied AC voltage—hence the name half wave rectifier.

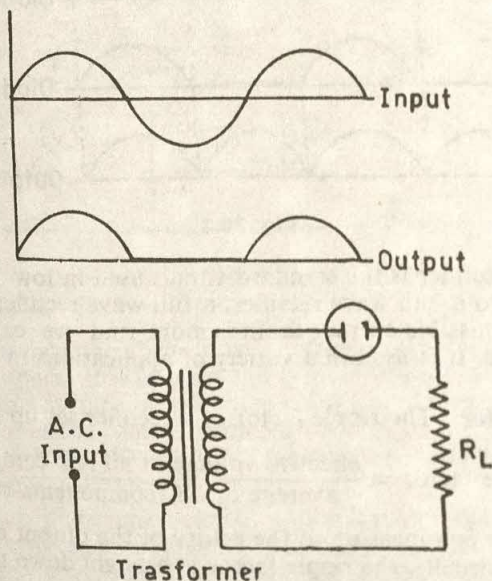


Fig. 20.2

In a half wave rectifier, as one half of the input voltage is effectively lost to the load, it has a low efficiency and is seldom used except in applications requiring a small current drain.

The Full-wave Rectifier A full-wave rectifier basically consists of two half-wave rectifiers and uses a power transformer provided with a centre tapping (Fig. 20.3). The load is connected between the centre tap and the common junction of the cathodes of the two diodes. The two diodes conduct during alternate half cycles but the current in the load always flows in the same direction.

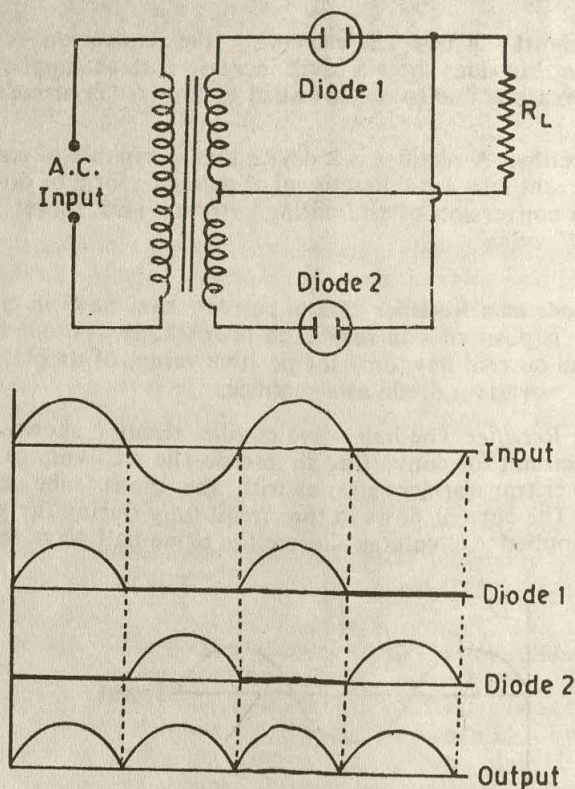


Fig. 20.3

The full wave rectifier is the standard circuit used in low power applications. Compared to a half wave rectifier, a full wave rectifier has a higher efficiency; the permissible current drain is more and we can get a higher DC output voltage. It is used in a variety of applications in electronics.

20.1.6 Ripple Factor The ripple factor of a rectifier set up is defined as

$$\text{Ripple factor} = \frac{\text{effective voltage of all AC components}}{\text{average or DC components voltage}}$$

The ripple factor is a measure of the purity of the output of a rectifier. In actual rectifier circuits, the ripple factor is brought down to suitable low values by using *filter circuits*. These are usually made by using choke coils and capacitors. The combination of a filter circuit with a rectifier constitutes a DC power supply.

20.1.7 The Triode The triode, developed in 1907 by Le de Forest, is an electron tube consisting of three electrodes. In addition to the cathode and

plate of a diode, it also has a wire mesh electrode, called the grid, between the cathode and the plate.

The triode is a simple *control valve* in that the conditions at one pair of terminals (the cathode and the grid) control the flow of energy in a circuit connected to another pair of terminals (the cathode and the plate).

The Grid The grid of a triode is made by winding a number of turns of fine wire in between the 'grooves' of two suitable supports. It is an open helical structure of metal kept close to the cathode. Because of its nearness to the cathode, it has a very effective control on the electric field at the cathode. This, in turn, produces a significant change in the number of electrons that are finally able to reach the plate. The 'open structure' of the grid causes but a few of the electrons to get intercepted in their flight towards the plate.

The grid is normally kept at a negative potential. This helps the triode to control the conditions in the plate-cathode circuit with negligible power dissipation in the grid circuit.

20.1.8 The Triode Characteristics The plate current in a triode depends upon both the plate potential and the grid potential, i.e.

$$I_b = f(E_b, E_c)$$

The actual form of this relation is studied experimentally by using a circuit of the type shown in Fig. 20.4.

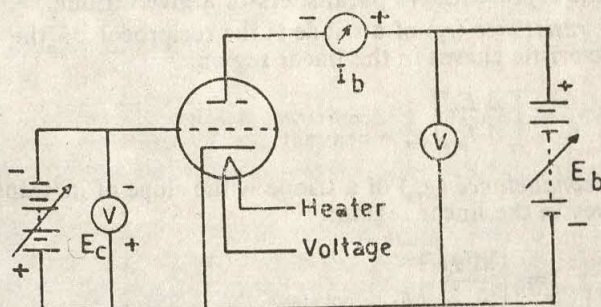


Fig. 20.4

We could represent the above relation graphically by using a system of co-ordinates in space. However, as this is inconvenient, we employ the three projections of this surface on the three co-ordinate planes. These 'projections' are known as the *characteristic curves* of a triode and are named as (i) *plate characteristics*, (ii) *mutual or transfer characteristics* and (iii) *constant current characteristics*.

The family of plate characteristics depicts the dependence of plate current on plate voltage for constant values of grid voltage (Fig. 20.5a). The family of transfer characteristics depicts the dependence of plate current on grid voltage for constant values of plate voltage (Fig. 20.5b).

The constant current characteristics show the relation between plate voltage and grid voltage that will keep the plate current constant. However, we generally plot only the first two families of characteristic curves.

The characteristic curves of a triode are curved in the lower portion near the plate current cut-off and are fairly linear in the upper portions. The

voltage range over which the curves are linear is referred to as the *linear region* of the characteristics.

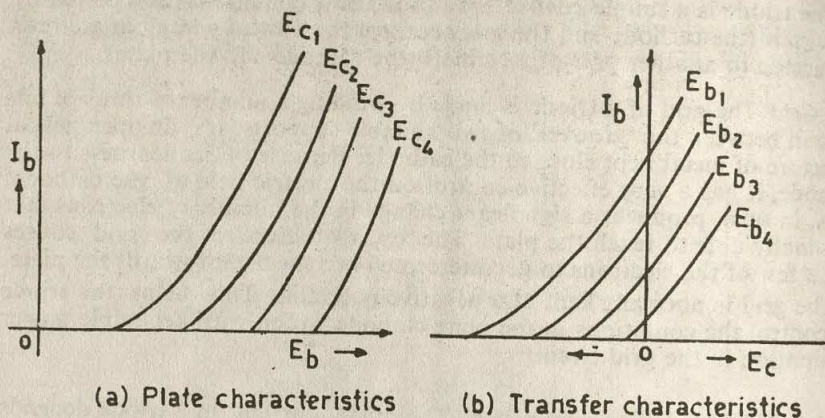


Fig. 20.5

20.1.9 Triode Parameters Over the linear region, the triode characteristic curves are not only nearly straight but also more or less parallel to each other. We, therefore, find it convenient to define the slopes (or their reciprocals) of each of the three family of characteristic curves (in the linear region) as the representative parameters of a given triode.

The *plate resistance* (r_p) of a triode is the reciprocal of the slope of the plate characteristic curves in the linear region:

$$r_p = \left[\frac{\Delta E_b}{\Delta I_b} \right]_{E_c} = \text{constant}$$

The *transconductance* (g_m) of a triode is the slope of its transfer characteristic curves in the linear region:

$$g_m = \left[\frac{\Delta I_c}{\Delta E_c} \right]_{E_b} = \text{constant}$$

The amplification factor (μ) is the slope of the constant current characteristic curves in the linear region. It is the parameter that is of greater interest and tells us as to how many times the grid is more effective than the plate in controlling the current

$$\mu = \left[\frac{\Delta E_b}{\Delta E_c} \right]_{I_b} = \text{constant}$$

It is easy to see that the three parameters are related as

$$\mu = r_p g_m$$

20.1.10 Inter-electrode Capacitances in a Triode The three electrodes of a triode give rise to three capacitances C_{pk} (plate-cathode capacitance), C_{gp} (grid-plate capacitance) and C_{gk} (grid-cathode capacitance). Of these, C_{gp} causes a sort of coupling between the grid and the plate at high frequencies and puts a limit on the high frequency range upto which a triode can be effectively used as an amplifier.

20.1.11 Triode as an Amplifier A small signal applied between the grid and the cathode of a triode gets 'reproduced' in an amplified form in the plate-cathode circuit. The basic circuit of a triode as an amplifier is shown in Fig. 20.6.

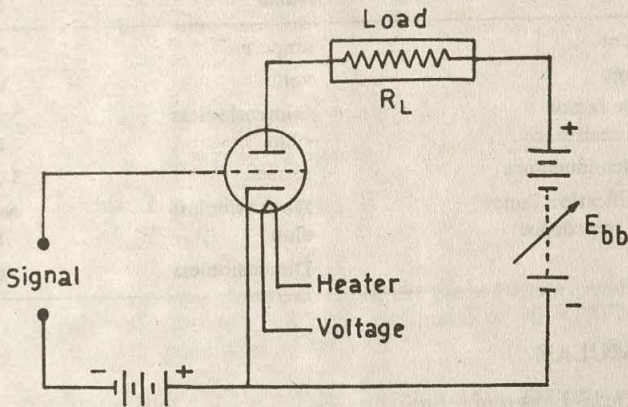


Fig. 20.6

It can be shown that the *gain* of this circuit is given by

$$A = \frac{-\mu r_p}{(r_p + R_L)}$$

For $r_p \gg R_L$, $A \rightarrow -\mu$; thus the maximum gain possible with a given triode equals its amplification factor.

The following points about the use of a triode as an amplifier are worth noting.

1. The value of the *bias* or steady voltage E_{cc} applied to the grid determines the *operating or quiescent point* of the given amplifier. This value has to be carefully chosen along with the value of the plate potential to make the amplification nearly *distortionless*. Of course, for special applications, we choose the operating point (i.e. the values of the bias voltage E_{cc} and the plate voltage E_{bb}) to introduce the desired degree and type of distortion in the amplified signal.
2. The behaviour of a triode in the presence of a load in the plate circuit is depicted by what are known as its *dynamic characteristics*. These are determined not only by the triode itself but also by the load used.
3. An increasing grid voltage results in a decrease in the plate-cathode voltage ($= E_{bb} - I_b R_L$) and, therefore, the grid voltage and plate voltage in a triode amplifier are said to be π radians out of phase with each other.
4. The maximum amplification possible with a single triode being μ , we have to carry out two or more stages of amplifications. For this, we *suitably couple* the output of a preceding stage to the input of the succeeding stage. The gain of a chain of such *cascaded* amplifiers is given by

$$A = A_1 A_2 \dots A_n$$

where $A_1, A_2, \dots A_n$ are the gains of the successive stages. We often do not use more than three stages of amplification because, otherwise, the *noise level* also goes up considerably.

20.2 SI UNITS

Table 20.1

Physical Quantity	Name	SI Unit	Symbol
Current	ampere		A
Voltage	volt		V
Ripple factor	Dimensionless		No unit
Plate resistance	ohm		Ω
Transconductance	siemen		S (or Ω^{-1})
Amplification factor	Dimensionless		No unit
Load impedance	ohm		Ω
Gain	Dimensionless		No unit

20.3 FORMULAE

1. The Child-Langmuir Law

$$I_p = k V_p^{3/2}$$

I_p = plate current in the space charge limited region of a diode

V_p = plate potential

$$2. \quad r = \frac{(E_{AC})_{\text{eff}}}{(E_{DC})_{\text{average}}}$$

r = ripple factor, $(E_{AC})_{\text{eff}}$ = effective value of all AC components, $(E_{DC})_{\text{average}}$ = average or DC component voltage.

$$3. \quad r_p = \left[\frac{\Delta E_b}{\Delta I_b} \right]_{E_c = \text{constant}}$$

r_p = plate resistance, E_b = plate voltage, I_b = plate current, E_c = grid voltage.

$$4. \quad g_m = \left[\frac{\Delta I_b}{\Delta E_c} \right]_{E_b = \text{constant}}$$

g_m = transconductance

$$5. \quad \mu = \left[\frac{\Delta E_b}{\Delta E_c} \right]_{I_b = \text{constant}}$$

μ = amplification factor

$$6. \quad \mu \approx r_p g_m$$

$$7. \quad A = \frac{-\mu}{\left(1 + \frac{r_p}{R_L}\right)}, \quad A = \text{gain of a single stage triode amplifier,}$$

R_L = load impedance.

8. $A = A_1 A_2 \dots A_n$, where A is the gain of a cascaded chain of n stages and $A_1, A_2, \dots$ etc. are the gains of successive stages.

20.4 NUMERICAL EXAMPLES

Example 1: The constant of proportionality in the Child-Langmuir law for a given diode is $2 \times 10^{-4} \Omega$. What is the plate current for a plate voltage of 45 V that lies in the space charge limited region of the diode?

Solution: According to Child-Langmuir law

$$I_p = k V_p^{3/2}$$

Substituting the numerical values, we have

$$\begin{aligned} I_p &= 2 \times 10^{-4} \Omega \times (45)^{3/2} \text{ V} \\ &= 603.7 \times 10^{-4} \text{ A} \\ &= 60.37 \text{ mA} \end{aligned}$$

Example 2: In a given two electrode tube, the plate current, with full space charge, has a value of 72 mA for a plate potential of 90 V. What will be the plate current for a potential of 60 V?

Solution: We know that for the space charge limited region

$$I_p = k V_p^{3/2}$$

$$\therefore 72 \text{ mA} = k (90)^{3/2}$$

$$\text{and } x = k (60)^{3/2}$$

Dividing the two equations we have

$$\frac{x}{72 \text{ mA}} = \left(\frac{60}{90}\right)^{3/2} = (2/3)^{3/2} = 0.544$$

$$\begin{aligned} \text{or } x &= 0.544 \times 72 \text{ mA} \\ &= 39.13 \text{ mA} \end{aligned}$$

Example 3: For a triode the plate current corresponding to a grid voltage of -10 V and a plate voltage of 180 V was found to be 5 mA . The amplification factor of the triode is 15. Assuming that the plate current is space charge limited all the time, calculate its value for a plate voltage of 150 V and a grid voltage of -5 V .

Solution: We know that the plate voltage V_p is as effective as a grid voltage of V_p/μ in controlling the plate current. Assuming the plate current to be space charge limited all the time, the dependence of the plate current on the plate and grid voltages can be written (in terms of the Child-Langmuir law) as

$$I_p = k \left(\frac{V_p}{\mu} + V_g \right)^{3/2}$$

Substituting the value given in the data and calling the required unknown current as x , we have

$$5 \text{ mA} = k \left(\frac{180}{15} - 10 \right)^{3/2} = k (2)^{3/2}$$

$$\text{and } x = k \left(\frac{150}{15} - 5 \right)^{3/2} = k (5)^{3/2}$$

Dividing the two equations we have

$$\frac{x}{5 \text{ mA}} = \left(\frac{5}{2}\right)^{3/2} = (2.5)^{3/2} = 3.953$$

$$x = 19.76 \text{ mA}$$

Example 4: The $I_p - V_g$ characteristic of a triode can be represented by the expression

$$I_p = 0.01 \left(\frac{V_p}{\mu} + V_g \right)^{3/2} \text{ mA}$$

The amplification factor of the triode is 20. Find the plate current when $V_g = -2 \text{ V}$ and $V_p = 220 \text{ V}$ and the plate resistance when $V_g = -3 \text{ V}$ and $V_p = 200 \text{ V}$.

Solution: We have

$$I_p = 0.01 \left(\frac{V_p}{20} + V_g \right)^{3/2} \text{ mA}$$

For $V_g = -2 \text{ V}$ and $V_p = 220 \text{ V}$,

$$\text{we have } I_p = 0.01 \left(\frac{220}{20} - 2 \right)^{3/2} \text{ mA}$$

$$= 0.01 (9)^{3/2} = 0.27 \text{ mA}$$

Also the change (ΔI_p) in plate current with a change (ΔV_p) in the plate voltage, at a constant grid voltage, is given by

$$\Delta I_p = 0.01 \times \frac{3}{2} \left(\frac{V_p}{\mu} + V_g \right)^{1/2} \frac{1}{\mu} \Delta V_p \text{ in milliampere}$$

$$\therefore \frac{\Delta V_p}{\Delta I_p} = \frac{2 \mu \times 10^3}{0.01 \times 3 \left(\frac{V_p}{\mu} + V_g \right)^{1/2}} \Omega$$

Therefore, the plate resistance r_p at $V_p = 200 \text{ V}$ and $V_g = -3 \text{ V}$ is

$$r_p = \frac{2 \times 20 \times 10^3}{0.03 \left(\frac{200}{20} - 3 \right)^{1/2}} = \frac{40 \times 10^3}{0.03 \times 7^{1/2}}$$

$$= \frac{40 \times 10^3}{0.079} \Omega = 506.3 \text{ k} \Omega$$

Example 5: For a certain triode, the following data was obtained:

Plate Voltage	Grid Voltage	Plate Current
150 V	-2 V	5 mA
150 V	-3.5 V	3.2 mA
195 V	-3.5 V	5 mA

Determine the plate resistance, mutual conductance and the amplification factor of the given triode.

Solution: The plate resistance r_p is given by

$$r_p = \left(\frac{\Delta V_p}{\Delta I_p} \right)_{V_g} = \left[\frac{195 - 150}{(5 - 3.2) \times 10^{-3}} \right]_{V_g = -3.5 \text{ V}} \Omega$$

$$= \frac{45 \times 10^3}{1.8} \Omega = 25 \text{ k}\Omega$$

The mutual conductance g_m is given by

$$g_m = \left(\frac{\Delta I_p}{\Delta V_g} \right)_{V_p} = \left[\frac{(5 - 3.2) \times 10^{-3}}{(-2) - (-3.5)} \right]_{V_p = 150 \text{ V}}$$

$$= \frac{1.8}{1.5} \times 10^{-3} \text{ siemen} = 1.2 \times 10^{-3} \text{ siemen}$$

The amplification factor μ is given by

$$\mu = \left(\frac{\Delta V_p}{\Delta V_g} \right)_{I_p} = \left[\frac{195 - 150}{-3.5 - (-2)} \right]_{I_p = 5 \text{ mA}} = \frac{45}{-1.5} = -30$$

We could have got μ also from the relation

$$\mu = -r_p g_m = -25 \times 10^3 \times 1.2 \times 10^{-3} = -30$$

Example 6: A triode with an amplification factor 20 and plate resistance $15 \text{ k}\Omega$ is used as an amplifier. If the load resistance is $75 \text{ k}\Omega$, draw the equivalent circuit and calculate the gain.

Solution: The equivalent circuit of a triode as an amplifier is shown in Fig. 20.7.

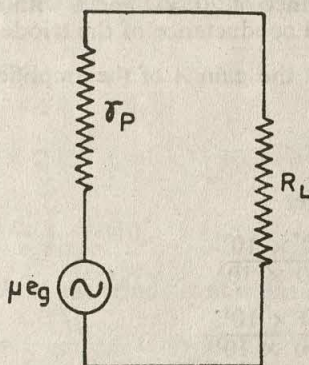


Fig. 20.7

The gain is given by

$$A = \left(\frac{\mu e_g}{r_p + R_L} \right) \frac{R_L}{e_g} = \frac{\mu R_L}{r_p + R_L}$$

$$= \frac{\mu}{\left(1 + \frac{r_p}{R_L} \right)}$$

Here $r_p = 15 \text{ k}\Omega$ and $R_L = 75 \text{ k}\Omega$. Therefore

$$A = \frac{\mu}{1 + \frac{r_p}{R_L}} = \frac{5}{1 + \frac{15}{75}} = \frac{5}{6} \mu = \frac{5}{6} \times 20 = 16.67$$

Example 7: A triode with an amplification factor 30 and a plate resistance $20 \text{ k}\Omega$ is used as an amplifier. The operating point is chosen as $I_p = 5 \text{ mA}$, $V_p = 100 \text{ V}$. Find the load resistance and the gain if the battery voltage is 300 V .

Solution: Let R_L be the load resistance. The voltage drop across it corresponding to a plate current of 5 mA is $5 R_L \times 10^{-3} \text{ V}$. Since the battery voltage is 300 V and $V_p = 100 \text{ V}$, we have

$$100 + 5 R_L \times 10^{-3} = 300$$

$$R_L = \frac{200}{5 \times 10^{-3}} \Omega = 40 \times 10^3 \Omega = 40 \text{ k}\Omega$$

The gain of the amplifier is given by

$$\begin{aligned} A &= \frac{\mu}{1 + \frac{r_p}{R_L}} \\ &= \frac{30}{1 + \frac{20}{40}} = 20 \end{aligned}$$

Example 8: In a resistance coupled single stage amplifier, the gain obtained is 5 with a load resistance of $20 \text{ k}\Omega$ and 6 with a load resistance of $40 \text{ k}\Omega$. Calculate the mutual conductance of the triode.

Solution: We know that the gain A of the amplifier is given by

$$A = \frac{\mu R_L}{r_p + R_L}$$

Substituting values, we get

$$5 = \frac{\mu \times 20 \times 10^3}{r_p + 20 \times 10^3}$$

$$6 = \frac{\mu \times 40 \times 10^3}{r_p + 40 \times 10^3}$$

Dividing the two equations, we have

$$\frac{6}{5} = \frac{40}{20} \times \frac{r_p + 20 \times 10^3}{r_p + 40 \times 10^3}$$

or
$$6 r_p + 240 \times 10^3 = 10 r_p + 200 \times 10^3$$

$$r_p = \frac{40}{4} \times 10^3 \Omega = 10 \text{ k}\Omega$$

$$\therefore \mu = \frac{5 \times 30 \times 10^3}{20 \times 10^3} = 7.5$$

But $\mu = r_p g_m$, so that $g_m = \frac{\mu}{r_p}$

$$g_m = \frac{7.5}{10 \times 10^3} \text{ siemen} = 7.5 \times 10^{-4} \text{ siemen}$$

Example 9: The plate current i_b of a triode is given by

$$i_b = (10 e_c + e_b)^2 \text{ for } e_c \leq 0 \text{ and } e_b \geq -10 e_c.$$

where i_b is in μA and e_c and e_b are the values of the grid and plate potential (in volt) respectively. What is the transconductance of the triode when operated at a plate potential of 200 V and a grid potential of -10 V? What would be the output signal if the input signal is 0.5 V rms, the load impedance being 10 k Ω ?

Solution: We have

$$i_b = (10 e_c + e_b)^2 \text{ in } \mu\text{A}$$

$$\text{or } i_b = 10^{-6} \times (10 e_c + e_b)^2 \text{ in } A$$

$$\text{now } g_m = \left(\frac{\partial i_b}{\partial e_c} \right)_{e_b}$$

$$\therefore g_m = 10^{-6} \times 2 (10 e_c + e_b) \times 10 \\ = 20 \times 10^{-6} (10 e_c + e_b)$$

At $e_c = -10$ V and $e_b = 200$ V (these values are consistent with the given conditions, namely, $e_c \leq 0$ and $e_b \geq -10 e_c$), we have

$$g_m = 20 \times 10^{-6} (-100 + 200) = 2 \times 10^{-3} \text{ siemen}$$

To find the gain of the triode, we need its amplification factor which is given by

$$\mu = - \left(\frac{\partial e_b}{\partial e_c} \right)_{i_b}$$

$$0 = 10^{-6} \times 2 (10 e_c + e_b) \left(10 + \frac{\partial e_b}{\partial e_c} \right)$$

$$\text{or } \mu = - \left(\frac{\partial e_b}{\partial e_c} \right)_{i_b} = 10$$

The plate resistance r_p of the triode is now easily obtained. It is given by

$$r_p = \frac{\mu}{g_m} = \frac{10}{2 \times 10^{-3}} \Omega = 5 \times 10^3 \Omega$$

The gain A is given by

$$A = \frac{\mu R_L}{r_p + R_L} = \frac{10 \times 10 \times 10^3}{(5 + 10) \times 10^3} \\ = \frac{100}{15} = 6.67$$

The output voltage corresponding to an input signal voltage of 0.5 V is, therefore,

$$e_o = 0.5 \times 6.67 V_{\text{rms}} = 3.33 V_{\text{rms}}$$

Example 10: A 6J5 triode having an amplification factor of 20 and a plate resistance of 10 k Ω is being used as an amplifier. The 'load' used is an inductance coil of resistance 500 Ω and an inductance of 1.0 mH. What will be the 'gain' of this circuit at an input signal frequency of $0.5 \times (10^6/\pi)$ Hz? Also obtain a formula for the 'varying component' of the plate current in terms of the input signal voltage (e_c) and signal frequency ν .

Solution: We first work out the load impedance R_L at the given frequency. This is given by

$$\begin{aligned} R_L &= \sqrt{r^2 + \omega^2 L^2} \\ &= \left[(500)^2 + \left(2\pi \times \frac{0.5 \times 10^6}{\pi} \times 1 \times 10^{-3} \right)^2 \right]^{1/2} \Omega \\ &= [0.25 \times 10^6 + 10^6]^{1/2} \Omega \\ &= \sqrt{1.25 \times 10^3} \Omega = 1.12 \times 10^3 \Omega \end{aligned}$$

The gain A of the amplifier is given by

$$\begin{aligned} A &= \frac{-\mu}{1 + \frac{r_p}{R_L}} \\ &= \frac{-20}{1 + \left(\frac{10 \times 10^3}{1.12 \times 10^3} \right)} = \frac{-20}{1 + 8.93} = -2.01 \end{aligned}$$

Thus, corresponding to an input voltage e_c , the output voltage across the load will be ($A e_c$). The load impedance being

$$\sqrt{r^2 + \omega^2 L^2} = \sqrt{r^2 + 4\pi^2 \nu^2 L^2},$$

the varying component of the plate current is given by

$$i_p = \frac{A e_c}{(r^2 + 4\pi^2 \nu^2 L^2)^{1/2}}$$

20.5 TRUE-FALSE STATEMENTS WITH REASONS

We give below a few statements. We have to decide whether they are true or false giving, in brief, the reasons in support of our answers.

STATEMENTS

1. The voltage upto which the current in a diode is space charge limited is proportional to the square of the temperature of the filament.
2. The observed slight increase in the plate current with the applied plate potential in the 'saturation region' of a diode can be understood in terms of 'field emission'.
3. The three families of characteristic curves of a triode are the projections of the 'surface'

$$I_b = f(E_b, E_c)$$

on the co-ordinate planes.

4. We cannot calculate all the three tube parameters of a triode by using only one of the three families of the characteristic curves of the triode.

5. A triode cannot be used over high frequencies because of the existence of the inter-electrode capacitance between the plate and the grid.
6. A load in a full wave rectifier is connected between the common junction of the cathodes of the two diodes and one end of the centre tapped power transformer.
7. The grid and plate voltages in a triode are in the opposite phase.
8. The control grid in an amplifier is biased by connecting a resistance R_k in between the cathode and the ground.

ANSWERS

1. *False* The current 'available' in a diode does increase with an increase in the temperature of the filament but the voltage at which saturation 'sets in' is almost independent of the filament temperature.
2. *True* The electrons in the cathode can be 'pulled out' by the field also in addition to those 'thrown out' by heating the cathode. The effect is small and is also known as 'cold-emission'.
3. *True* The plate characteristics, the transfer characteristics and the constant current characteristics of a triode are the projections of the surface $I_b = f(E_b, E_a)$ on the three planes corresponding to $E_c = \text{constant}$, $E_b = \text{constant}$ and $I_b = \text{constant}$ respectively.
4. *False* We can calculate both r_p and μ from the plate characteristics and both g_m and μ from the transfer characteristics. The third tube parameter can then be obtained by using the relation $\mu = r_p g_m$.
5. *True* The impedance of a capacitor goes down at high frequencies. The inter-electrode capacitance between the grid and the plate of a triode can, therefore, act as a 'short' connection between these two electrodes at sufficiently high frequencies.
6. *False* The load is connected between the common junction of the cathodes of the two diodes and the 'centre tap' point of the centre tapped transformer. This ensures that the current in the load is in the same direction for the full cycle of the input AC voltage.
7. *True* The plate-cathode voltage for a triode is $(E_b - I_b R_L)$ where E_b is the 'high tension battery voltage' applied in the plate-cathode circuit. As I_b increases, with an increase in the grid-cathode voltage, there will be a fall in the plate cathode voltage.
8. *False* The biasing is done by having a high value capacitor in parallel with the resistance R_k . This is necessary, as otherwise, the AC component of the plate current will cause a form of 'negative feed back' and the gain of the amplifier will be reduced.

20.6 MULTIPLE-CHOICE QUESTIONS

Choose the correct answer from the given alternatives in each of the following:

1. The 'load' in a triode amplifier is a pure resistance of value R . The relation between the input and output voltages is then represented by the
 - (a) static plate characteristic curves of the triode
 - (b) static transfer characteristic curves of the triode
 - (c) dynamic transfer characteristic curves of the triode that depend on the value of R
 - (d) dynamic transfer characteristic curves of the triode that are independent of the value of R .

2. When an input signal voltage e_s is applied to the grid of a triode, then for the AC component of its plate current, the tube behaves as a
 - (a) voltage generator generating a voltage μe_s and having an internal resistance r_p
 - (b) voltage generator generating a voltage e_s/μ and having an internal resistance r_p
 - (c) current generator generating a current $g_m e_s$ and having an internal resistance r_p/μ
 - (d) current generator generating a current μe_s and having an internal resistance $g_m r_p$
3. The output from a full wave rectifier is
 - (a) an AC voltage
 - (b) a DC voltage
 - (c) zero
 - (d) a pulsating voltage.
4. A half wave rectifier is being used to rectify an alternating voltage of frequency 50 Hz. The number of pulses of rectified current obtained in one second is
 - (a) 50
 - (b) 25
 - (c) 100
 - (d) 200,
5. It is found that the cut-off voltage in a triode is -5 V when the plate voltage is 150 V. If the plate voltage is increased to 200 V, the cut-off voltage will be
 - (a) -5.0 V
 - (b) 6.0 V
 - (c) 6.06 V
 - (d) -6.66 V
6. If the grid of a triode is made more negative, the plate resistance
 - (a) remains unaffected
 - (b) increases
 - (c) decreases
 - (d) may increase or decrease depending on the value of the negative grid voltage.
7. When the plate voltage of a triode is increased from 225 V to 250 V, the control grid voltage must be changed from -4 V to -4.7 V to maintain a constant plate current. The amplification factor of the triode is
 - (a) 35.8
 - (b) 25
 - (c) 70
 - (d) 20
8. In a triode, the change in the plate current is the same whether the anode voltage alone is increased from 60 V to 100 V or the grid voltage alone is increased from -0.5 V to $+2$ V. The amplification factor of the triode is
 - (a) 26.6
 - (b) 16
 - (c) 20
 - (d) 100
9. In a triode amplifier, the load resistance is equal to the plate resistance r_p . If μ is the amplification factor of the triode, the stage gain of the amplifier is
 - (a) $\mu/2$
 - (b) μ
 - (c) 2μ
 - (d) 4μ

20.7 NUMERICAL EXERCISES

1. In a diode, the plate current under space charge limited condition is 120 mA for a plate voltage of 100 V. Find the plate current for a plate voltage of (a) 200 V and (b) 300 V under the same condition.
2. A triode has a mutual conductance of 3 mA V^{-1} and an anode resistance of $14 \text{ k}\Omega$. The valve is used as a voltage amplifier with a load resistance of $20 \text{ k}\Omega$. Calculate the 'stage gain' obtained,

3. A load resistance of $50\text{ k}\Omega$ is used in a triode amplifier. The valve has an amplification factor of 16 and an anode resistance of $30\text{ k}\Omega$. Calculate the rms value of the output voltage when 2.5 V rms is applied between the grid and the cathode.

4. The anode current characteristic of a triode is represented by the expression

$$I_a = 2 \times 10^{-6} (V_a + 10 V_g)^2$$

where I_a is the anode (plate) current in ampere, and V_a and V_g are the anode and control grid voltages respectively. Determine mathematically the value of (a) the amplification factor μ , (b) the mutual conductance g_m and (c) the anode resistance R_a for this triode at the point where $V_a = 200\text{ V}$ and $V_g = -10\text{ V}$.

(AMIE)

5. The slope of the anode current-grid potential curve of a triode is 2 mA V^{-1} and the slope of the anode current-anode potential curve is 0.25 mA V^{-1} . Calculate the voltage amplification the triode will give for an anode resistance of $12,000\Omega$.

(AMIE)

6. A vacuum tube has an amplification factor of 7. If the grid voltage is changed from -12 V to -3 V , what will be the effect on the plate current? Must the plate voltage be increased to regain the original value of the plate current? What is the actual change in the plate voltage required to do this?
7. A tube has an amplification factor of 45. What grid voltage will produce a cut-off if the following plate voltages are applied: (a) 150 V , (b) 200 V and (c) 250 V .
8. The plate resistance of a 6J5 triode is $7.7\text{ k}\Omega$ and the transconductance is 2.6×10^{-3} siemen. If only the plate voltage is increased by 50 V , what happens to the plate current? What change must be brought about in the grid voltage so as to regain the original value of the plate current?
9. The following observations were obtained with a triode valve:

Plate voltage (V)	200	200	250	250	250
Grid voltage (V)	-1	-1.5	-1	-1.8	-2.5
Plate current (mA)	5	2	9.8	5	2

Assuming that the observations are taken in the linear region of the characteristic curves of the triode, obtain the values for the three tube parameters.

10. The following data was obtained for the plate characteristics of a certain triode valve. Here V_g stands for the grid voltage.

Plate Voltage	Plate Current at	
	$V_g = 0\text{ V}$	$V_g = -2\text{ V}$
80 V	8 mA	2.2 mA
100 V	10.9 mA	5.5 mA
120 V	13.8 mA	8.0 mA
140 V	17.0 mA	10.8 mA

Plot the plate characteristics and obtain the transconductance, the plate resistance and the amplification factor of the given triode for a plate voltage of 100 V .

ANSWERS

MULTIPLE CHOICE QUESTIONS

- | | | | |
|--------|--------|--------|--------|
| 1. (c) | 2. (a) | 3. (d) | 4. (a) |
| 5. (d) | 6. (b) | 7. (a) | 8. (b) |
| 9. (a) | | | |

NUMERICAL EXERCISES

- (a) 339.36 mA, (b) 623.54 mA
- 28.4
- 25 V
- (a) $\mu = -10$, (b) $g_m = 4 \times 10^{-3}$ siemen, (c) $R_p = 2500$ ohm
- 6
- increase; no; 63 V decrease
- (a) -3.33 V, (b) -4.44 V, (c) -5.55 V
- increases; 2.5 V more negative
- $r_p = 10.42$ k Ω , $g_m = 6 \times 10^{-3}$ siemen, $\mu = 62.25$
- $g_m = 2.7 \times 10^{-3}$ siemen, $r_p = 6.9 \times 10^3 \Omega$, $\mu = 18.6$

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